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Equazioni a derivate parziali. — *Abstract nonlinear Volterra integrodifferential equations with nonsmooth kernels.* Nota (*) di MAURIZIO GRASSELLI e ALFREDO LORENZI, presentata dal Socio E. MAGENES.

ABSTRACT. — A Cauchy problem for an abstract nonlinear Volterra integrodifferential equation is considered. Existence and uniqueness results are shown for any given time interval under weak time regularity assumptions on the kernel. Some applications to the heat flow with memory are presented.

KEY WORDS: Integrodifferential Volterra equations; Monotone operators; Contraction principle; Heat flow in materials with memory.

RIASSUNTO. — *Equazioni integrodifferenziali astratte nonlineari di Volterra con nuclei non regolari.* Si studia un problema di Cauchy per un'equazione integrodifferenziale astratta nonlineare di Volterra. Si provano risultati di esistenza e unicità supponendo il nucleo debolmente regolare rispetto al tempo. Si presentano alcune applicazioni dei risultati ottenuti a modelli di conduzione del calore con memoria.

0. INTRODUCTION

Here we want to study the following nonlinear Cauchy problem

$$u'(t) + A(u(t)) = \int_0^t K(t,s,u(s),u'(s))ds + f(t), \qquad \text{for a.e. } t \in (0,T), T>0, \tag{0.1}$$

$$u(0) = u_0, \tag{0.2}$$

where A, K are nonlinear operators, f is a given function defined on $(0,T)$ and u_0 is a given element.

Similar problems were extensively studied by many authors using maximal monotone operator techniques (see, *e.g.* [2, 5, 7, 12, 14]), analytic semigroup theory (see, *e.g.* [1, 4, 6, 8, 9, 13, 16, 17]) or classical Picard iteration (see [3]). Nevertheless, in the quoted works the assumptions on the time regularity of kernel K are generally rather strong (see however [1,6] for milder hypotheses).

Aim of this paper is to show that the Cauchy problem (0.1)-(0.2) is well-posed in a variational setting, for any $T>0$, under weak time regularity assumptions on K (*e.g.*, $K \in L^1(0,T)$ if it is of convolution type). Such a result is merely obtained by a careful application of the Contraction Mapping Principle step by step in time.

As is well known equations like (0.1) occur in heat flow in material with memory (see, *e.g.* [13,14]). Concerning this context it is physically meaningful to have deal with operators K singular in time. Taking this fact into account, we will show how our abstract results apply to a nonlinear model describing heat flow in a material with memory.

(*) Pervenuta all'Accademia il 30 luglio 1990.

1. Main results

Before stating the assumptions on A, K, f, u_0, and our main results we need some notations. Let $(V, \|\cdot\|)$ be a real, separable, reflexive Banach space with topological dual $(V^*, \|\cdot\|_*)$, $\langle\cdot,\cdot\rangle$ denoting the duality pairing in $V \times V^*$. Moreover, let $(H, (\cdot,\cdot), |\cdot|)$ be a real Hilbert space identified with its dual, in which V is densely and continuously embedded, *i.e.* $V \hookrightarrow H \hookrightarrow V^*$. $L^p(0,T;X)$, X being a Banach space and $p \in [1, +\infty)$, denotes the space consisting of p-summable functions defined on $(0,T)$ and taking their values in X. This space becomes a Banach space when it is endowed with the following norm

$$\|u\|_{L^p(0,T;X)} := \left\{ \int_0^T \|u(t)\|_X^p \, dt \right\}^{1/p}, \qquad \forall u \in L^p(0,T;X).$$

Besides, we set

$$L^\infty(0,T;X) := \{u\colon (0,T) \to X \ \textit{measurable}\colon \ \|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t\in(0,T)} \|u(t)\|_X < +\infty\}.$$

We now can state our hypotheses on A, K, f, and u_0. Let us assume that the non-linear operators $A\colon V \to V^*$, $K\colon Q_T \times V \times V^* \to V^*$, where $Q_T := \{(t,s) \in \mathbf{R}^2 : 0 < s < t < T\}$, satisfy the following assumptions:

(*i*) *there exists a constant* M *such that* $\|A(v)\|_* \leq M\|v\|$, $\forall v \in V$;

(*ii*) *there exists a constant* $\tilde{M}$ *such that* $\|A(u) - A(v)\|_* \leq \tilde{M}\|u - v\|$, $\forall u, v \in V$;

(*iii*) *there exists a positive constant* α *such that*

$$\langle A(u) - A(v), u - v\rangle \geq \alpha\|u - v\|^2, \ \forall u, v \in V;$$

(*iv*) *there exists a function* $\mathcal{K}\colon Q_T \to \mathbf{R}_+$ *such that*

$$\|K(t,s,u,v) - K(t,s,\tilde{u},\tilde{v})\|_* \leq \mathcal{K}(t,s)\{\|u - \tilde{u}\| + \|v - \tilde{v}\|_*\},$$

for a.e. $(t,s) \in Q_T$, *for any* u, $\tilde{u} \in V$ *and* v, $\tilde{v} \in V^*$;

(*v*) *there exists* $\gamma \in [0,1]$ *such that*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|(\mathcal{K}(t,\cdot))^{2\gamma}\|_{L^1(0,t)} < +\infty, \qquad \operatorname*{ess\,sup}_{s\in(0,T)} \|(\mathcal{K}(\cdot,s))^{2(1-\gamma)}\|_{L^1(s,T)} < +\infty;$$

(*vi*) *the function* $m\colon \mathbf{R}_+ \to \mathbf{R}_+$ *defined by*

$$m(\delta) := \sup_{t\in(0,T)} \left\{ \operatorname*{ess\,sup}_{r\in(t,t+\delta)} \|(\mathcal{K}(r,\cdot))^{2\gamma}\|_{L^1(t,r)} \operatorname*{ess\,sup}_{s\in(t,t+\delta)} \|(\mathcal{K}(\cdot,s))^{2(1-\gamma)}\|_{L^1(s,t+\delta)} \right\},$$

where $\mathcal{K}$ *has been extended by zero outside* Q_T, *is such that* $\lim_{\delta\to 0^+} m(\delta) = 0$.

Moreover, for the sake of simplicity, we assume

$$K(t,s,0,0) = 0. \qquad \textit{for a.e.}\ (t,s) \in Q_T. \tag{1.1}$$

As far as f and u_0 are concerned we require

$$f \in L^2(0,T;V^*); \tag{1.2}$$

$$u_0 \in H. \tag{1.3}$$

Our first result is

THEOREM 1.1. *Assume* (*i*) - (*vi*), (1.1)-(1.3). *Then the Cauchy problem* (0.1)-(0.2) *admits a unique solution* $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ *with* $u' \in L^2(0,T;V^*)$.

Let us consider now the case in which

$$K(t,s,u,v) \equiv k(t,s,u), \qquad \textit{for a.e.}\ (t,s) \in Q_T, \tag{1.4}$$

for any $u \in V$, $v \in V^*$.

Let $(V_i, \|\cdot\|_i)$, $(i=1,\dots,q)$ be real, separable, reflexive Banach spaces such that $V_i \hookrightarrow H \hookrightarrow V_i^*$, $(V_i^*, \|\cdot\|_i^*)$ being the topological dual of V_i. Moreover, let $A_i: V_i \to V_i^*$, $(i=1,\dots,q)$ be nonlinear operators such that

(*vii*) *there exists a constant* M_i *such that* $\|A_i(v)\|_i^* \leqslant M_i \|v\|^{p_i-1}$, $\forall v \in V_i$;

(*viii*) $A_i: V_i \to V_i^*$ *is monotone and hemicontinuous,*

(*ix*) *there is a positive constant* α_i *such that* $\langle A_i(u), u\rangle_i \geqslant \alpha_i \|u\|^{p_i}$, $\forall u \in V_i$.

Here $p_i \in (1,+\infty)$ $\forall i \in \{1,\dots,q\}$ and $\langle\cdot,\cdot\rangle_i$ denotes the duality pairing on $V_i \times V_i^*$.

Let us set now

$$A(v) := \sum_{i=0}^{q} A_i(v), \qquad \forall v \in \bigcap_{i=0}^{q} V_i, \tag{1.5}$$

where $A_0: V_0 := V \to V^*$ is a nonlinear operator satisfying (*i*)-(*iii*), and

$$\mathcal{V}(0,T) := \bigcap_{i=0}^{q} L^{p_i}(0,T;V_i), \qquad p_0 := 2.$$

Besides, let us assume, in place of (1.2),

$$f \in \mathcal{W}(0,T) := L^2(0,T;V^*) + \sum_{i=1}^{q} L^{p_i'}(0,T;V_i^*), \qquad \frac{1}{p_i} + \frac{1}{p_i'} = 1. \tag{1.6}$$

Then we have

THEOREM 1.2. *Assume* (*iv*) - (*ix*), (1.1), (1.3), (1.5), (1.6). *Then the Cauchy problem* (0.1)-(0.2) *admits a unique solution* $u \in L^\infty(0,T;H) \cap \mathcal{V}(0,T)$ *with* $u' \in \mathcal{W}(0,T)$.

REMARK 1.1. The solution u given by Theorem 1.1 (1.2) also belongs to $C^0([0,T];H)$ (see, *e.g.* [11, Lemma 8.1]).

REMARK 1.2. If $\gamma = 1$ in (*v*), then the assumptions (*v*)-(*vi*) can be replaced by the following (cf. [6, Proof of Thm. 3] and Sections 3, 4)

$$\int_{Q_T} \mathcal{K}(t,s)^2\, dt\, ds < +\infty.$$

REMARK 1.3. The assumptions (*iii*), (*ix*) can be weakened replacing by suitable seminorms the norms appearing at the right hand sides (cf. [10, Chap. 2, Thm. 1.2 bis, Rem. 1.13]).

We conclude by observing that if $\mathcal{K}(t,s) = \tilde{\mathcal{K}}(t-s)$ and $\tilde{\mathcal{K}} \in L^1(0,T)$, then (*v*)-(*vi*) are easily satisfied taking $\gamma = 1/2$. Hence we have

COROLLARY 1.1. *Under the assumptions of Theorem* 1.1, *if* $\mathcal{K}(t,s)=\tilde{\mathcal{K}}(t-s)$, *where* $\tilde{\mathcal{K}}\in L^1(0,T)$, *then there exists a unique solution* $u\in L^\infty(0,T;H)\cap L^2(0,T;V)$ *to the Cauchy problem* (0.1)-(0.2). *Moreover* $u'\in L^2(0,T;V^*)$.

COROLLARY 1.2. *Under the assumptions of Theorem* 1.2, *if* $\mathcal{K}(t,s)=\tilde{\mathcal{K}}(t-s)$, *where* $\tilde{\mathcal{K}}\in L^1(0,T)$, *then there exists a unique solution* $u\in L^\infty(0,T;H)\cap \mathcal{V}(0,T)$ *to the Cauchy problem* (0.1)-(0.2). *Moreover* $u'\in \mathcal{W}(0,T)$.

2. PROOF OF THEOREM 1.1

Let $t_0\in(0,T]$ and let $\delta\in(T-t_0,T)$ be a positive real number to be chosen later on. We consider the following Cauchy problem

$$u'(t)+A(u(t))=\int_{t_0}^{t}K(t,s,u(s),u'(s))ds+\int_0^{t_0}K(t,s,v(s),v'(s))ds+f(t), \tag{2.1}$$

$$\text{for a.e. } t\in(0,T),T>0,$$

$$u(t_0)=v_0, \tag{2.2}$$

where $v\in L^2(0,t_0;V)\cap H^1(0,t_0;V^*)$ and $v_0\in H$ are given. Here $H^1(0,t_0;V^*)$ denotes as usual the first-order Sobolev space related to $L^2(0,t_0;V^*)$.

By using the Contraction Mapping Principle we will show that the Cauchy problem (2.1)-(2.2) has a unique solution $u\in L^\infty(t_0,t_0+\delta;H)\cap L^2(t_0,t_0+\delta;V)$ for some $\delta\in(T-t_0,T)$. Then, as our estimates do not depend on t_0, a step by step (in time) procedure will give the assertion.

First let us set $X(t_0,t):=L^2(t_0,t;V)\cap H^1(t_0,t;V^*)$, $t\in(t_0,T]$. Clearly $X(t_0,t)$ is a Banach space when it is endowed with the following norm

$$\|u\|_{X(t_0,t)}:=\max\{\|u\|_{L^2(t_0,t;V)},\|u'\|_{L^2(t_0,t;V^*)}\}.$$

We now consider the Cauchy problem

$$u'(t)+A(u(t))=G(w)(t)+F_0(t),\qquad \text{for a.e. } t\in(t_0,t_0+\delta), \tag{2.3}$$

$$u(t_0)=v_0, \tag{2.4}$$

where $G(w)$ and F_0 are defined, respectively, by

$$G(w)(t):=\int_{t_0}^{t}K(t,s,w(s),w'(s))ds,\qquad \text{for a.e. } t\in(t_0,T), \tag{2.5}$$

$$F_0(t):=\int_0^{t_0}K(t,s,v(s),v'(s))ds,\qquad \text{for a.e. } t\in(t_0,T), \tag{2.6}$$

and $w\in X(t_0,T)$ is given.

Observe that $F_0 \in L^2(t_0, T; V^*)$. Indeed, owing to (*iv*)-(*vi*) and (1.1), we have

$$\left\| \int_0^{t_0} K(t,s,v(s),v'(s))ds \right\|_*^2 \leqslant \left\{ \int_0^{t_0} \mathcal{K}(t,s)[\|v(s)\| + \|v'(s)\|_*]ds \right\}^2 \leqslant$$

$$\leqslant 2\left\{ \int_0^{t_0} (\mathcal{K}(t,s))^{2\gamma} ds \right\} \left\{ \int_0^{t_0} (\mathcal{K}(t,s))^{2(1-\gamma)} [\|v(s)\|^2 + \|v'(s)\|_*^2] ds \right\} \qquad \textit{for a.e. } t \in (t_0, T).$$

Hence

$$(2.7) \qquad \int_{t_0}^{t} \left\| \int_0^{t_0} K(r,s,v(s),v'(s))ds \right\|_*^2 dr \leqslant 2 \operatorname*{ess\,sup}_{r \in (t_0,t)} \|(\mathcal{K}(r,\cdot))^{2\gamma}\|_{L^1(0,t_0)} \times$$

$$\times \int_0^{t_0} [\|v(s)\|^2 + \|v'(s)\|_*^2] ds \int_{t_0}^{t} (\mathcal{K}(r,s))^{2(1-\gamma)} dr \leqslant$$

$$\leqslant 2 \operatorname*{ess\,sup}_{r \in (t_0,t)} \|(\mathcal{K}(r,\cdot))^{2\gamma}\|_{L^1(0,t_0)} \operatorname*{ess\,sup}_{s \in (0,t_0)} \|(\mathcal{K}(\cdot,s))^{2(1-\gamma)}\|_{L^1(t_0,t)} \times$$

$$\times \int_0^{t_0} [\|v(s)\|^2 + \|v'(s)\|_*^2] ds, \qquad \forall t \in (t_0, T].$$

Thus (*v*) and (2.7) imply $F_0 \in L^2(t_0, T; V^*)$. We now prove that $G(w) \in L^2(t_0, T; V^*)$. Reasoning as above we get

$$\|G(w)(t)\|_*^2 \leqslant 2\left\{ \int_{t_0}^{t} (\mathcal{K}(t,s))^{2\gamma} ds \right\} \left\{ \int_{t_0}^{t} (\mathcal{K}(t,s))^{2(1-\gamma)} [\|w(s)\|^2 + \|w'(s)\|_*^2] ds \right\},$$

$$\textit{for a.e. } t \in (t_0, T).$$

Then we have

$$(2.8) \qquad \int_{t_0}^{t} \|G(w)(r)\|_*^2 dr \leqslant 2 \operatorname*{ess\,sup}_{r \in (t_0,t)} \|(\mathcal{K}(r,\cdot))^{2\gamma}\|_{L^1(t_0,r)} \times$$

$$\times \int_{t_0}^{t} [\|w(s)\|^2 + \|w'(s)\|_*^2] ds \int_{s}^{t} (\mathcal{K}(r,s))^{2(1-\gamma)} dr \leqslant$$

$$\leqslant 2 \operatorname*{ess\,sup}_{r \in (t_0,t)} \|(\mathcal{K}(r,\cdot))^{2\gamma}\|_{L^1(t_0,r)} \operatorname*{ess\,sup}_{s \in (t_0,t)} \|(\mathcal{K}(\cdot,s))^{2(1-\gamma)}\|_{L^1(s,t)} \times$$

$$\times \int_{t_0}^{t} [\|w(s)\|^2 + \|w'(s)\|_*^2] ds, \qquad \forall t \in (t_0, T].$$

Therefore, from (*v*) and (2.8) it follows that $G(w) \in L^2(t_0, T; V^*)$ if $w \in X(t_0, T)$. Since $G(w) + F_0 \in L^2(t_0, T; V^*)$, the problem (2.3)-(2.4) admits a unique solution $u \in X(t_0, T)$ (cf. (*i*)-(*iii*) and [10, Chap. 2, Thm. 1.2 with $p = 2$]). Hence the mapping $J: X(t_0, T) \to X(t_0, T)$, $J(w) := u$, is well defined.

Let now $w_1, w_2 \in X(t_0, T)$ be given. Note that $u_2 - u_1 := J(w_2) - J(w_1)$ satisfies the equations

$$(2.9) \qquad (u_2 - u_1)'(t) + A(u_2(t)) - A(u_1(t)) = G(w_2)(t) - G(w_1)(t), \quad \textit{for a.e. } t \in (t_0, T),$$

$$(2.10) \qquad (u_2 - u_1)(t_0) = 0.$$

A standard energy estimate, obtained by multiplying (2.9) by $2(u_2 - u_1)$ and using (*iii*), (2.10), gives

$$(2.11) \qquad |(u_2 - u_1)(t)|^2 + \int_{t_0}^{t} \|(u_2 - u_1)(r)\|^2 dr \leqslant C(\alpha) \int_{t_0}^{t} \|G(w_2)(r) - G(w_1)(r)\|_*^2 dr, \qquad \forall t \in (t_0, T),$$

where $C(\alpha)$ denotes from now on a positive constant depending on α only. Moreover from (2.9), taking (*ii*) and (2.11) into account, we get

$$(2.12) \qquad \int_{t_0}^{t} \|(u_2 - u_1)'(r)\|_*^2 dr \leqslant C(\alpha) \int_{t_0}^{t} \|G(w_2)(r) - G(w_1)(r)\|_*^2 dr, \qquad \forall t \in (t_0, T).$$

On the other hand, reasoning as in (2.8), we easily obtain the estimate

$$(2.13) \qquad \int_{t_0}^{t} \|G(w_2)(r) - G(w_1)(r)\|_*^2 dr \leqslant$$

$$\leqslant 2 \operatorname*{ess\,sup}_{r \in (t_0, t)} \|(\mathcal{K}(r, \cdot))^{2\gamma}\|_{L^1(t_0, r)} \operatorname*{ess\,sup}_{s \in (t_0, t)} \|(\mathcal{K}(\cdot, s))^{2(1-\gamma)}\|_{L^1(t_0, t)} \times$$

$$\times \int_{t_0}^{t} [\|(w_2 - w_1)(s)\|^2 + \|(w_2' - w_1')(s)\|_*^2] ds, \qquad \forall t \in (t_0, T].$$

Then from (2.11)-(2.13) we derive

$$(2.14) \qquad \|J(w_2) - J(w_1)\|_{X(t_0, t)} \leqslant C(\alpha) \Big\{ \operatorname*{ess\,sup}_{r \in (t_0, t)} \|(\mathcal{K}(r, \cdot))^{2\gamma}\|_{L^1(t_0, r)} \times$$

$$\times \operatorname*{ess\,sup}_{s \in (t_0, t)} \|(\mathcal{K}(\cdot, s))^{2(1-\gamma)}\|_{L^1(s, t)} \Big\}^{1/2} \|w_2 - w_1\|_{X(t_0, t)}, \qquad \forall t \in (t_0, T].$$

Recalling now (*vi*), (2.14) gives

$$(2.15) \qquad \|J(w_2) - J(w_1)\|_{X(t_0, t)} \leqslant C(\alpha) m(\delta)^{1/2} \|w_2 - w_1\|_{X(t_0, t)}, \qquad \forall t \in (t_0, t_0 + \delta).$$

Finally, choosing, *e.g.*, $\delta > 0$ such that (cf. (*vi*)) $C(\alpha) m(\delta)^{1/2} \leqslant 1/2$, J is a contraction mapping from $X(t_0, t_0 + \delta)$ into itself and the theorem is proved.

3. Proof of theorem 1.2

Observe that, by virtue of (1.4), from the previous proof we deduce that $G(w) \in L^2(t_0, T; V^*) \subset \mathcal{W}(t_0, T)$ whenever $w \in L^2(t_0, T; V)$ and $F_0 \in L^2(t_0, T; V^*) \subset \mathcal{W}(t_0, T)$ if $v \in \mathcal{V}(0, t_0)$ is given. Hence the Cauchy problem (2.3)-(2.4) has a unique solution $u \in L^\infty(t_0, T; H) \cap \mathcal{V}(t_0, T)$ with $u' \in \mathcal{W}(t_0, T)$ (cf. (1.1), (1.5), (1.6), (*vii*)-(*ix*), and

[10, Chap. 2, Thm. 1.4]). Thus we can define a mapping $\tilde{J}: L^2(t_0, t; V) \to \mathcal{V}(t_0, t) \subseteq L^2(t_0, t; V)$, $t \in (t_0, T]$ by setting $\tilde{J}(w) := u$.

Using now the same reasoning as in Section 2 and looking for the fixed point of $\tilde{J}$ in $L^2(t_0, t_0 + \delta; V)$ we conclude the proof showing that the Cauchy problem (2.1)-(2.2) admits a unique solution $u \in L^\infty(t_0, t_0 + \delta; H) \cap \mathcal{V}(t_0, t_0 + \delta)$, where $\delta > 0$ does not depend on t_0.

4. Applications

Let Ω be a bounded and connected subset of $\mathbf{R}^n$ (*e.g.*, $n = 1, 2$ or 3). We assume that Ω represents a rigid body in which the heat conduction phenomena are affected by memory effects.

Denoting by $u(x, t)$ its temperature at a point $x \in \Omega$, at time t, a mathematical model describing heat propagation in Ω is given by the following constitutive assumptions (cf., *e.g.* [12, 14, 15, 16])

$$\varepsilon(x, t) = \varepsilon_0 + a_0 u(x, t) + \int_0^{+\infty} a(x, s) u(x, t - s) ds, \tag{4.1}$$

$$q(x, t) = -\chi(\nabla u(x, t)) - \int_0^{+\infty} \sigma(s, \nabla u(x, t - s)) ds, \tag{4.2}$$

where $-\infty < t < +\infty$, $x \in \Omega$ and ∇ denotes as usual the gradient operator with respect to space variables. Here ε, q represent the *internal energy* and the *heat flux*, respectively. Moreover, ε_0, a_0 are given positive constants and $a: \Omega \times \mathbf{R}_+ \to \mathbf{R}$, $\chi: \mathbf{R}^n \to \mathbf{R}^n$, $\sigma: \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$ are known functions satisfying suitable assumptions which will be specified later.

Recalling the energy balance

$$\frac{\partial \varepsilon}{\partial t}(x, t) = -\operatorname{div} q(x, t) + h(x, t), \qquad x \in \Omega,\ t \in \mathbf{R}, \tag{4.3}$$

where h denotes the *heat supply*, we find that the evolution of u is governed by the equation

$$\begin{aligned} a_0 \frac{\partial u}{\partial t}(x, t) + \int_0^{+\infty} a(x, s) \frac{\partial u}{\partial t}(x, t - s) ds = \operatorname{div} \chi(\nabla u(x, t)) +& \\ + \int_0^{+\infty} \operatorname{div} \sigma(s, \nabla u(x, t - s)) ds + h(x, t), &\qquad x \in \Omega,\ t \in \mathbf{R}. \end{aligned} \tag{4.4}$$

Let us assume that (cf., *e.g.* [12, 14])

$$u(x, t) = \tilde{u}(x, t), \qquad \forall x \in \overline{\Omega}, \forall t \in (-\infty, 0],$$

where $\tilde{u}$ is a given, smooth enough, history function satisfying equation (4.4) for $t \leq 0$. Setting

$$(4.5) \qquad F(x,t) := h(x,t) + \int_t^{+\infty} \operatorname{div} \sigma(s, \nabla\tilde{u}(x,t-s))ds - \int_t^{+\infty} a(x,s) \frac{\partial \tilde{u}}{\partial t}(x,t-s)ds, \qquad x \in \Omega,\ t \geq 0,$$

equation (4.4) turns out to be

$$(4.6) \qquad a_0 \frac{\partial u}{\partial t}(x,t) - \operatorname{div} \chi(\nabla u(x,t)) = F(x,t) + $$
$$+ \int_0^t \left[\operatorname{div} \sigma(s, \nabla u(x,t-s)) - a(x,s) \frac{\partial u}{\partial t}(x,t-s) \right] ds, \qquad x \in \Omega,\ t \geq 0.$$

A standard initial-boundary value problem which can be associated with (4.6) is the following (cf., *e.g.* [4, 5, 12, 14])

(P_1) *Find* $u: \Omega \times (0,T) \to \mathbf{R}$ $(T > 0)$ *solving* (4.6) *and satisfying the conditions*

$$(4.7) \qquad u(x,0) = u_0(x) := \tilde{u}(x,0), \qquad x \in \Omega,$$

$$(4.8) \qquad u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T).$$

We now show as problem (P_1) can be reformulated in the abstract form (0.1)-(0.2). Let us set $H := L^2(\Omega)$, $V := H_0^1(\Omega)$. Then $V^* = H^{-1}(\Omega)$ and $V \hookrightarrow H \hookrightarrow V^*$. Assume that a, χ and σ satisfy the following hypotheses

($h1$) $a \in L^1(0, +\infty; W^{1,\infty}(\Omega))$;

($h2$) $\chi \in C^1(\mathbf{R}^n; \mathbf{R}^n)$, $\chi(0) = 0$;

($h3$) *there exists* $M > 0$ *such that* $|||D\chi(\xi)||| \leq M$, $\forall \xi \in \mathbf{R}^n$;

($h4$) *there exists* $\alpha > 0$ *such that* $(D\chi(\zeta)\xi, \xi) \geq \alpha|\xi|^2$ $\forall \zeta, \xi \in \mathbf{R}^n$;

($h5$) $\sigma(t, \cdot) \in C^1(\mathbf{R}^n; \mathbf{R}^n)$, $\sigma(t,0) = 0$, *for a.e.* $t \in (0, +\infty)$;

($h6$) $\sigma(\cdot, \xi) \in L^1(0, +\infty)$, *for a.e.* $\xi \in \mathbf{R}^n$;

($h7$) *there exists* $\tilde{\sigma} \in L^1(0,T)$ *such that* $|\sigma(t,\xi_1) - \sigma(t,\xi_2)| \leq \tilde{\sigma}(t)|\xi_1 - \xi_2|$, *for a.e.* $t \in (0,T)$, $\forall \xi_1, \xi_2 \in \mathbf{R}^n$.

Here $D\chi(\xi)$ denotes the Jacobian matrix of χ evaluated at a point $\xi \in \mathbf{R}^n$ and $|||\cdot|||$ denotes its norm. Besides, $(\cdot,\cdot)$ is the usual scalar product in $\mathbf{R}^n$. Set now

$$A(u)v := a_0^{-1} \int_\Omega (\chi(\nabla u), \nabla v)dx, \qquad \forall v \in V,$$

for any $u \in V$,

$$K(t,s,u,v)w := a_0^{-1} \int_\Omega (\sigma(t-s, \nabla u), \nabla w)dx - a_0^{-1} \langle a(t-s)v, w \rangle,$$
$$\textit{for a.e. } (t,s) \in Q_T, \quad \forall w \in V,$$

for any $u \in V$, $v \in V^*$, where $\langle \cdot,\cdot \rangle$ denotes the duality pairing on $V^* \times V$. It is then an easy task to check that problem (P_1) can be set into the form (0.1)-(0.2) and A and K

satisfy conditions (*i*)-(*vi*), (1.1). Hence, assuming

$$F \in L^2(0,T;V^*); \tag{4.9}$$

$$u_0 \in H; \tag{4.10}$$

Corollary 1.1 applies and yields the following result

THEOREM 4.1. *Under the assumptions* (*h*1)-(*h*7), (4.9), (4.10) *problem* (P_1) *has a unique solution* $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ *with* $\partial u/\partial t \in L^2(0,T;V^*)$.

REMARK 4.1. Note that, recalling (4.5), the assumption (4.9) holds when assuming, *e.g.*,

$$h \in L^2(0,T;V^*), \qquad \tilde{u} \in L^2(-\infty,0;V) \cap H^1(-\infty,0;V^*).$$

The same remark can be made for the next application.

We conclude by showing an application of Theorem 1.2 or, more precisely, of Corollary 1.2.

Recalling (4.2), let us consider the following constitutive assumption

$$q(x,t) = -\left[\sum_{i=1}^{q} c_i |\nabla u(x,t)|^{p_i-2}\right]\nabla u(x,t) - \chi(\nabla u(x,t)) - \int_0^{+\infty} \sigma(s,\nabla u(x,t-s))ds,$$

where $(x,t) \in \Omega \times \mathbf{R}$ and $c_i > 0$, $p_i \in (1,+\infty)$ are given $(i=1,\dots,q)$. Moreover, let us assume

(*h*8) $\quad a \in W^{1,1}(0,+\infty;L^\infty(\Omega))$.

Reasoning as above (cf. (4.1), (4.3), (4.6)), and using the additional regularity of a (cf. (*h*8)) to differentiate (4.1) with respect to t, we obtain the following equation

$$\begin{aligned}a_0 \frac{\partial u}{\partial t}(x,t) - \operatorname{div}\left\{\left[\sum_{i=1}^{q} c_i |\nabla u(x,t)|^{p_i-2}\right]\nabla u(x,t) + \chi(\nabla u(x,t))\right\} + a(x,0)u(x,t) = \\ = \int_0^t \left[\operatorname{div}\sigma(t-s,\nabla u(x,s)) - \frac{\partial a}{\partial t}(x,t-s)u(x,s)\right]ds + \tilde{F}(x,t),\end{aligned} \tag{4.11}$$

where $x \in \Omega$, $t \geqslant 0$, and $\tilde{F}$ is defined by (cf. (4.5))

$$\tilde{F}(x,t) := h(x,t) + \int_t^{+\infty}\left[\operatorname{div}\sigma(s,\nabla\tilde{u}(x,t-s)) - \frac{\partial a}{\partial t}(x,s)\tilde{u}(x,t-s)\right]ds, \qquad x \in \Omega, t \geqslant 0.$$

Let us consider the problem

(P_2) *Find* $u: \Omega \times (0,T) \to \mathbf{R}$ $(T>0)$ *solving* (4.11) *and satisfying* (4.7)-(4.8).

We will show that Corollary 1.2 can be applied to (P_2). Taking $H := L^2(\Omega)$, $V := H_0^1(\Omega)$, and $V_i := W_0^{1,p_i}(\Omega) \cap L^2(\Omega)$ we have $V^* = H^{-1}(\Omega)$, $V_i^* = W^{-1,p_i'}(\Omega) + L^2(\Omega)$, $(1/p_i + 1/p_i' = 1)$ and $V \hookrightarrow H \hookrightarrow V^*$, $V_i \hookrightarrow H \hookrightarrow V_i^*$, $\forall i \in \{1,\dots,q\}$.

Assume hypotheses ($h2$)-($h8$), (4.10) and set

$$A_0(u)v := a_0^{-1}\int_\Omega [(\chi(\nabla u), \nabla v) + a(0)uv]\,dx, \qquad \forall v \in V,$$

$$A_i(u)v := a_0^{-1}\int_\Omega (c_i|\nabla u|^{p_i-2}\nabla u, \nabla v)\,dx, \qquad \forall v \in V,\ \forall i \in \{1,\dots,q\},$$

$$k(t,s,u)v := a_0^{-1}\int_\Omega \left[(\sigma(t-s,\nabla u),\nabla v) - \frac{\partial a}{\partial t}(t-s)uv\right]dx,$$

$$\text{for a.e. } (t,s)\in Q_T, \quad \forall v \in V,$$

for any $u \in V$.

Moreover, assume, for the sake of simplicity, that

$$a(x,0) \geqslant 0, \qquad \text{for a.e. } x \in \Omega. \tag{4.12}$$

Then we can easily prove that problem (P_2) has an abstract formulation (0.1)-(0.2) and conditions (iv)-(ix), (1.1), (1.4) hold.

Finally, assuming

$$\tilde{F} \in L^2(0,T;V^*) + \sum_{i=1}^{q} L^{p_i'}(0,T;V_i^*), \tag{4.13}$$

and (4.10), Corollary 1.2 and Remark 1.3 give

THEOREM 4.2. *Under the assumptions* ($h2$)-($h8$), (4.10), (4.12), (4.13) *problem* (P_2) *has a unique solution* $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ *such that*

$$u \in \bigcap_{i=1}^{q} L^{p_i}(0,T;V_i), \qquad \frac{\partial u}{\partial t} \in L^2(0,T;V^*) + \sum_{i=1}^{q} L^{p_i'}(0,T;V_i^*),$$

REMARK 4.2. The Corollary 1.2 can be also applied to a model in which the heat flux $\boldsymbol{q}$ is given by the constitutive equations (cf., *e.g.* [10, Chap. 2, Ex. 1.7.1])

$$q^i(x,t) = -c_i\left|\frac{\partial u}{\partial x_i}(x,t)\right|^{p_i-2}\frac{\partial u}{\partial x_i}(x,t) - \chi^i(\nabla u(x,t)) - \int_0^{+\infty} \sigma^i(s,\nabla u(x,t-s))\,ds,$$

where $(x,t)\in\Omega\times\mathbf{R}$ and $c_i>0$, $p_i\in(1,+\infty)$ are given ($i=1,\dots,n$).

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