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Meccanica dei solidi. — *Unconditionally stable mid-point time integration in elastic-plastic dynamics.* Nota di ALBERTO CORIGLIANO e UMBERTO PEREGO, presentata (*) dal Corrisp. G. MAIER.

ABSTRACT. — The dynamic analysis of elastoplastic systems discretized by finite elements is dealt with. The material behaviour is described by a rather general internal variable model. The unknown fields are modelled in terms of suitable variables, generalized in Prager's sense. Time integrations are carried out by means of a generalized mid-point rule. The resulting nonlinear equations expressing dynamic equilibrium of the finite step problem are solved by means of a Newton-Raphson iterative scheme. The unconditional stability of the adopted integration method is proved according to two nonlinear stability criteria.

KEY WORDS: Dynamics; Time integration; Elastoplasticity; Stability.

RIASSUNTO. — *Integrazione nel tempo incondizionatamente stabile con il metodo del punto medio generalizzato in dinamica non lineare.* Si considera l'analisi dinamica di sistemi elastoplastici discretizzati ad elementi finiti. Il comportamento del materiale è descritto da un modello alquanto generale a variabili interne. I campi incogniti sono modellati in funzione di opportune variabili, generalizzate nel senso di Prager. Le integrazioni nel tempo sono effettuate per incrementi finiti adottando il metodo detto del *punto medio generalizzato*. Le equazioni non lineari di equilibrio dinamico che così si formulano vengono risolte per mezzo di uno schema iterativo tipo Newton-Raphson. Si dimostra che il metodo di integrazione adottato è incondizionatamente stabile secondo due criteri di stabilità validi in campo non lineare.

1. INTRODUCTION

This paper is concerned with the small displacement dynamic analysis of elastoplastic systems discretized by finite elements. The material is assumed to be an elastoplastic *standard material* as defined in [1, 2]. According to this definition, the hardening behaviour is described by means of a suitable set of conjugate static and kinematic internal variables. The existence of a convex potential of the kinematic variables is also postulated. The governing relations are formulated in terms of *generalized variables*. The use of these variables presents some computational advantages which have been discussed *e.g.* in [3-5].

In nonlinear dynamics, the stability of the algorithm used to perform numerical time integrations is a crucial issue. Several stability criteria have been proposed in the literature [6-9]. The most appropriate ones appear to be related to the notion of *stability in energy*. Simo and Govindjee [10] recently considered the time integration of elastoplastic constitutive laws. Starting from the contractive nature of the relations which govern the continuum problem, they proposed a stability condition resting on the contractivity of a suitable energy norm. They also showed how a generalized mid-point time integration can satisfy this condition.

(*) Nella seduta del 14 giugno 1990.

The same mid-point integration scheme has been adopted by Simo and Wong [11] in rigid body dynamics.

In the present paper, the generalized mid-point rule is adopted for the integration of both the elastoplastic constitutive relations and the equation of motion. The resulting algorithm turns out to satisfy, for a certain choice of a parameter, a stability condition which appears to be the natural extension to dynamics of the one proposed in [10]. The generalized mid-point rule is also shown to comply with the stability in energy criterion employed in [8].

2. Formulation of the elastic-plastic dynamic problem in generalized variables

Consider a finite element model of the system to be analyzed. Let all fields be modelled over each element by making use of Prager's notion of generalized variables. These are variables which govern the interpolation of each field and are endowed with the following noteworthy properties: *i*) each generalized variable can be interpreted as a weighted average over the relevant element of the corresponding field; *ii*) the scalar product of dual static and kinematic generalized variable vectors equals the integral over the element of the scalar product of the corresponding fields. In the present paper, the modelled displacements are taken as compatible interpolations of nodal values. The other kinematic fields are approximated by interpolating values at the Gauss points used for the numerical integrations over the element.

The material behaviour is described in an average sense by relations which involve generalized variables pertaining to the whole discretized system. The following constitutive relations express the local elastoplastic constitutive behaviour in an average sense over finite elements, for all elements simultaneously:

$$\boldsymbol{\varepsilon} = \boldsymbol{e} + \boldsymbol{p}, \tag{1}$$

$$\boldsymbol{\sigma} = \partial U(\boldsymbol{e})/\partial \boldsymbol{e}, \qquad \boldsymbol{\chi} = \partial W(\boldsymbol{\eta})/\partial \boldsymbol{\eta}, \tag{2}$$

$$\dot{\boldsymbol{p}} = (\partial \boldsymbol{\phi}^T(\boldsymbol{\sigma},\boldsymbol{\chi})/\partial \boldsymbol{\sigma})\,\dot{\boldsymbol{\lambda}}, \qquad \dot{\boldsymbol{\eta}} = -(\partial \boldsymbol{\phi}^T(\boldsymbol{\sigma},\boldsymbol{\chi})/\partial \boldsymbol{\chi})\,\dot{\boldsymbol{\lambda}}, \tag{3}$$

$$\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{\sigma},\boldsymbol{\chi}) \leqslant \mathbf{0}, \qquad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \qquad \boldsymbol{\phi}^T \dot{\boldsymbol{\lambda}} = 0, \tag{4}$$

$$\dot{D} = \boldsymbol{\sigma}^T \dot{\boldsymbol{p}} - \boldsymbol{\chi}^T \dot{\boldsymbol{\eta}} \geqslant 0. \tag{5}$$

In eq. (1) $\boldsymbol{\varepsilon}$, $\boldsymbol{e}$ and $\boldsymbol{p}$ denote total, elastic and plastic strains respectively (the term *generalized* will henceforth be dropped for brevity). In eqs. (2) $\boldsymbol{\sigma}$ are stresses, while $\boldsymbol{\chi}$ and $\boldsymbol{\eta}$ are conjugate static and kinematic internal variables respectively; U is the elastic strain energy and W is the stored strain energy due to structural rearrangements at the microscale. The sum $H \equiv U + W$ represents the Helmholtz free energy. The evolution equations for $\boldsymbol{p}$ and $\boldsymbol{\eta}$ are given in (3), where $\boldsymbol{\phi}$ denotes the vector of continuously differentiable yield functions and $\dot{\boldsymbol{\lambda}}$ denotes the vector of plastic multipliers. The rates of $\boldsymbol{p}$ and $\boldsymbol{\eta}$ are assumed to be normal to the yield surface. In correspondence to a corner they are contained in the cone of the outward normals to that point. The loading-unloading conditions are enforced in (4). Finally eq. (5) defines the nonnegative rate of dissipation.

As it is discussed in detail in [5], under mild restrictions, the above defined relations preserve the essential properties (*e.g.*, sign definiteness, normality and complementarity) characterizing the local material constitutive law. Hence, if U is strictly convex and W and each component of $\boldsymbol{\phi}$ are convex functions of their arguments, the relations (1)-(5) define the generalized variable version of the so called *standard elastoplastic material* as specified in [1]. For the subsequent developments, reference will be made to a standard material with quadratic Helmholtz free energy, *i.e.*

(6) $$H = (\boldsymbol{e}^T \boldsymbol{C} \boldsymbol{e} + \boldsymbol{\eta}^T \boldsymbol{A} \boldsymbol{\eta})/2.$$

The convexity assumptions on U and W entail positive definiteness of $\boldsymbol{c}$ and positive semidefiniteness of $\boldsymbol{A}$.

For convenience of notation, use will be made of the following symbols (see [1]):

(7) $$\boldsymbol{E} \equiv \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{0} \end{Bmatrix}, \quad \boldsymbol{\Theta} \equiv \begin{Bmatrix} \boldsymbol{e} \\ \boldsymbol{\eta} \end{Bmatrix}, \quad \boldsymbol{\Gamma} \equiv \begin{Bmatrix} \boldsymbol{p} \\ -\boldsymbol{\eta} \end{Bmatrix}, \quad \boldsymbol{\Sigma} \equiv \begin{Bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\chi} \end{Bmatrix},$$

(8) $$\boldsymbol{G} \equiv \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}.$$

The global vectors $\boldsymbol{\Sigma}$, $\boldsymbol{E}$, $\boldsymbol{\Theta}$ and $\boldsymbol{\Gamma}$ of static and kinematic quantities will be briefly referred to as stress and total, elastic and plastic strains respectively. Accounting for assumption (6) and making use of the definitions (7) and (8), the constitutive relations (1)-(5) can be given the more compact form:

(9) $$\boldsymbol{E} = \boldsymbol{\Theta} + \boldsymbol{\Gamma},$$

(10) $$\boldsymbol{\Sigma} = \boldsymbol{G}\boldsymbol{\Theta},$$

(11) $$\dot{\boldsymbol{\Gamma}} = (\partial \boldsymbol{\phi}^T(\boldsymbol{\Sigma})/\partial \boldsymbol{\Sigma})\,\dot{\boldsymbol{\lambda}}$$

(12) $$\boldsymbol{\phi}(\boldsymbol{\Sigma}) \leqslant \boldsymbol{0}, \qquad \dot{\boldsymbol{\lambda}} \geqslant \boldsymbol{0}, \qquad \boldsymbol{\phi}^T \dot{\boldsymbol{\lambda}} = 0,$$

(13) $$\dot{D} = \boldsymbol{\Sigma}^T \dot{\boldsymbol{\Gamma}} \geqslant 0.$$

The convexity and differentiability of $\boldsymbol{\phi}$ imply that, for any $\boldsymbol{\Sigma}^1$ and $\boldsymbol{\Sigma}^2$,

(14) $$\boldsymbol{\phi}(\boldsymbol{\Sigma}^1) - \boldsymbol{\phi}(\boldsymbol{\Sigma}^2) \geqslant \left.\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\Sigma}^T}\right|_{\boldsymbol{\Sigma}^2} (\boldsymbol{\Sigma}^1 - \boldsymbol{\Sigma}^2).$$

Let now $\boldsymbol{\Sigma}^2$ be a point on the surface $\boldsymbol{\phi}(\boldsymbol{\Sigma}) = \boldsymbol{0}$ and let $\boldsymbol{\Sigma}^1$ be any vector such that $\boldsymbol{\phi}(\boldsymbol{\Sigma}^1) \leqslant \boldsymbol{0}$. From (11), (12) and (14) it follows that:

(15) $$(\boldsymbol{\Sigma}^2 - \boldsymbol{\Sigma}^1)^T \dot{\boldsymbol{\Gamma}}^2 \geqslant \boldsymbol{0}.$$

Inequality (15) can be regarded as a manifestation of the principle of maximum dissipation. Furthermore, if the origin belongs to the elastic domain ($\phi(\boldsymbol{0}) \leqslant \boldsymbol{0}$), (15) also implies positiveness of the dissipation rate (13).

Let the displacements $\boldsymbol{u}$ denote the dynamical degrees of freedom of the discretized system. The small strain compatibility equations and the equations of motion take the form

$$E(t) = \boldsymbol{B}\boldsymbol{u}(t), \tag{16}$$

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{V}\dot{\boldsymbol{u}}(t) + \boldsymbol{B}^T \boldsymbol{\Sigma}(t) = \boldsymbol{F}(t), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \dot{\boldsymbol{u}}(0) = \dot{\boldsymbol{u}}_0. \tag{17}$$

In eqs. (16) and (17), $\boldsymbol{M}$ is the mass matrix and $\boldsymbol{V}$ is the viscous damping matrix. Both $\boldsymbol{M}$ and $\boldsymbol{V}$ are symmetric, $\boldsymbol{M}$ is positive definite and $\boldsymbol{V}$ is positive semidefinite. $\boldsymbol{F}$ denotes the vector of equivalent nodal forces and $\boldsymbol{B}$ is defined as

$$\boldsymbol{B} \equiv \begin{bmatrix} \hat{\boldsymbol{B}} \\ \boldsymbol{0} \end{bmatrix}, \tag{18}$$

where $\hat{\boldsymbol{B}}$ is the linear compatibility operator such that $\boldsymbol{\varepsilon} = \hat{\boldsymbol{B}}\boldsymbol{u}$.

3. Decay of perturbation energy in the elastic plastic dynamic problem

Consider the discretized structural system defined in the previous section and subject to a given history of external actions $\boldsymbol{F}(t)$. At time $t = 0$, let $\{\boldsymbol{u}_0^1, \dot{\boldsymbol{u}}_0^1\}$ and $\{\boldsymbol{u}_0^2, \dot{\boldsymbol{u}}_0^2\}$ be two different sets of initial conditions. Let $\boldsymbol{\Sigma}_0^1$ and $\boldsymbol{\Sigma}_0^2$ be the corresponding initial states of stress. At time $t = \tau$ the stress distribution $\boldsymbol{\Sigma}_\tau^1$ is in equilibrium with $\boldsymbol{F}_\tau - \boldsymbol{M}\ddot{\boldsymbol{u}}_\tau^1 - \boldsymbol{V}\dot{\boldsymbol{u}}_\tau^1$. Analogously, $\boldsymbol{\Sigma}_\tau^2$ is in equilibrium with $\boldsymbol{F}_\tau - \boldsymbol{M}\ddot{\boldsymbol{u}}_\tau^2 - \boldsymbol{V}\dot{\boldsymbol{u}}_\tau^2$. The application of the virtual work principle yields:

$$\begin{aligned} -(\boldsymbol{\Sigma}_\tau^1 - \boldsymbol{\Sigma}_\tau^2)^T (\dot{\boldsymbol{\Gamma}}_\tau^1 - \dot{\boldsymbol{\Gamma}}_\tau^2) = (\boldsymbol{\Sigma}_\tau^1 - \boldsymbol{\Sigma}_\tau^2)^T (\dot{\boldsymbol{\Theta}}_\tau^1 - \dot{\boldsymbol{\Theta}}_\tau^2) + \\ +(\ddot{\boldsymbol{u}}_\tau^1 - \ddot{\boldsymbol{u}}_\tau^2)^T \boldsymbol{M}(\dot{\boldsymbol{u}}_\tau^1 - \dot{\boldsymbol{u}}_\tau^2) + (\dot{\boldsymbol{u}}_\tau^1 - \dot{\boldsymbol{u}}_\tau^2)^T \boldsymbol{V}(\dot{\boldsymbol{u}}_\tau^1 - \dot{\boldsymbol{u}}_\tau^2), \end{aligned} \tag{19}$$

where $\dot{\boldsymbol{E}}_\tau = \dot{\boldsymbol{\Theta}}_\tau + \dot{\boldsymbol{\Gamma}}_\tau$ is compatible with $\dot{\boldsymbol{u}}_\tau$. The *l.h.s.* of eq. (19) is nonpositive, as it can be shown by noting that $\boldsymbol{\Sigma}_\tau^1$ and $\boldsymbol{\Sigma}_\tau^2$ both satisfy condition (12*a*) and by applying twice inequality (15) with reversed indices. The last term on the *r.h.s.* is nonnegative due to the assumed semi-positiveness of $\boldsymbol{V}$. Therefore, taking into account eq. (10), eq. (19) gives rise to the following inequality:

$$\frac{d}{dt}[(\boldsymbol{\Theta}_\tau^1 - \boldsymbol{\Theta}_\tau^2)^T \boldsymbol{G}(\boldsymbol{\Theta}_\tau^1 - \boldsymbol{\Theta}_\tau^2)/2 + (\dot{\boldsymbol{u}}_\tau^1 - \dot{\boldsymbol{u}}_\tau^2)^T \boldsymbol{M}(\dot{\boldsymbol{u}}_\tau^1 - \dot{\boldsymbol{u}}_\tau^2)/2] \leqslant 0. \tag{20}$$

The term in brackets represents the total energy of the motion $[\boldsymbol{u}^1(t) - \boldsymbol{u}^2(t)]$, *i.e.* the sum of its Helmholtz free energy and kinetic energy.

Remarks

i) Inequality (20) can be interpreted as a manifestation of the dissipative nature of the problem. Due to plastic dissipation and damping, the energy associated to the (free) motion $[\boldsymbol{u}^1(t) - \boldsymbol{u}^2(t)]$ decays with time.

ii) Let $\boldsymbol{\Sigma}^2(t)$ be interpreted as the response of the system to perturbed initial conditions $\{\boldsymbol{u}_0^1 + (\boldsymbol{u}_0^2 - \boldsymbol{u}_0^1), \dot{\boldsymbol{u}}_0^1 + (\dot{\boldsymbol{u}}_0^2 - \dot{\boldsymbol{u}}_0^1)\}$. Accordingly, let the term in brackets on the *l.h.s.* of (20) be interpreted as a measure of the consequent perturbation of the response at $t = \tau$. Then inequality (20) can be thought of as expressing the continuum dependence of the motion on the initial conditions.

iii) Inequality (20) represents the natural extension to dynamics of the similar property proved by Nguyen [12] in statics with reference to the same class of standard materials. Simo and Govindjee [10] employed the corresponding property at constitutive level as the basis for the definition of nonlinear stability of midpoint algorithms. In the next section the same concept will be applied to the present more general context.

4. Unconditional stability of mid-point time integration

Let $\overline{0\ T}$ be the time interval over which the dynamic response of the discrete structural system to the given initial conditions and history of external actions is to be computed. Select a suitable number N of instants t_0, t_1, $t_2 \dots t_n$, $t_{n+1} = t_n + \Delta t, \dots t_N = T$. Let subscripts n and $n+1$ denote values of variables at $t = t_n$ and $t = t_{n+1}$ respectively. Focus now on a finite interval $\overline{t_n\ t_{n+1}}$. At t_n the state of the system is assumed to be known. Denote by the symbol Δ the increments of all quantities over the considered time step. As usual, the problem of computing the response of the system to the given increments of external actions $\Delta \boldsymbol{F}$, is tackled by adopting an approximate time integration scheme. Use is made here of the mid-point rule in the form recently put forward in [10] in a different context (in statics and at a constitutive level only).

In general terms, the mid-point scheme can be summarized as follows. Given a system of first-order ordinary differential equations of the type (y_n and $\dot{y}_n$ being initial conditions at time t_n):

$$\dot{y} = f(y, t), \qquad y(t_n) = y_n, \qquad \dot{y}(t_n) = \dot{y}_n, \tag{21}$$

the unknown value y_{n+1} is obtained in the following approximated implicit form:

$$y_{n+1} = y_n + \Delta t f(y_{n+\alpha}, t_{n+\alpha}), \tag{22}$$

having set:

$$y_{n+\alpha} \equiv (1-\alpha) y_n + \alpha y_{n+1}, \qquad t_{n+\alpha} \equiv (1-\alpha) t_n + \alpha t_{n+1}, \qquad 0 \leqslant \alpha \leqslant 1. \tag{23}$$

The application of the mid-point rule to the relations which govern the elastoplastic dynamic problem eqs. (9)-(12) and eqs. (16), (17), leads to the formulation of the following algebraic problem:

$$\boldsymbol{\Sigma}_{n+1} = \boldsymbol{\Sigma}_n + \boldsymbol{G}(\Delta \boldsymbol{E} - \Delta \boldsymbol{\Gamma}), \tag{24}$$

$$\Delta \boldsymbol{\Gamma} = \left. \frac{\partial \boldsymbol{\phi}^T}{\partial \boldsymbol{\Sigma}} \right|_{\boldsymbol{\Sigma}_{n+\alpha}} \Delta \boldsymbol{\lambda}, \tag{25}$$

$$\boldsymbol{\phi}(\boldsymbol{\Sigma}_{n+\alpha}) \leqslant \boldsymbol{0}, \qquad \Delta \boldsymbol{\lambda} \geqslant \boldsymbol{0}, \qquad \boldsymbol{\phi}^T (\boldsymbol{\Sigma}_{n+\alpha}) \Delta \boldsymbol{\lambda} = 0, \tag{26}$$

$$\Delta \boldsymbol{E} = \boldsymbol{B} \Delta \boldsymbol{u}, \tag{27}$$

$$\dot{\boldsymbol{u}}_{n+1} = \dot{\boldsymbol{u}}_n + \Delta t \ddot{\boldsymbol{u}}_{n+\alpha}, \qquad \boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \Delta t \dot{\boldsymbol{u}}_{n+\alpha}, \tag{28}$$

$$\boldsymbol{M} \ddot{\boldsymbol{u}}_{n+\alpha} + \boldsymbol{V} \dot{\boldsymbol{u}}_{n+\alpha} + \boldsymbol{B}^T \boldsymbol{\Sigma}_{n+\alpha} = \boldsymbol{F}(t_{n+\alpha}). \tag{29}$$

Here all quantities with the subscript $n+\alpha$ are defined according to eqs. (23). Notice that $\boldsymbol{F}(t_{n+\alpha}) \neq \boldsymbol{F}_{n+\alpha}$ unless $\boldsymbol{F}$ is linear in time. It is also worth noting that normality, eq. (25), plastic consistency, eq. (26*a*), and complementarity, eq. (26*c*), are enforced at

$t_{n+\alpha}$ [10]. By making use of these conditions and of the convexity of $\boldsymbol{\phi}$ at $\boldsymbol{\Sigma}_{n+\alpha}$, the finite step counterpart of inequality (15) reads:

$$(\boldsymbol{\Sigma}^2_{n+\alpha} - \boldsymbol{\Sigma}^1_{n+\alpha})^T \, \Delta\boldsymbol{\Gamma}^2 \geq \mathbf{0}, \tag{30}$$

for each $\boldsymbol{\Sigma}^1_{n+\alpha}$ such that $\boldsymbol{\phi}(\boldsymbol{\Sigma}^1_{n+\alpha}) \leq \mathbf{0}$.

In eq. (29), which consistently enforces equilibrium at $t_{n+\alpha}$, $\boldsymbol{\Sigma}_{n+\alpha}$ is to be intended as nonlinearly dependent on displacements through the constitutive and compatibility relations. To solve the resulting system of nonlinear equations, first rewrite eqs. (28) and recall the definition of $\boldsymbol{\Sigma}_{n+\alpha}$:

$$\ddot{\boldsymbol{u}}_{n+\alpha} = \alpha^{-1}\Delta\boldsymbol{u}/\Delta t^2 - \alpha^{-1}\dot{\boldsymbol{u}}_n/\Delta t, \qquad \dot{\boldsymbol{u}}_{n+\alpha} = \Delta\boldsymbol{u}/\Delta t, \qquad \alpha \neq 0, \tag{31}$$

$$\boldsymbol{\Sigma}_{n+\alpha} = \boldsymbol{\Sigma}_n + \alpha\Delta\boldsymbol{\Sigma}. \tag{32}$$

Substitute now eqs. (31) and (32) into eq. (29), rearrange and adopt a Newton-Raphson iterative scheme for numerical solution. In this way, at each iteration one is led to solve the following linear algebraic system:

$$(\alpha^{-1}\boldsymbol{M}/\Delta t^2 + \boldsymbol{V}/\Delta t + \alpha\boldsymbol{K}_T)\,(\Delta\boldsymbol{u}^{i+1} - \Delta\boldsymbol{u}^i) = \boldsymbol{F}^i_{err}. \tag{33}$$

Here i is an iteration index and we have set:

$$\boldsymbol{K}_T \equiv \boldsymbol{B}^T \left.\frac{\partial\Delta\boldsymbol{\Sigma}}{\partial\Delta\boldsymbol{E}}\right|_{\Delta\boldsymbol{E}^i} \boldsymbol{B}, \tag{34}$$

$$\boldsymbol{F}^i_{err} \equiv (\alpha^{-1}\boldsymbol{M}/\Delta t^2 + \boldsymbol{V}/\Delta t)\,\Delta\boldsymbol{u}^i + \boldsymbol{B}^T\boldsymbol{\Sigma}^i_{n+\alpha} - (\boldsymbol{F}(t_{n+\alpha}) + \alpha^{-1}\boldsymbol{M}/\Delta t\,\dot{\boldsymbol{u}}_n). \tag{35}$$

The structure tangent matrix $\boldsymbol{K}_T$ defined in eq. (34) has to be computed directly from the integrated constitutive relations (24)-(26). A detailed derivation can be found in [10]. A new estimate $\Delta\boldsymbol{u}^{i+1}$ of the displacement increment can be obtained from eq. (33). Equations (24)-(27) are then solved with respect to the stress and plastic strain increments for the given increment $\Delta\boldsymbol{u}^{i+1}$. In the rest of the paper, it will be assumed that the iterative procedure is always convergent with the desired accuracy, so that equilibrium can be considered as exactly satisfied at $t_{n+\alpha}$.

The stability property (20) which has been shown to hold for the continuum (in time) problem, can be reformulated in the present finite-step context as follows:

$$[(\boldsymbol{\Theta}^{1-2}_{n+1})^T\,\boldsymbol{G}\boldsymbol{\Theta}^{1-2}_{n+1} + (\dot{\boldsymbol{u}}^{1-2}_{n+1})^T\,\boldsymbol{M}\dot{\boldsymbol{u}}^{1-2}_{n+1}]/2 \leq [(\boldsymbol{\Theta}^{1-2}_n)^T\,\boldsymbol{G}\boldsymbol{\Theta}^{1-2}_n + (\dot{\boldsymbol{u}}^{1-2}_n)^T\,\boldsymbol{M}\dot{\boldsymbol{u}}^{1-2}_n]/2. \tag{36}$$

In eq. (36), the superscript $1-2$ means difference between the relevant quantities in the states 1 and 2, *e.g.*,

$$\boldsymbol{\Theta}^{1-2}_{n+1} = \boldsymbol{\Theta}^1_{n+1} - \boldsymbol{\Theta}^2_{n+1}. \tag{37}$$

As in the previous section, the states 1 and 2 correspond to different initial conditions at time t_n.

The desirable property (36) is shown below to be satisfied for $1/2 \leq \alpha \leq 1$ by the time integration scheme described in (24)-(29). By enforcing equilibrium at $t_{n+\alpha}$ as in eq. (29), $\boldsymbol{\Sigma}^{1-2}_{n+\alpha}$ is in equilibrium with $-[\boldsymbol{M}\ddot{\boldsymbol{u}}^{1-2}_{n+\alpha} + \boldsymbol{V}\dot{\boldsymbol{u}}^{1-2}_{n+\alpha}]$. From eq. (27), it turns out that the difference between the strain increments $\Delta\boldsymbol{E}^{1-2} = \Delta\boldsymbol{E}^1 - \Delta\boldsymbol{E}^2 = \boldsymbol{E}^{1-2}_{n+1} - \boldsymbol{E}^{1-2}_n$ is compatible with homogeneous displacement boundary conditions. Therefore, the virtual work principle yields:

$$(38) \qquad -(\boldsymbol{\Sigma}_{n+\alpha}^{1-2})^T \Delta\boldsymbol{\Gamma}^{1-2} = (\boldsymbol{\Sigma}_{n+\alpha}^{1-2})^T \Delta\boldsymbol{\Theta}^{1-2} + (\ddot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M}\Delta\boldsymbol{u}^{1-2} + (\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{V}\Delta\boldsymbol{u}^{1-2}.$$

By adding to the *r.h.s.* the zero term $[(\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M}\Delta\dot{\boldsymbol{u}}^{1-2} - (\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M}\Delta\dot{\boldsymbol{u}}^{1-2}]$ and by noticing that the *l.h.s.* is nonpositive due to inequality (30) applied twice, one obtains:

$$(39) \qquad (\boldsymbol{\Sigma}_{n+\alpha}^{1-2})^T \Delta\boldsymbol{\Theta}^{1-2} + (\ddot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M}\Delta\boldsymbol{u}^{1-2} + (\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M}\Delta\dot{\boldsymbol{u}}^{1-2} - (\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M}\Delta\dot{\boldsymbol{u}}^{1-2} + (\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{V}\Delta\boldsymbol{u}^{1-2} \leqslant \boldsymbol{0}.$$

The following identities hold:

$$(40) \qquad \boldsymbol{\Sigma}_{n+\alpha}^{1-2} = (\boldsymbol{\Sigma}_{n+1}^{1-2} + \boldsymbol{\Sigma}_{n}^{1-2})/2 + (\alpha - 1/2)\,\Delta\boldsymbol{\Sigma}^{1-2},$$

$$(41) \qquad \dot{\boldsymbol{u}}_{n+\alpha}^{1-2} = (\dot{\boldsymbol{u}}_{n+1}^{1-2} + \dot{\boldsymbol{u}}_{n}^{1-2})/2 + (\alpha - 1/2)\,\Delta\dot{\boldsymbol{u}}^{1-2},$$

Substituting eqs. (40) and (41) in eq. (39) gives:

$$(42) \qquad (\boldsymbol{\Sigma}_{n+1}^{1-2} + \boldsymbol{\Sigma}_{n}^{1-2})^T \Delta\boldsymbol{\Theta}^{1-2}/2 + (\ddot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M}\Delta\boldsymbol{u}^{1-2} + (\dot{\boldsymbol{u}}_{n+1}^{1-2} + \dot{\boldsymbol{u}}_{n}^{1-2})^T \boldsymbol{M}\Delta\dot{\boldsymbol{u}}^{1-2}/2 - (\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M}\Delta\dot{\boldsymbol{u}}^{1-2} + (\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{V}\Delta\boldsymbol{u}^{1-2} \leqslant (1/2 - \alpha)\,(\Delta\boldsymbol{\Sigma}^{1-2})^T \Delta\boldsymbol{\Theta}^{1-2} + (1/2 - \alpha)\,(\Delta\dot{\boldsymbol{u}}^{1-2})^T \boldsymbol{M}\Delta\dot{\boldsymbol{u}}^{1-2}.$$

By noting that $(\Delta\boldsymbol{\Sigma}^{1-2})^T \Delta\boldsymbol{\Theta}^{1-2} = (\Delta\boldsymbol{\Theta}^{1-2})^T \boldsymbol{G}\Delta\boldsymbol{\Theta}^{1-2} \geqslant \boldsymbol{0}$, it follows that the *r.h.s.* of inequality (42) is nonpositive for $\alpha \geq 1/2$. Hence, inequality (42) reduces to:

$$(43) \qquad (\boldsymbol{\Theta}_{n+1}^{1-2} + \boldsymbol{\Theta}_{n}^{1-2})^T \boldsymbol{G}\Delta\boldsymbol{\Theta}^{1-2}/2 + (\dot{\boldsymbol{u}}_{n+1}^{1-2} + \dot{\boldsymbol{u}}_{n}^{1-2})^T \boldsymbol{M}\Delta\dot{\boldsymbol{u}}^{1-2}/2 + (\ddot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M}\Delta\boldsymbol{u}^{1-2} - (\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M}\Delta\dot{\boldsymbol{u}}^{1-2} + (\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{V}\Delta\boldsymbol{u}^{1-2} \leqslant \boldsymbol{0}.$$

Accelerations and velocities at $n+\alpha$ can be expressed in terms of their increments over the time step by inverting the midpoint approximations (28):

$$(44) \qquad \ddot{\boldsymbol{u}}_{n+\alpha}^{1-2} = \Delta\dot{\boldsymbol{u}}^{1-2}/\Delta t, \qquad \dot{\boldsymbol{u}}_{n+\alpha}^{1-2} = \Delta\boldsymbol{u}^{1-2}/\Delta t.$$

Substituting eqs. (44) into (43) and taking into account the positive semidefiniteness of $\boldsymbol{V}$ and the symmetry of $\boldsymbol{M}$, one obtains:

$$(45) \qquad (\boldsymbol{\Theta}_{n+1}^{1-2} + \boldsymbol{\Theta}_{n}^{1-2})\, \boldsymbol{G}\Delta\boldsymbol{\Theta}^{1-2}/2 + (\dot{\boldsymbol{u}}_{n+1}^{1-2} + \dot{\boldsymbol{u}}_{n}^{1-2})^T \boldsymbol{M}\Delta\dot{\boldsymbol{u}}^{1-2}/2 \leqslant \boldsymbol{0}.$$

Inequality (45) coincides with (36), as it can be easily recognized by developing the quadratic forms on the *l.h.s.*

Remarks

i) According to eq. (36), the total energy associated to the motion $[\boldsymbol{u}^1(t) - \boldsymbol{u}^2(t)]$ integrated by the mid-point rule, cannot increase over a time step independently of the step size. This can be interpreted as the discrete counterpart of eq. (20) which has been shown to hold for the continuum (in time) process.

ii) Inequality (36) appears to be the dynamic counterpart of the contractivity property proved in [10] at a constitutive level, for statical situations.

iii) The mid-point rule is stable in energy in the sense specified in [8], *i.e.* «the sum of the kinetic and internal energies are bounded within each time step relative to the external work and kinetic and internal energies in the previous time step». The mid-point approximation of the work performed by the external forces over a time step can be expressed in the form:

$$L_e = \int_{t_n}^{t_{n+1}} \boldsymbol{F}^T(\tau)\, \dot{\boldsymbol{u}}(\tau)\, d\tau \cong \boldsymbol{F}^T(t_{n+\alpha})\, \Delta \boldsymbol{u}. \tag{46}$$

By the virtual work principle, it follows that

$$-\boldsymbol{\Sigma}_{n+\alpha}^T \Delta \boldsymbol{\Gamma} = \boldsymbol{\Sigma}_{n+\alpha}^T \Delta \boldsymbol{\Theta} + \ddot{\boldsymbol{u}}_{n+\alpha}^T \boldsymbol{M} \Delta \boldsymbol{u} + \dot{\boldsymbol{u}}_{n+\alpha}^T \boldsymbol{V} \Delta \boldsymbol{u} - \boldsymbol{F}^T(t_{n+\alpha})\, \Delta \boldsymbol{u}. \tag{47}$$

By noting that the *l.h.s.* is nonpositive due to (30) and by following the same path of reasoning as for the proof on inequality (36), the above defined criterion for stability in energy turns out to be satisfied, *i.e.*:

$$\boldsymbol{\Theta}_{n+1}^T \boldsymbol{G} \boldsymbol{\Theta}_{n+1}/2 + \dot{\boldsymbol{u}}_{n+1}^T \boldsymbol{M} \dot{\boldsymbol{u}}_{n+1}/2 \leqslant \boldsymbol{\Theta}_n^T \boldsymbol{G} \boldsymbol{\Theta}_n/2 + \dot{\boldsymbol{u}}_n^T \boldsymbol{M} \dot{\boldsymbol{u}}_n/2 - \boldsymbol{F}^T(t_{n+\alpha})\, \Delta \boldsymbol{u}. \tag{48}$$

iv) By setting $\alpha = 1$, the backward difference scheme commonly used for integrating the constitutive law is recovered. The backward difference time integration leads to the formulation of a stepwise holonomic problem for which several extremum and convergence properties have been proven [5, 13]. Most of these properties can be shown to hold also when the mid-point rule is adopted, both in statics and in dynamics. These results will be presented in a forthcoming paper.

v) By setting $\alpha = 1/2$, the popular *average acceleration method* is recovered. This can also be viewed as a particular case ($\beta = 1/4$, $\gamma = 1/2$) of the more general Newmark's scheme:

$$\dot{\boldsymbol{u}}_{n+1} = \dot{\boldsymbol{u}}_n + \Delta t \ddot{\boldsymbol{u}}_{n+\gamma}\,, \qquad \boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \Delta t(\dot{\boldsymbol{u}}_n + \ddot{\boldsymbol{u}}_{n+2\beta}\, \Delta t/2), \tag{49}$$

where $\ddot{\boldsymbol{u}}_{n+\gamma}$ and $\ddot{\boldsymbol{u}}_{n+2\beta}$ are defined according to eq. (23). The unconditional stability of the mid-point rule for $\alpha \geqslant 1/2$ therefore implies that Newmark's method for $\beta = 1/4$ and $\gamma = 1/2$ is also unconditionally stable.

vi) Inequality (36) has been obtained by enforcing equilibrium at $t_{n+\alpha}$. In practical applications in nonlinear structural dynamics, equilibrium is usually enforced at t_{n+1} (see *e.g.* [7, 8]). In this case, $(\boldsymbol{\Sigma}_{n+1}^{1-2} + \boldsymbol{\Sigma}_n^{1-2})$ represents a state of stress in equilibrium with $-[\boldsymbol{M}(\ddot{\boldsymbol{u}}_{n+1}^{1-2} + \ddot{\boldsymbol{u}}_n^{1-2}) + \boldsymbol{V}(\dot{\boldsymbol{u}}_{n+1}^{1-2} + \dot{\boldsymbol{u}}_n^{1-2})]$. The application of the virtual work principle yields:

$$(\boldsymbol{\Sigma}_{n+1}^{1-2} + \boldsymbol{\Sigma}_n^{1-2})^T \Delta \boldsymbol{E}^{1-2} + (\ddot{\boldsymbol{u}}_{n+1}^{1-2} + \ddot{\boldsymbol{u}}_n^{1-2})^T \boldsymbol{M} \Delta \boldsymbol{u}^{1-2} + (\dot{\boldsymbol{u}}_{n+1}^{1-2} + \dot{\boldsymbol{u}}_n^{1-2})^T \boldsymbol{V} \Delta \boldsymbol{u}^{1-2} = \boldsymbol{0}. \tag{50}$$

By exploiting identities of the type of eqs. (40) and (41) for $\boldsymbol{\Sigma}_{n+\alpha}$, $\dot{\boldsymbol{u}}_{n+\alpha}$ and $\ddot{\boldsymbol{u}}_{n+\alpha}$, eq. (50) can be rewritten as:

$$\begin{aligned} &2[(\boldsymbol{\Sigma}_{n+\alpha}^{1-2})^T \Delta \boldsymbol{E}^{1-2} + (\ddot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{M} \Delta \boldsymbol{u}^{1-2} + (\dot{\boldsymbol{u}}_{n+\alpha}^{1-2})^T \boldsymbol{V} \Delta \boldsymbol{u}^{1-2}] + \\ &\qquad (1-2\alpha)\, [(\Delta \boldsymbol{\Sigma}^{1-2})^T \Delta \boldsymbol{E}^{1-2} + (\Delta \ddot{\boldsymbol{u}}^{1-2})^T \boldsymbol{M} \Delta \boldsymbol{u}^{1-2} + (\Delta \dot{\boldsymbol{u}}^{1-2})^T \boldsymbol{V} \Delta \boldsymbol{u}^{1-2}] = \boldsymbol{0}. \end{aligned} \tag{51}$$

The term in the second brackets vanishes since $\Delta \boldsymbol{\Sigma}^{1-2}$ is in equilibrium with $-[\boldsymbol{M} \Delta \ddot{\boldsymbol{u}}^{1-2} + \boldsymbol{V} \Delta \dot{\boldsymbol{u}}^{1-2}]$. Hence, eq. (51) coincides with eq. (38). As a consequence, the property (36) holds also when equilibrium is enforced at t_{n+1}.

5. Conclusions and future prospects

The finite element dynamic analysis of elastoplastic systems has been dealt with in the following context: *i*) the considered elastoplastic material model with internal variables belongs to a class which is contained in the definition of *standard material* given in [1]; *ii*) the relations governing the spatially discretized problem have been formulated in terms of generalized variables [5]; *iii*) the time integrations have been carried out by applying the generalized mid-point rule as suggested in [10, 11].

In this framework, a Newton-Raphson scheme has been devised in order to solve the nonlinear algebraic equations arising from time discretization. The mid-point rule has been shown to be unconditionally stable in the sense specified in [10], for $1/2 \leqslant \alpha \leqslant 1$, *i.e.*, the total energy associated to the difference between the perturbed and original motions cannot increase over a time step independently of the step size. The integration algorithm has also been shown to satisfy the condition of stability in energy employed *e.g.* in [8]. This circumstance implies that the increment of the total (internal plus kinetic) energies over a time step is bounded by the work performed by the external forces over the same time step.

The finite step elastoplastic dynamic problem as formulated in this paper and integrated by the mid-point rule can be given some extremum characterizations. Convergence criteria can also be established for a modified Newton-Raphson scheme. These aspects will be pursued in a forthcoming paper.

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