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Constrained elasticity

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Meccanica dei continui. — *Constrained Elasticity.* Nota di Paolo Podio-Guidugli, presentata (*) dal Socio G. Grioli.

Abstract. — Some foundational aspects of the constitutive theory of finite elasticity are considered in the case, regarded here as general, when internal kinematical constraints are imposed. The emphasis is on the algebraic-geometric structure induced by constraints. In particular, old and new examples of internal constraints are reviewed, and the material symmetry issue in the presence of constraints is discussed.

Key words: Finite elasticity; Internal constraints; Material symmetries.

Riassunto. — *Elasticità vincolata.* Si espongono alcuni aspetti di fondamento della teoria costitutiva dell'elasticità finita nel caso in cui si impongano vincoli cinematici interni, caso che è qui riguardato come generale. L'attenzione è concentrata sulla struttura algebrico-geometrica indotta dai vincoli. In particolare, si passano in rassegna vecchi e nuovi esempi di vincoli interni e si discute la questione delle simmetrie materiali in presenza di vincoli.

1. Introduction

In a customary presentation, the constitutive theory of elasticity is the study of the mapping

$$\hat{S}: \mathscr{D} \to \mathscr{C}, \qquad S = \hat{S}(F) \tag{1.1}$$

transforming the displacement gradient F from a given reference placement into the stress S.

Typically, one takes the domain $\mathscr{D}$ of the response mapping $\hat{S}$ to be all of Lin^+ (the collection of second-order tensors with positive determinant), and stipulates that the codomain $\mathscr{C}$ coincides with Sym (the space of symmetric tensors), therefore interpreting S as the Cauchy stress. One then proceeds to discuss the two main topics of the constitutive theory, namely, the classification problem, the problem of finding the symmetry group corresponding to a given response mapping, and the representation problem, the problem of finding a representation formula for all response mappings sharing a given symmetry group. Remarkably, the algebraic (group) structure of the domain of the response mapping turns out to play a crucial role in dealing with both problems.

When internal constraints, *i.e.*, limitations on possible displacement gradients, are introduced, so that $\mathscr{D} \subset\subset \mathrm{Lin}^+$, then the geometric (manifold) structure of $\mathscr{D}$ becomes important. In customary presentations, for constrained materials the classification problem is more or less ignored, whereas the representation problem is solved only in the easy case of an incompressible isotropic material, by a straightforward adaptation of the argument used for unconstrained isotropic materials. However, consistency between reasonings based on both the group and the manifold structure of

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$\mathscr{D}$ is in general not at all automatic: for the classification problem, *e.g.*, such consistency must be assured in order to avoid unnatural constructs such as isotropic solids inextensible in certain material directions (1).

It would appear that the qualifier in our title implies that constrained elasticity should be viewed as a special theory, to be deduced from the general unconstrained theory by use of *ad hoc* assumptions and techniques. In sharp contrast with this traditional view, we sketch in sect. 3 a presentation of the constitutive theory of elasticity where the underlying mathematical structure is consistently exploited in full, as the constrained case is regarded as typical. Section 4 is devoted to illustrating the general theory by reviewing many old and new examples of internal constraints, with special attention to their geometrical interpretation. Finally, in sect. 5 we complete and sharpen our description of the stress response by discussing the issue of material symmetries in the presence of internal constraints.

2. Notation

We adopt the notation of [1]. Thus $\mathscr{V}$ is the three-dimensional Euclidean space. We stipulate to consider in $\mathscr{V}$ cartesian orthogonal coordinates. We let Lin = the space of all second-order tensors (linear transformations) on $\mathscr{V}$. As usual, Lin can be made into an inner-product space by defining, for all $A, B \in \text{Lin}$, $A \cdot B = tr(AB^T)$, with tr the trace and B^T the transpose of B. All subspaces of Lin are given the metric induced by the norm associated to this inner product, *i.e.*, $\|A\| := (A \cdot A)^{1/2}$. In particular, we consider $\text{Sym} = \{H \in \text{Lin} | H = H^T\}$, $\text{Dev} = \{H \in \text{Sym} | H \cdot I = 0\}$, where the identity of Lin is denoted by I; we define the subspaces Skw of Lin, and Sph of Sym by

$$\text{Lin} = \text{Sym} \oplus \text{Skw} \quad \text{and} \quad \text{Sym} = \text{Dev} \oplus \text{Sph}, \tag{2.1}$$

respectively; moreover, we let $\text{Pos} = \{H \in \text{Sym} | v \cdot Hv \geqslant 0 \text{ for all } v \in \mathscr{V}\}$, $\text{Rot} = \{R \in \text{Lin}^+ | RR^T = R^T R = I\}$, $\text{Unim} = \{H \in \text{Lin}^+ | \det H = 1\}$.

For $H \in \text{Lin}$, H^*, the cofactor of H, is defined by

$$H^* H^T = H H^{*T} = (\det H)\, I. \tag{2.2}$$

It follows from this definition that

$$H \cdot H^* = 3 \det H \quad \text{for all } H \in \text{Lin}, \tag{2.3}$$

so that, in particular,

$$H \in \text{Unim} \Leftrightarrow H \cdot H^* = 3. \tag{2.4}$$

It also follows from (2.2) that

$$F^* = (\det F)\, F^{-T} \quad \text{for all } F \in \text{Lin}^+. \tag{2.5}$$

(1) The consistency issue of the classification problem is treated at length in [1]; see also [2] and [3].

Finally, the following well-known formulae define uniquely the polar factors of $F \in \mathrm{Lin}^+$:

(2.6) $$V_F := (FF^T)^{1/2}\,; \quad U_F := (F^T F)^{1/2}\,; \quad R_F := V_F^{-1} F = F U_F^{-1}\,.$$

3. The Stress Response

Our delineation of the stress response of an elastic material consists of three assumptions, the third of which we postpone until sect. 5. We begin by assuming that

(**I**) *the stress response is described by a pair* $(\mathcal{A}, \mathcal{R})$ *of set-valued mappings, both defined on* $\mathcal{D}$, *a connected smooth manifold of* Lin$^+$ *such that*

$$(M_1) \quad I \in \mathcal{D}\,; \qquad (M_2) \quad \mathrm{Rot}\,\mathcal{D} = \mathcal{D}\,. \ (^2)$$

Let $\dot{\mathcal{D}}(F)$ *denote the tangent space at* $F \in \mathcal{D}$. *To each* $F \in \mathcal{D}$, *the action mapping* $\mathcal{A}$ *assigns a (non-empty) set* $\mathcal{A}(F) \subset \mathrm{Lin}$, *with*

(3.1) $$\mathcal{A}(F) = \dot{\mathcal{D}}(F)\,F^{-1}\,;$$

the reaction mapping $\mathcal{R}$, *the orthogonal complement of* $\mathcal{A}(F)$ *in* Lin:

(3.2) $$\mathcal{R}(F) = (\dot{\mathcal{D}}(F)\,F^{-1})^{\perp}\,.$$

In view of (M_2), the prescription of $\mathcal{D}$ agrees with the axiom of invariance under change in observer (cf. *e.g.* [4, Chapter VII]). On the other hand, (M_1) is the requirement that the gradient I of the identity displacement from the reference placement is an element of $\mathcal{D}$.

Remark 3.1. At first sight, (M_1) may look as a technical assumption that could have been safely left tacit, but it is not quite like so. To see this, let us suspend it provisionally (as is always done in [3]). Then, if $\mathcal{D} \subset\subset \mathrm{Lin}^+$, (M_2) presents us with the following alternative: either $I \in \mathcal{D}$ and then $\mathrm{Rot} \subset \mathcal{D}$, or $I \notin \mathcal{D}$, and then $\mathrm{Rot} \cap \mathcal{D} = \emptyset$. Choosing the first possibility: (i) allows one to consider $\hat{S}(I)$, the stress in the reference placement, as well as the stress $\hat{S}(R)$ in all placements obtained from the reference one by means of a rotation $R \in \mathrm{Rot}$; (ii) is mandatory whenever, as frequently happens, $\mathcal{D}$ has to have the group structure; (iii) is likewise mandatory if one accepts Noll's description of continuum kinematics [5], where the classes of displacements D and placements P are supposed to be such that, among other things, $\boldsymbol{\kappa} \circ \boldsymbol{\kappa}^{-1} \in D$ for any $\boldsymbol{\kappa} \in P$ (cf. the final note of [1]). □

We further specify the stress response by assuming that

(**II**) *the stress can be split into an active part* S_A *and a reactive part* S_R:

(3.3) $$S = S_A + S_R\,.$$

(²) Thus, $\mathcal{D}$ is a constraint manifold in the sense of [2], provided «smooth» is read as «of class C^1»; to make the smoothness assumption precise is irrelevant to our present developments, so much so that one can safely think of $\mathcal{D}$ as being a C^∞–manifold.

The active stress accounts for the mechanical response to admissible displacements from the reference placement; in the class of materials sharing a given domain $\mathscr{D}$, a particular elastic material is specified by a mapping $\hat{S}$ such that

$$\mathscr{D} \ni F \mapsto S_A = \hat{S}(F) \in \mathscr{A}(F) \cap \mathrm{Sym}. \quad (^3) \tag{3.4}$$

The reactive stress, which accounts for the internal reaction necessary to maintain the kinematical constraints that the displacement gradient F belong to $\mathscr{D}$, is such that

$$S_R \in \mathscr{R}(F) \quad \textit{for all } F \in \mathscr{D}, \tag{3.5}$$

but is otherwise arbitrary.

It is proved in [1] that

$$\dot{\mathscr{D}}(RF) = R\dot{\mathscr{D}}(F) \quad \text{for all } R \in \mathrm{Rot} \quad \text{and} \quad F \in \mathscr{D}; \tag{3.6}$$

$$\mathscr{R}(F) \subset \mathrm{Sym} \quad \text{for all } F \in \mathscr{D}. \tag{3.7}$$

Thus, the reactive stress is symmetric. Using (3.6) and (3.7) one can also show that

$$\mathscr{R}(F) = F(\dot{\mathscr{D}}(F)^{\perp})^T, \tag{3.8}$$

a formula for $\mathscr{R}(F)$ alternative to (3.2) (cf. [3, formula (3.8)]). In addition, it follows from (2.4), (3.2) and (3.5) that

$$\pi_R := S_R F^* \cdot \dot{F} = 0 \quad \text{for all } F \in \mathscr{D} \quad \text{and} \quad \dot{F} \in \dot{\mathscr{D}}(F); \tag{3.9}$$

in words, the stress power π_R associated with the reactive stress vanishes in every possible motion (4).

The material is **constrained** whenever $\dim \mathscr{D} \leq 8$; otherwise, the unconstrained case obtains. If there are no constraints, we have that $\mathscr{A}(F) \equiv \mathrm{Lin}$, $\mathscr{R}(F) \equiv \{0\}$ for all F, and the familiar setting of the unconstrained theory is recovered.

REMARK 3.2. It is often the case that $\mathscr{D}$ is a subgroup of Lin^+ (so that, as it turns out, $\mathscr{D}$ is independent of the reference placement [1]). Then

$$\dot{\mathscr{D}}(F)F^{-1} \equiv \dot{\mathscr{D}}(I) \tag{3.10}$$

and, rather than consisting of a distribution of sets $\mathscr{A}(F) \cap \mathrm{Sym}$ parametrized on F, the codomain $\mathscr{C}$ of $\hat{S}$ is such that

$$\mathscr{C} = \dot{\mathscr{D}}(I) \cap \mathrm{Sym}. \quad \square \tag{3.10}$$

REMARK 3.3. It is worth-noticing that, within the general thermodynamical context considered in [2], our present assumptions on the stress response can be given the status of a proved theorem. $\square$

(3) As anticipated in the Introduction, a constraint manifold $\mathscr{D}$ and a response mapping for the active stress $\hat{S}$ should be compatible, in a sense that we shall make precise in sect. 5 by means of our assumption (**III**).

(4) This is the starting assumption of the mechanical theory of constrained materials proposed by Noll in [6. §30].

4. Examples

A classical border example is provided by **rigid** materials, for which $\mathscr{D} = \mathrm{Rot}$, so that $\dot{\mathscr{D}}(R) = \mathrm{Skw}\, R$, $\mathscr{A}(R) = \mathrm{Skw}$ and $\mathscr{R}(R) = \mathrm{Sym}$ for all $R \in \mathscr{D}$; consequently, (3.8) reduces to $\mathscr{C} = \{0\}$, and the stress is purely reactive. Generalizing slightly, we may consider materials capable only of **conformal** displacements (5), namely, such that $\mathscr{D} = \{F \in \mathrm{Lin}^+ \,|\, F = \alpha R, \text{ with } \alpha \in \mathbb{R}^+ \text{ and } R \in \mathrm{Rot}\}$. It is easily found that $\mathscr{A}(F) \equiv \mathrm{Sph} = \mathscr{C}$ and $\mathscr{R}(F) = \mathrm{Dev}$, so that the active stress is a pressure. Consideration of the conformality constraint puts in a somewhat unusual perspective another classical example, namely, **incompressible** materials (6). Here $\mathscr{D} = \mathrm{Unim}$; it follows that $\dot{\mathscr{D}}(F) = \{\dot{F} \in \mathrm{Lin} | \dot{F} \cdot F^* = 0\}$, $\mathscr{A}(F) \cap \mathrm{Sym} \equiv \mathrm{Dev} = \mathscr{C}$ and $\mathscr{R}(F) \equiv \mathrm{Sph}$, so that now the reactive stress is a pressure.

Rigidity, conformality and incompressibility are internal constraints independent of the reference placement (cf. Remark 3.3). Next, we consider various examples of constraint manifolds that have not the group property.

A material is **inextensible** in the direction e in the reference placement if $\mathscr{D} = \{F \in \mathrm{Lin}^+ \,|\, Fe \cdot Fe = 1\}$; then, $\mathscr{A}(F) = \{A \in \mathrm{Lin} | A \cdot (Fe \otimes Fe) = 0\}$ and, by use of either (3.2) or (3.8), $\mathscr{R}(F) = \mathrm{span}\,\{Fe \otimes Fe\}$, and the reaction is an arbitrary pure traction in the direction Fe (7). A material is capable only of displacements **preserving orthogonality** of the directions e and f of the reference placement if, for e and f orthogonal unit vectors, $\mathscr{D} = \{F \in \mathrm{Lin}^+ \,|\, Fe \cdot Ff = 0\}$. With this constraint, $\mathscr{R}(F) = \mathrm{span}\{Fe \otimes Ff + Ff \otimes Fe\}$, and the reactive stress is an arbitrary pure shear in the plane of Fe and Ff (8).

Incompressibility, inextensibility and preserving ortyhogonality enforce, respectively conservation of volume, arc length and angle. By imposing other, less studied internal constraints, we may require that other quantities of direct geometrical interpretation be displacement invariants. For example, conservation of area for a family of material surfaces follows from the requirement that $\mathscr{D} = \{F \in \mathrm{Lin}^+ \,|\, F^* n \cdot F^* n = 1\}$: only displacements **preserving area** of surfaces in the plane of normal n in the reference placement are allowed. With the formula $\dot{F}^* = \partial_F F^* [\dot{F}] = (\dot{F} \cdot F^*) F^{-T} - F^{-T} \dot{F}^T F^*$, the construction of ($\dot{\mathscr{D}}$, $\mathscr{A}(F)$ and) $\mathscr{R}(F)$ becomes straightforward; one gets

(5) A conformality constraint on microstructural deformations has been used in [7].

(6) As is well-known, an incompressible material is generally simpler in its behavior than corresponding unconstrained materials: on occasions, the incompressibility constraint may allow for explicit solution of conceptually and practically relevant model problems. In particular, incompressible materials play an important role in both the early and current development of a rational theory of finite elasticity. In this connection, a number of pioneering papers by A. Signorini, R. S. Rivlin and J. L. Ericksen must be mentioned; complete references can be found in [6].

(7) The inextensibility constraint can be traced back to the first successful theories of structural mechanics, such as the theory of the catenary and the elastica. Recently, it has been used to idealize the behavior of fiber-reinforced materials.

(8) The constraint of orthogonality preserving is an essential ingredient in the classical theories of plates and shells (recall the so-called Kirchhoff-Love hypotheses; cf. [8]); in a fairly recent paper, Ericksen [9] has used it in modelling phase transformations of certain elastic crystals.

$\mathcal{R}(F) = \text{span}\{\mathrm{I} - \mathrm{F}^*\mathrm{n} \otimes \mathrm{F}^*\mathrm{n}\}$, *i.e.*, the reactive stress is an arbitrary pressure in the plane normal to $F^* n$.

Finally, we list some internal constraints that have recently been proposed, either in the framework of a constitutive theory of constrained materials or for their relevance in certain applications; our main purpose here is to clarify the geometrical interpretation, which is less transparent than for the constraints considered so far.

We begin by writing the inextensibility constraint in the equivalent forms

$$F^T F \cdot e \otimes e = 1 \Leftrightarrow (F^T F - I) \cdot e \otimes e = 0. \tag{4.1}$$

Then we observe that: (i) for a fixed unit vector n, the mean value of $e(\theta) \otimes e(\theta)$, with $e(\theta) \cdot n = 0$, over the interval $[0, 2\pi]$ is the tensor $(I - n \otimes n)$, the orthogonal projection on the plane normal to n; (ii) the mean value of $e \otimes e$ over the unit sphere in $\mathcal{V}$ is the identity tensor I. Now, as for e, f and n an orthogonal triad of unit vectors we have that $I - n \otimes n = e \otimes e + f \otimes f$, the internal constraints expressed by

$$Fe \cdot Fe + Ff \cdot Ff = 1 \quad \text{and} \quad Fe \cdot Fe + Ff \cdot Ff + Fn \cdot Fn = 1, \tag{4.2}$$

can be equivalently written as, respectively,

$$F^T F \cdot (I - e \otimes e) = 2 \Leftrightarrow (F^T F - I) \cdot (I - e \otimes e) = 0, \tag{4.3}$$

and

$$F^T F \cdot I = 3 \Leftrightarrow (F^T F - I) \cdot I = 0. \tag{4.4}$$

Then we conclude, using (i) for $(4.2)_1$ and (ii) for $(4.2)_2$, that both constraints impose **inextensibility in the mean**, the first one in the plane of normal n in the reference placement (9). We leave it to the reader to determine the codomain of the action and reaction mappings for these constraints.

Ericksen uses $(4.4)_1$ in [9] to complete his model of phase transformations for crystals alluded at in footnote 6; he does not bother to find names, but describes this constraint as one of the infinitely many possible extrapolations of the incompressibility constraint of the linear theory of deformations, namely, in our present notations,

$$F \cdot I = 3. \tag{4.5}$$

Remark 4.1. In the nonlinear theory, (4.5) is not acceptable as a prescription of internal constraint, as it is incompatible with the axiom of invariance under change in observer. Indeed, the axiom would require that $(R - I) \cdot F = 0$ for all $R \in \text{Rot}$; but, as Rot spans Lin (cf. [10]), this in turn would imply, absurdly, that $F \equiv 0$. (10) □

Remark 4.2. Another extrapolation of (4.5), different from both $(4.4)_1$, and the

(9) The terminology used in [3] for these constraints is «total inextensibility»; our present geometrical interpretation makes clear why we find this terminology inappropriate.

(10) More generally, it can be shown along the same lines that no constraint equations involving the trace of the displacement gradient can possibly agree with (M_2).

obvious one, which is $(2.4)_2$, is important in idealizing by means of an internal constraint certain experimental findings about large plastic deformations of ordered solids: as Bell authoritatively reports in [11], when plastic flow occurs, it is rather the trace of V_F (or of U_F, which is the same) than the determinant of F to be conserved. Bell then credits Ericksen the suggestion of introducing the constraint manifold

$$\mathscr{D} = \{F \in \mathrm{Lin}^+ \,|\, F \cdot R_F = 3\} \tag{4.6}$$

(here use of (2.6) has been made). We can easily compute the corresponding reactive stress, starting from the observation that $\dot{F} \cdot R_F = [\dot{V}_F R_F + V_F \dot{R}_F] \cdot R_F = \dot{V}_F \cdot I + V_F \cdot R_F \dot{R}_F^T = 0$ because, in view of the constraint equation and (2.6), $\dot{V}_F \cdot I = 0$, $V_F \in \mathrm{Sym}$ and $R_F \dot{R}_F^T \in \mathrm{Skw}$. Thus, we have that $\mathscr{A}(F) = \dot{\mathscr{D}}(F)\, F^{-1} = \{A \in \mathrm{Lin} \,|\, A \cdot R_F F^T = 0\}$; as by definition $V_F \in \mathrm{Sym}$, or rather, $R_F F^T = F R_F^T$, we also have that $\mathscr{R}(F) = \mathrm{span}\,\{F R_F^T\}$ (cf. [11, §5]). □

5. Consistency of Symmetry Groups

Let a constraint manifold $\mathscr{D}$ and a response mapping for the active stress $\hat{S}$ be given.

Adapting a definition of Noll [12], the symmetry group Syg associated with $\hat{S}$ consists of all elements $H \in \mathrm{Unim} \cap \mathscr{D}$ such that

$$\hat{S}(FH) = \hat{S}(F) \qquad \text{for all } F \in \mathscr{D}. \tag{5.1}$$

Similarly, the set Myg of all elements $H \in \mathscr{D}$ such that

$$\mathscr{D} H = \mathscr{D}, \tag{5.2}$$

is the symmetry group associated with $\mathscr{D}$.

Syg is the collection of all displacements from the given reference placement which are mechanically undetectable; Myg is the collection of all displacements leaving $\mathscr{D}$ unaltered. Shortly, we call Syg and Myg the **response** and the **constraint group**, respectively.

Noll [12] has demonstrated the importance for the classification problem of demanding that Syg be included into Unim; later, Gurtin and Williams [13] have supplied thermodynamical arguments supporting Noll's assumption. However, we share with [1] and [3] the view that there are no reasons why Myg should satisfy the same inclusion relationship. Indeed, the direct inspection of $\mathscr{D}$ rather leads to the opposite conclusion, as exemplified in the following remark.

Remark 5.1. Consider the inextensibility constraint. It follows easily from $(4.1)_2$ that all admissible $F \in \mathrm{Lin}^+$ must be such that

$$F^T F = e \otimes e + A, \tag{5.3}$$

with

$$A \in \mathrm{Pos}, \qquad A \cdot e \otimes e = 0, \qquad A \cdot f \otimes f > 0 \qquad \text{for all } f \text{ such that } e \cdot f = 0. \tag{5.4}$$

Now, for each $H \in$ Myg, we have that

$$(H^T H - I) \cdot e \otimes e = 0 \quad \text{and} \quad F^T F \cdot He \otimes He = 1. \tag{5.5}$$

From (5.3) and (5.5) we deduce that

$$(H \cdot e \otimes e)^2 + A \cdot He \otimes He = 1. \tag{5.6}$$

In view of $(5.4)_2$, all $H \in \mathscr{D}$ such that

$$He = \pm e, \tag{5.7}$$

solve (5.6). On the other hand, there cannot be other solutions: if He were not parallel to e, it would follow from (5.4) and the arbitrariness left in the choice of A that the positive number $A \cdot He \otimes He$ could be made arbitrarily large, thus violating (5.6). In conclusion, for inextensibility the constraint group is

$$\text{Myg} = \{H \in \mathscr{D} \,|\, He = \pm e\}, \tag{5.8}$$

a subgroup of Lin that is not included, in Unim [11]. □

A constraint manifold and a response mapping for the active stress must be consistently assigned, in the sense that any mechanically undetectable change in the reference placement should not change the constraint manifold as well. Formally, we complete the specification of the stress response given in sect. 3 by assuming that

(**III**) *the constraint and the response groups are consistent,* i.e.,

$$\text{Myg} \supset \text{Syg}. \tag{5.9}$$

In this connection, given a response mapping for the active stress $\hat{S}$, and therefore given its symmetry group Syg, one may ask what internal constraints are compatible with $\hat{S}$. We close this section by stating two theorems in this vein; the interested reader may consult [1] and [2] for two different proofs of the first theorem; the second one is proved in [1].

THEOREM 5.2. A constrained fluid must be incompressible.

REMARK 5.3. In the light of (5.9), the last result is plausible because, if $\hat{S}$ describes a fluid, by definition Syg = Unim, whereas incompressibility requires that Myg = Unim. □

Recall that a standard result in dealing with the classification problem is that a material is an isotropic solid if Syg = Rot. Our next theorem lists all possible instances of constrained isotropic solids.

THEOREM 5.4. Let Myg ⊃ Rot. Then, one of the following is satisfied: (i) $\mathscr{D}$ is rigid; (ii) $\mathscr{D}$ is conformal; (iii) $\mathscr{D}$ has maximal dimension, *i.e.*, $\dim \mathscr{D} = 8$.

Thus, for a constrained isotropic solid, beside the more or less trivial possibilities

[11] Cf. [3] for a different proof of this result.

(rigidity and conformality) and the obvious one (incompressibility) ([12]), infinitely many other constraints may be considered.

REMARK 5.5. An example of constraint manifold of maximal dimension compatible with both isotropy and solidity is provided by the constraint of inextensibility in the mean specified by either one of (4.4). As a straightforward analysis shows, if F satisfies (4.4), F must be such that

$$F^T F = I + A, \tag{5.10}$$

with

$$A \in \mathrm{Dev}, \quad (I+A) \cdot v \otimes v > 0 \quad \text{for all } v \in \mathcal{V} \quad \text{with } v \neq 0. \tag{5.11}$$

Moreover, for each $H \in \mathrm{Myg}$, we have that

$$H \cdot H = 3 \quad \text{and} \quad 1 = (F^T F - I) \cdot HH^T. \tag{5.12}$$

It follows that $A \cdot HH^T = 0$ for all $A \in \mathrm{Dev}$, or rather

$$HH^T = \alpha I \quad \text{for some } \alpha \in \mathbb{R}. \tag{5.13}$$

Using (5.12), we deduce from (5.13) that $\alpha = 1$; thus, we conclude that the constraint group Myg equals Rot (cf. also [3]). Curiously, the case of an isotropic solid inextensible in the mean as prescribed by (4.4) parallels the case of an incompressible fluid, in that for both materials Myg = Syg. □

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([12]) Notice that incompressibility is the only possible constraint of maximal dimension if the manifold $\mathcal{D}$ has the group property (cf. [1]).

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