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Extension of CR functions to «wedge type» domains

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Equazioni a derivate parziali. — *Extension of CR functions to «wedge type» domains.* Nota di Andrea D'Agnolo, Piero D'Ancona e Giuseppe Zampieri, presentata (*) dal Socio G. Scorza Dragoni.

Abstract. — Let X be a complex manifold, S a generic submanifold of X^R, the real underlying manifold to X. Let Ω be an open subset of S with $\partial\Omega$ analytic, Y a complexification of S. We first recall the notion of Ω-tuboid of X and of Y and then give a relation between; we then give the corresponding result in terms of microfunctions at the boundary. We relate the regularity at the boundary for $\bar\partial_b$ to the extendability of CR functions on Ω to Ω-tuboids of X. Next, if X has complex dimension 2, we give results on extension for some classes of hypersurfaces (which correspond to some $\bar\partial_b$ whose Poisson bracket between real and imaginary part is ≥ 0). The main tools of the proof are the complex $\mathcal{C}_{\Omega|Y}$ by Schapira and the theorem of Ω-regularity of Schapira-Zampieri and Uchida-Zampieri.

Key words: Partial differential equations on manifolds; Several complex variables and analytic spaces; Boundary value problems.

Riassunto. — *Estensione di funzioni CR a domini di tipo «wedge».* Siano X una varietà complessa, S una sottovarietà generica di X^R, Ω un aperto di S, Y una complessificazione di $\partial\Omega$, $\mathcal{O}_X$ le funzioni olomorfe su X, $\mathcal{O}_Y^{\bar\partial_b}$ le soluzioni in $\mathcal{O}_Y$ del sistema di Cauchy-Riemann tangenziale. Si mette in relazione l'estendibilità a domini di tipo «wedge» con base Ω, per funzioni di $\mathcal{O}_X$ e di $\mathcal{O}_Y^{\bar\partial^b}$; ciò collega il microsupporto in $\partial\Omega$ di iperfunzioni $C.R.$ e di soluzioni iperfunzioni di $\bar\partial^b$. Si dà infine un criterio di regolarità al bordo per sistemi $\bar\partial^b$ che assicura la precedente estendibilità. A tal fine si utilizzano i risultati di Schapira-Zampieri e Uchida-Zampieri.

1. The system $\bar\partial_b$

Let X be a complex manifold of complex dimension n, S a real analytic submanifold of X^R of dimension m (X^R being the real underlying manifold to X), Y a complexification of S. Due to the complex structure of X we get a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d] & X \\ Y \arrow[ur, "\tilde\phi"'] & \end{tikzcd}$$

In this article we will assume S to be a generic submanifold of X, *i.e.* $S\times_X TX = TS + {}+_S\sqrt{-1}\,TS$. In particular a hypersurface is always generic.

Remark 1.1. The genericity of S implies that $\tilde\phi$ is smooth. In fact one has: $\tilde\phi'(S\times_Y TY)=\tilde\phi'(TS\oplus_S\sqrt{-1}\,TS)=\tilde\phi'(TS)+_S\sqrt{-1}\,\tilde\phi'(TS)=TS+_S\sqrt{-1}\,TS=S\times_X TX$. Where the third equality follows from $\tilde\phi|_S=\phi$.

Due to Remark 1.1 ${}^t\tilde\phi'(T^*X)=Y\times_X T^*X$ is a sub-bundle of T^*Y.

(*) Nella seduta dal 14 giugno 1990.

One defines $\overline{\partial}_b$ as the system of complex vector fields on Y which annihilate $Y \times_X T^* X$.

Remark 1.2. One has

(1) $\tilde{\phi}^{-1}(\mathcal{O}_X) = \mathcal{O}_Y^{\overline{\partial}_b}$,

(2) $\operatorname{char}(\overline{\partial}_b) = Y \times_X T^* X$.

(Here $\mathcal{O}_Y^{\overline{\partial}_b}$ is the sheaf of germs of holomorphic functions annihilated by $\overline{\partial}_b$.) In fact, according to Remark 1.1 one can take as a system of coordinates in Y $(z_i)_{i=1,\dots,m}$ with $z_i = \tilde{\phi}_i$, $i = 1, \dots, n$. Then clearly $\overline{\partial}_b = (\partial/\partial z_{n+1}, \dots, \partial/\partial z_m)$ and the claim follows. In particular, since TS is preserved by $\tilde{\phi}'$, one has

$$(\operatorname{char}(\overline{\partial}_b)) \cap T_S^* Y \cong T_S^* X. \tag{1.1}$$

2. A brief review on the language of tuboids

Let $S \subset X$ be C^2-manifolds, $\Omega \subset X$ an open set with $N(\Omega) \neq \emptyset$ (here $N(\Omega)$ denotes the normal cone to Ω in S of [4, § 1.2.3]).

Definition 2.1. Let γ be an open convex cone of $\overline{\Omega} \times_S T_S X$. A set $U \subset X$ is said to be an Ω-tuboid of X with profile γ iff $\rho(TX \setminus C(X \setminus U, \overline{\Omega})) \supset \gamma$. (Where ρ: $TX \to T_S X$.)

Remark 2.2. If one chooses a local coordinate system $(x, y) \in X, S = \{(x, y) : y = 0\}$ then U is an Ω-tuboid with profile γ iff for every $\gamma' \subset\subset \gamma$ there exists $\varepsilon = \varepsilon_{\gamma'}$, so that

$$U \supset \{(x, y) \in \Omega \times_S \gamma' : |y| < \varepsilon \operatorname{dist}(x, \partial\Omega) \wedge 1\}.$$

(Here we identify $T_S X \cong X$ in local coordinates.)

3. A link between tuboids in Y and in X

Let S, X, Y be as in § 1, let $\Omega \subset S$ be an open set with analytic boundary.

Our aim is to give a relation between Ω-tuboids in Y and in X.

Let $U \subset X$ be an open set, $\gamma \subset T_S X$, $U' = \tilde{\phi}^{-1}(U) \subset Y$, $\gamma' = \tilde{\phi}'^{-1}(\gamma) \subset T_S Y$ (we still denote by $\tilde{\phi}'$ the induced map $\tilde{\phi}' : T_S Y \to T_S X$).

Lemma 3.1. *The open set U is an Ω-tuboid of X with profile γ iff U' is an Ω-tuboid of Y with profile γ'.*

Proof. Since $\Omega \subset S$, we have $\overline{\Omega} = \tilde{\phi}(\overline{\Omega})$.

If $\rho(TY \setminus C(Y \setminus U', \overline{\Omega})) \supset \gamma'$, then $\rho(TX \setminus C(X \setminus U, \overline{\Omega})) =$
$= \rho(TX \setminus C(X \setminus \tilde{\phi}(U'), \overline{\Omega})) = \rho(\tilde{\phi}'(TY \setminus C(Y \setminus U', \overline{\Omega}))) =$
$= \tilde{\phi}'(\rho(TY \setminus C(Y \setminus U', \overline{\Omega}))) \supset \tilde{\phi}'(\tilde{\phi}'^{-1}(\gamma)) = \gamma$.

If $\rho(TX \setminus C(X \setminus U, \overline{\Omega})) \supset \gamma$, then $\rho(TY \setminus C(Y \setminus U', \overline{\Omega})) =$
$= \rho(TY \setminus C(Y \setminus \tilde{\phi}^{-1}(U), \overline{\Omega})) = \rho(\tilde{\phi}'^{-1}(TX \setminus C(X \setminus U, \overline{\Omega}))) =$
$= \tilde{\phi}'^{-1}(\rho(TX \setminus C(X \setminus U, \overline{\Omega}))) = \tilde{\phi}'^{-1}(\gamma) = \gamma'$. ■

Using this lemma and 1, 2 of Remark 1.2 we can then claim

PROPOSITION 3.2. *Let U be an Ω-tuboid of X with profile γ, $U' = \tilde{\phi}^{-1}(U), \gamma' = \tilde{\phi}'^{-1}(\gamma)$. We have $f \in \mathcal{O}_X(U)$ iff $f \circ \tilde{\phi} \in \mathcal{O}_Y^{\bar{\partial}_b}(U')$.*

4. A MICROLOCAL APPROACH

Let S, X, Y as before, $\Omega \subset S$ an open set with analytic boundary (Ω locally on one side of $\partial\Omega$).

The framework of this paragraph is the microlocal study of sheaves by Kashiwara and Schapira [4].

We will still denote by $\bar{\partial}_b$ the coherent $\mathcal{D}_Y$-module asociated to the system of complex vector fields, *i.e.* $\bar{\partial}_b = \tilde{\phi}^*(\mathcal{D}_X)$.

In [6] Schapira defined the complex of microfunctions at the boundary $\mathcal{C}_{\Omega|Y} = \mu hom(\mathbf{Z}_\Omega, \mathcal{O}_Y) \otimes or_{S|Y}[m]$, similarly we set $\mathcal{C}_{\Omega|X} = \mu hom(\mathbf{Z}_\Omega, \mathcal{O}_X) \otimes or_{S|X}[2n-m]$. To give a relation between $\mathcal{C}_{\Omega|X}$ and $\mathcal{C}_{\Omega|Y}$ we first need to translate in the language of derived categories the results of section 1.

PROPOSITION 4.1. *One has $\tilde{\phi}^{-1}(\mathcal{O}_X) = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\bar{\partial}_b, \mathcal{O}_Y)$.*

PROOF. $\tilde{\phi}^{-1}(\mathcal{O}_X) = \tilde{\phi}^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{O}_X) = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\bar{\partial}_b, \mathcal{O}_Y)$, where the second equality is the Cauchy-Kovalevsky-Kashiwara's theorem which holds since $\tilde{\phi}$ is non-characteristic for $\mathcal{D}_X$. ■

We then have

THEOREM 4.2.

$$\mathcal{C}_{\Omega|X} \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\bar{\partial}_b, \mathcal{C}_{\Omega|Y}). \tag{4.1}$$

PROOF. One has $\mu hom(\mathbf{Z}_\Omega, \mathcal{O}_X) \cong \mu hom(\mathbf{Z}_\Omega, \tilde{\phi}^! \mathcal{O}_X)$ due to [4, Corollary 5.5.6]. Here one notices that both complexes are supported by $Y \times_X T^*X$.

On the other hand by [4, Proposition 1.3.1] $\tilde{\phi}^! \mathcal{O}_X = \tilde{\phi}^{-1}\mathcal{O}_X \otimes or_{Y|X}[2m-2n] = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\bar{\partial}_b, \mathcal{O}_Y) \otimes or_{Y|X}[2m-2n]$, and the claim follows. ■

Next, similarly to the sheaf of Sato's hyperfunctions $\mathcal{B}_S = H^0(\boldsymbol{R}\Gamma_S(\mathcal{O}_Y) \otimes or_{S|Y}[m])$, one sets (*e.g.* cf. [7]) $\mathcal{B}_{S|X} = H^0(\boldsymbol{R}\Gamma_S(\mathcal{O}_X) \otimes or_{S|X}[2n-m])$. Recall that, S being generic, $H^j(\boldsymbol{R}\Gamma_S\mathcal{O}_X) = 0$ $\forall j < 2n-m$, then by applying $\boldsymbol{R}^0\pi_*$ in Theorem 4.2 we get

$$\mathcal{H}om_{\mathcal{D}_Y}(\bar{\partial}_b, \Gamma_\Omega(\mathcal{B}_S)) \cong \Gamma_\Omega(\mathcal{B}_{S|X}). \tag{4.2}$$

Let

$$\alpha\colon \pi^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\bar{\partial}_b, \Gamma_\Omega(\mathcal{B}_S)) \to H^0(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\bar{\partial}_b, \mathcal{C}_{\Omega|Y})), \qquad \beta\colon \pi^{-1}\Gamma_\Omega(\mathcal{B}_{S|X}) \to H^0(\mathcal{C}_{\Omega|X}),$$

be the canonical maps and define

$$SS_{\Omega|Y}^{\bar{\partial}_b, 0}(f) = \operatorname{supp}(\alpha(f)), \quad f \in \mathcal{H}om_{\mathcal{D}_Y}(\bar{\partial}_b, \Gamma_\Omega(\mathcal{B}_S)),$$

$$SS_{\Omega|X}(g) = \operatorname{supp}(\beta(g)), \quad g \in \Gamma_\Omega(\mathcal{B}_{S|X}).$$

COROLLARY 4.3. *Let* $u \in \Gamma_\Omega(\mathcal{B}_{S|X})$ *then* $SS_{\Omega|X}(u) = SS^{\bar\partial_b,0}_{\Omega|Y}(u \circ \tilde\phi)$.

Note that, after [12], there is a tight relation between this Corollary and Proposition 3.2.

REMARK 4.4. Note that $\mathcal{H}om_{\mathcal{D}_Y}(\bar\partial_b, \Gamma_\Omega(\mathcal{B}_S))$ are nothing but the CR functions in Ω (*i.e.* hyperfunction solutions of the system $\bar\partial_b$).

5. THE CASE OF A HYPERSURFACE

Let X, S, Y, Ω as before; from now on assume moreover S being a hypersurface of X^R.

In this case $\dot T_S X$ is the union of two half rays, say $\pm\gamma$; set $\pm\gamma' = \tilde\phi'^{-1}(\pm\gamma)$.

Fix a point $x_0 \in \partial\Omega$ and call $X^\pm$ the two connected components of $X \setminus S$ near x_0.

Let U be a neighborhood of Ω at x_0 and let $f \in \mathcal{O}_X(U \cap X^+)$. In this case, using Proposition 3.2, we then get an equivalent of (4.1), (4.2) without using the results of § 4:

PROPOSITION 5.1. *f extends to an Ω-tuboid of X with profile $\bar\Omega \times_S \gamma$ iff $f \circ \tilde\phi$ extends, as a solution of $\bar\partial_b$, to an Ω-tuboid of Y with profile $\bar\Omega \times_S \gamma'$.*

To prove this statement, recall that, by using [12] we get that f (resp $f \circ \tilde\phi$) extends to a tuboid with profile γ (resp $\gamma' = \tilde\phi'^{-1}\gamma$) iff $\gamma^* \notin SS_{\Omega|X}(b(f))$ (resp $\gamma'^* \notin SS^{\bar\partial_b,0}_{\Omega|Y}(b(f \circ \tilde\phi))$).

In fact the latter is equivalent to $b(f) \in \pi_* \Gamma_{\gamma^{*a}}((\mathcal{C}_{\Omega|X})_{T^*_S X})$ (resp $b(f \circ \tilde\phi) \in \pi_* \Gamma_{\gamma'^{*a}}((\mathcal{C}_{\Omega|Y})_{T^*_S Y})$) (We recall that $H^j(\mathcal{C}_{\Omega|X})_{T^*_S X} = 0$ $\forall j < 0$.)

This last remark, together with Proposition 5.1, gives the following:

$$SS_{\Omega|X}(b(f)) = SS^{\bar\partial_b,0}_{\Omega|Y}(b(f \circ \tilde\phi)). \tag{5.1}$$

We will make use of the following mixed version of (5.1) and Proposition 5.1:

PROPOSITION 5.2. *f extends to a tuboid of X with profile $\bar\Omega \times_S \gamma$ iff* $\gamma'^* \cap SS^{\bar\partial_b,0}_{\Omega|Y}(b(f \circ \tilde\phi)) = \emptyset$.

6. Ω-REGULARITY

Let S be a real analytic manifold, Y a complexification of S, $\Omega \subset S$ an open set with analytic boundary (Ω locally on one side of $\partial\Omega$). Let ω be the canonical 1-form.

We shall endow T^*Y of a real symplectic structure by $\mathrm{Re}\, d\omega$ and $T^*_S Y$ by $\mathrm{Im}\, d\omega$. We shall denote by H^R and H^I the corresponding hamiltonian isomorphisms.

Choose coordinates $(x; \partial/\partial x) \in TS$, and the dual coordinates $(x; \sqrt{-1}\,\eta) \in T^*_S Y$; assume $\Omega = \{x: \varphi(x) > 0\}$.

Take a pseudodifferential operator $P(x; \partial/\partial x) \in (\mathcal{E}_Y)_\lambda$, $\lambda \in \partial\Omega \times_S \dot T^*_S Y$. Set $p = \mathrm{Re}\, \sigma(P)|_{T^*_S Y}$, $q = \mathrm{Im}\, \sigma(P)|_{T^*_S Y}$. We assume that $\{p, \varphi\} \equiv 1$ (and $p(\lambda) = q(\lambda) = \varphi(\lambda) = 0$).

It follows that $dp \wedge d\varphi \wedge \operatorname{Im} \omega \neq 0$ and thus one can divide $q = a + \varphi b$ with $\{p, a\} \equiv 0$.

PROPOSITION 6.1. *Assume that in a neighborhood of* λ:

$$(6.1) \qquad \begin{cases} \{p, \varphi\} \equiv 1, \\ \{\varphi, q\}|_{\{\varphi=0\}} \equiv 0, \\ da \neq 0 \text{ or } da \equiv 0, \\ \{b, a\} \equiv 0. \end{cases}$$

Assume also

$$(6.2) \qquad b \geqslant 0 \quad \text{for } \varphi \geqslant 0.$$

Then P is Ω-regular at λ (i.e.

$$(6.3) \qquad \mathcal{H}om(P, \Gamma_{\dot{\pi}^{-1}(\overline{S \setminus \Omega})} \mathcal{C}_{\Omega|Y})_\lambda = 0).$$

(Here we still denote by P the module $\mathcal{M} = \mathcal{D}_Y/\mathcal{D}_Y P$.)

PROOF. We first choose coordinates $x = (x_1, x')$, $x' = (x_2, x'')$ in S, $(x; \sqrt{-1}\,\eta) \in T^*_S Y$ so that $p = \eta_1, \varphi = x_1$. We observe that (6.2) implies $\{\varphi, a\} \equiv 0$. Thus: $q(x; \sqrt{-1}\,\eta) = a(x'; \sqrt{-1}\,\eta') + x_1 b(x; \sqrt{-1}\,\eta)$. Assume $da \neq 0$; by the trick of the dummy variable (that do not affect the conclusion of the theorem) it is not restrictive to assume $da \wedge \omega \neq 0$.

One can then change the coordinates $(x'; \sqrt{-1}\,\eta')$ so that $a = \eta_2, b = b(x_1, x''; \sqrt{-1}\,\eta), \lambda = (0; \sqrt{-1}\,\eta_0), \eta_0 = (0, \dots, 0, 1)$. Let $N = \{x: \varphi(x) = 0\}, V = \{(x; \sqrt{-1}\,\eta): \eta_2 = 0\}$. We note that $N \times_S V$ is regular involutive. We also recall that $b \geqslant 0$ when $x_1 \geqslant 0$.

We claim that then

$$(6.4) \qquad -H^R(-d\varphi) \notin C_\lambda(\operatorname{char}(\mathcal{M}), \tilde{V}_{\bar{\Omega}}),$$

$\tilde{V}_{\bar{\Omega}}$ being the union of the leaves of V^C issued from $\Omega \times_S V$ and $C(\cdot, \cdot)$ the normal cone in the sense of [4]. In fact let $(z; \zeta)$, $z = x + \sqrt{-1}y$, $\zeta = \xi + \sqrt{-1}\,\eta$ be coordinates on T^*Y. If $\operatorname{Im} \sigma(p + \sqrt{-1}q) = 0$ then $\xi_1 = \eta_2 + x_1 b^R - y_1 b^I$. We have

$$b^R = b|_{T^*_S Y} + O((|y_1| + |y''|)|\eta| + |\xi|),$$

thus we have for some c:

$$x_1 b^R + c((|y_1| + |y''|)|\eta| + |\xi|) \geqslant \begin{cases} 0, & x_1 \geqslant 0 \\ -c|x_1|\,|\eta|, & x_1 \leqslant 0. \end{cases}$$

It follows for a new c:

$$\xi_1 \geqslant -c[|\zeta_2| + |\xi''| + (|y_1| + |y''| + Y(-x_1)|x_1|)|\eta|],$$

and hence (6.4).

Finally (6.4) implies (6.3) by [9], [11].

As for the case $a \equiv 0$ it can be handled by using the results on $\bar{\Omega}$-hyperbolicity instead of $\bar{\Omega} - V$-hyperbolicity (*i.e.* for $V = T^*_S Y$). (cf [9, § 3].) ■

7. An application

Let $X \cong \boldsymbol{C}^2 \ni (u_1, u_2)$, $S \ni (x_1, x_2, x_3)$ a real hypersurface of X, Y a complexification of $S, \Omega = \{x: \varphi(x) > 0\} \subset S$ an open set with analytic boundary. Let $x_0 \in \partial\Omega$, U a neighborhood of Ω at x_0, $X^{\pm}$ the two components of $X \setminus S$ near x_0.

In this case $\bar{\partial}_b$ is a vector field $p(x; \partial/\partial x) + \sqrt{-1}\, q(x; \partial/\partial x)$. We still denote by $p = \sqrt{-1} q$ the symbol $\sigma(\bar{\partial}_b)|_{T^*_S Y}$.

Let γ be the half space $N(X^+)$ and γ'^* the half ray $\gamma'^* = {}^t\tilde{\varphi}'(\gamma^*)$. Let U be a neighborhood of Ω at x_0.

Proposition 7.1. *Assume that the functions p, q, φ satisfy (6.1), (6.2) at $\lambda = \gamma'^*_{x_0}$ and let $f \in \mathcal{O}_X(X^+ \cap U)$. Then f extends to a tuboid of X with profile $\bar{\Omega} \times_S \gamma$.*

Proof. Clearly $b(f \circ \tilde{\varphi}) \in \mathcal{H}om(\bar{\partial}_b, \Gamma_{\tilde{\pi}^{-1}(\overline{S\setminus\Omega})}(\mathcal{C}_{\Omega|Y}))_\lambda$. By Theorem 6.1, $\lambda \notin SS^{\bar{\partial}_b, 0}_{\Omega|Y}(b(f \circ \tilde{\varphi}))$. Then f extends to U verifying (2.5) on account of Corollary 4.3. ∎

Example 7.2. Assume that

(i) $S = \{(u_1, u_2) \in X: u_j = \chi_j(x) + \sqrt{-1}\,\psi_j(x),\ j = 1, 2, x \in S\}$,

(ii) $\varphi = \psi_1$,

(iii) $d\chi_1 \wedge d\chi_2 \wedge d\varphi \neq 0$,

(iv) $\partial_{x_2}\psi_2 + \partial_{x_1}\psi_2\, \partial_{x_3}\psi_2 = 0$.

By (ii), (iii), $\|\partial\chi_j/\partial x_i\|_{j=1,2;\, i=2,3}$ is non singular; one can then set $\chi_1 = x_2$, $\chi_2 = x_3$, $\psi_1 = x_1$.

In such a case we have: $\bar{\partial}_b = \partial_{x_1} - \sqrt{-1}[\partial_{x_2} + \beta(x_1, x_2, x_3)\, \partial_{x_3}]$, for β solving: $\sqrt{-1}\partial_{x_1}\psi_2 + \partial_{x_2}\psi_2 - \sqrt{-1}\beta + \beta\partial_{x_3}\psi_2 = 0$. Setting $\beta = \partial_{x_1}\psi_2$, we get:

$$\bar{\partial}_b = \partial_{x_1} - \sqrt{-1}[\partial_{x_2} + \partial_{x_1}\psi_2\, \partial_{x_3}]. \tag{7.1}$$

Write $\psi_2 = x_1\, a(x_2, x_3) + x_1^2\, c(x_1, x_2, x_3)$ and set $b = 2c + x_1 \partial_{x_1} c$. Assume $\{\xi_2 + a\xi_3, b\xi_3\} \equiv 0$ (for instance take $a(x_2, x_3) = a$ and $c(x_1, x_2, x_3) = c(x_1)$, or any $a(x_2, x_3)$ and $c(x_1, x_2, x_3) = 0$).

Under such hypotheses (6.1) is satisfied. If we then assume $b \leqslant 0$ for $x_1 \geqslant 0$ and $\eta \sim \eta_0$, we get Ω-regularity at $(x_0; \sqrt{-1}\eta_0)$.

Remark 7.3. Note that if $b \leqslant 0$ for $x_1 \leqslant 0$, we get $S \setminus \bar{\Omega}$-regularity at $(x_0; \sqrt{-1}\eta_0)$.

Thus for instance for $S = \{u: u_1 = x_2 + \sqrt{-1}x_1, u_2 = x_3 + \sqrt{-1}x_1^2\}$, $\Omega = \{x: x_1 > 0\}$ and $\gamma^+ = N(\{u: \operatorname{Im} u_2 > \operatorname{Im} u_1^2\})$ then any f^+ (resp g^+) defined in $X^+ \cap W^+$ (resp $X^+ \cap W^-$) for $W^{\pm}$ a neighborhood of $S \cap \{\pm \operatorname{Im} u_1 > 0\}$, extends to a domain of type $\{u: \operatorname{Im} u_1 > 0, (\text{resp} \operatorname{Im} u_1 < 0) \operatorname{Im} u_1^2 < \operatorname{Im} u_2 < \varepsilon \operatorname{Im} u_1\}$ (for $\gamma^{+*} = -\sqrt{-1}d \operatorname{Re} u_2$ in the duality $T_M X \times T^*_M X \to \boldsymbol{R}$ associated to $-\operatorname{Im}\omega$).

This is of course classical by Bochner's theorem.

On the contrary for $S = \{u: u_1 = x_2 + \sqrt{-1}x_1, u_2 = x_3 + \sqrt{-1}x_1^3\}$ and for $W^{\pm}$ a neighborhood of $S \cap \{u; \sqrt{-1}u_1 > 0\}$, one has extension for f^+ (resp g^-) from

$X^+ \cap W^+$ (resp $X^- \cap W^-$) to a domain of type $\{u; \operatorname{Im} u_1 > 0, \operatorname{Im} u_1^2 < \operatorname{Im} u_2 < \varepsilon \operatorname{Im} u_1\}$ (resp. $\{u; \operatorname{Im} u_1 < 0, -\varepsilon \operatorname{Im} u_1 < \operatorname{Im} u_2 < -\operatorname{Im} u_1^2\}$).

REMARK 7.4. Let $S = \{u: u_1 = x_2 + \sqrt{-1}x_1, u_2 = x_3 + \sqrt{-1}a(x_2, x_3)x_1\}$, with $\partial a/\partial x_2 + a\partial a/\partial x_3 = 0$ and $\Omega = \{x: x_1 > 0\}$.

We have

$$\bar{\partial}_b = \partial/\partial x_1 - \sqrt{-1}[\partial/\partial x_2 + a(x_2, x_3)\,\partial/\partial x_3],$$

(which corresponds to the case $b \equiv 0$ in Proposition 6.1). Then one gets Ω and $S \setminus \bar{\Omega}$-regularity at both points in $T^*_S Y \cap \operatorname{char} \bar{\partial}_b$.

8. REMOVABLE SINGULARITIES

Let $S \subset X \cong \mathbf{C}^2$ be a generic hypersurface, Y a complexification of S. Let $N \subset S$ be an hypersurface, generic on X, given by $N = \{x; \varphi(x) = 0\}$. Let $N^{\mathbf{C}}$ be a complexification of N. Assume that, for $\bar{\partial}_b = p + \sqrt{-1}q$, one has $\{p, \varphi\} \equiv 1$. For $q = a + \varphi b$ ($\{p, a\} \equiv 0$), set $V = \{x; a(x) = 0\}$. Assume (6.1) to hold and moreover:

(6.2)' $\qquad b \geq 0$ on $T^*_S X$ (for any φ).

Let $\Sigma \subset N$ be such that $\sqrt{-1}N^*(\Sigma) \subset \rho\varpi(V)$ (here we denoted by ρ and ϖ the maps: $T^* N^{\mathbf{C}} \xleftarrow{\rho} N^{\mathbf{C}} \times_Y T^* Y \xrightarrow{\varpi} T^* Y$).

Take $u \in \Gamma_{S \setminus \Sigma}(\mathcal{B}_{S|Y})_{x_0}$, $x_0 \in \partial\Sigma$.

PROPOSITION 8.1. *Take $u \in \Gamma_{S \setminus \Sigma}(\mathcal{B}_{S|X})_{x_0}$, $x_0 \in \partial\Sigma$. If $\pm\lambda \notin SS(u|_{S \setminus \Sigma})$ then u extends to S at x_0 to a function $\tilde{u}$ with $\pm\lambda \notin SS(\tilde{u})$.*

SKETCH OF THE PROOF. We can look at u as being a section of $\mathcal{H}om_{\mathcal{D}_Y}(\bar{\partial}_b, \Gamma_{S \setminus \Sigma}\mathcal{B}_{S|Y})_{x_0}$. Let $\varphi = x_1$, let $\Omega^{\pm} = \{\pm x_1 > 0\}$ and denote by $\gamma^{\pm}(u)$ the traces of u on N. We have $SS(\gamma^{\pm}(u)) \subset \rho\varpi^{-1} SS_{\Omega}^{\bar{\partial}_b, 0}(u)$ and so, by Proposition 6.1, $\rho(\lambda^{\pm}) \notin SS(\gamma^{\pm}(u))$. Hence also $\rho\varpi^{-1}(V) \cap SS(\gamma^{\pm}(u)) = \emptyset$.

Since $\operatorname{char}(\bar{\partial}_b) \cap \rho^{-1}\rho\varpi^{-1} V^{\mathbf{C}} \subset T^*_S Y$, then $SS(\gamma^{\pm}) \cap \rho\varpi^{-1}(V) = \emptyset$. Since $\gamma^+ - \gamma^- = 0$ on $S \setminus \Sigma$, we can propagate by the classical sweeping-out theorem. ∎

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