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Group actions on rational homology spheres

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Topologia. — *Group actions on rational homology spheres.* Nota (*) di Stefano De Michelis, presentata dal Corrisp. E. Arbarello.

Abstract. — We study the homology of the fixed point set on a rational homology sphere under the action of a finite group.

Key words: Finite group action; Homology spheres; Borel spectral sequence.

Riassunto. — *Azioni di gruppo su spazi aventi l'omologia razionale di una sfera.* Studiamo l'omologia dell'insieme dei punti fissi dell'azione di un gruppo finito su di uno spazio avente l'omologia razionale di una sfera.

Introduction

In a previous paper, [2], we proved a result, conjectured by Smith, on group actions on S^4. More generally, we proved that every finite group acting on an integral homology sphere has fixed point set an integral homology spheres. The main tool was Smith's theorem on actions of Z/p on Z/p homology sphere. In this paper we study finite group actions on rational homology spheres. It is known that Smith's theorem does not hold in this category for higher dimensional manifolds and in section one we exhibit some examples of this failure in the lowest possible dimension, which is three. In section two we quickly review the Borel spectral sequence and the localization theorem for cyclic group actions and we show how to use them to get information on the homology of the fixed point set. In section three we study the homology of the fixed point set assuming we know the action of the group on the cohomology of the total space. We give some bounds on the genus of the fixed surfaces, depending on the number of trivial and cyclotomic factors appearing in the representation of the group. In section four we prove that it is not possible to better these estimates. In particular we construct actions of cyclic groups of prime order p on $Z[1/p]$ homology spheres such that the fixed point set consist of points and surfaces of arbitrarily large genus. More generally, we prove that any combination of points and surfaces can be realized in this way provided that its Euler characteristic is two.

1. Group actions on rational homology spheres

In this section and in the following ones we will show how to extend the results of [2] in another direction: we will allow the homology of the total space to contain some torsion, as we will see this will have very strong consequences on the result, due

(*) Pervenuta all'Accademia il 17 luglio 1990.

to the impossibility of applying Smith theorem. Recall that it asserts that the fixed point set of a Z/p action on a Z/p homology sphere is a Z/p homology sphere if p is prime.

It is well known that the assumptions on the group and the homology in this theorem cannot be relaxed in any way; the two following examples should convince the reader:

EXAMPLE 1: Let X be RP^3, so that it is a Z/p homology sphere for any p odd but not a $Z/2$ homology sphere, and let $Z/2$ act on X according to the law: $(X_0 : X_1 : X_2 : X_3 :) \to (-X_0 : X_1 : X_2 : X_3 :)$ where X_i are homogeneous coordinate on X; this action is locally linear and smooth. The fixed point set is described in homogeneous coordinates by $(X_0, 0, 0, 0) \cup (0 : X_1 : X_2 : X_3)$, the first set is just a point, the origin, the second is a real projective plane, the plane at infinity; we can also take equivariant connected sums of X with itself along the fixed point or the fixed plane, so as to find counterexamples with more complicated fixed point set.

EXAMPLE 2: With X as above let $Z/2$ act according to $(X_0 : X_1 : X_2 : X_3) \to$ $\to (X_0 : X_1 : -X_2 : -X_3)$ in this case the fixed point set is $(X_0 : X_1 : 0 : 0) \cup$ $\cup (0 : 0 : X_2 : X_3)$, that is the disjoint union of two circles in this case the action is orientation preserving.

These examples seem to suggest that the fixed point set can become arbitrarily complicated, provided that the torsion in the homology groups is large enough; the only restriction being the one on the Euler characteristic given by Lefschetz theorem. We will make these remarks more precise later in the setting of four dimensional manifolds.

2. THE BOREL SPECTRAL SEQUENCE AND THE LOCALIZATION THEOREM

A powerful tool in the computation of the homology of the fixed point set is given by the Borel spectral sequence. If X is any Hausdorf space with an action of a Lie group G, consider the product or X and EG, a contractible space on which G acts freely (such a space exists for any Lie group), this is a natural G-space with the product action moreover this action is free, hence the quotient is well defined, call it X_G. The projection $X \times EG \to EG$ is compatible with the G action and its quotient gives a map $X_G \to BG$, where $BG = EG/G$ is the classifying space of G; it is easy to check that this map is a fibration. X_G is called the Borel construction on the G-space X and if the action is free it has the same homotopy type as the quotient X/G, this explains the terminology «homotopy quotient» which is sometimes used. The fibration $X_G \to BG$ gives a (co)-homology spectral sequence called the Borel spectral sequence. Its $E_2^{p,q}$ term is $H^p(BG, H^q(X))$ where $H^q(X)$ is a G module for the obvious action of G on the (co)-homology of X. The E_∞ term is the graded group associated to a filtration of the (co)-homology of X. Observe that the cohomology of the latter is naturally a module over the cohomology ring of BG. This module or rather its torsion free part is strictly related to the cohomology of the fixed point set, this relation is described more precisely in the localization theorem which holds for any Lie group G and any generalized cohomology theory. We will state it in the form we will need, *i.e.*, for a Z/p action for p a prime.

First recall that the homology and the cohomology of $B(Z/p)$ can be matched together in the Z-graded Tate cohomology groups $\hat{H}^i$, they are isomorphic to the usual cohomology groups in strictly positive dimensions and for the negative signs they correspond to homology groups shifted by one. If u is the generator of $\hat{H}^2(B(Z/p))$, cup product with u gives the periodicity isomorphisms for even and odd cohomology.

Theorem 2.1. Let X^G the fixed point set of the action, so that $H^*(X^G \times BG)$ and $H^*(X^G)$ are $H^i(BG)$ modules, then after localization at u, they become isomorphic $\hat{H}^i(BG)$ modules.

According to this theorem the Borel spectral sequence for Tate cohomology can be used to compute the homology and cohomology of the fixed point set.

Observe that in the process of localization we loose the information on the grading, so that it may be impossible to compute the Betti numbers.

Using the Borel spectral sequence and the localization theorem we will deduce as many positive results as possible on the topology of the fixed point set of Z/p-actions on certain four manifolds. Since the action will be orientation preserving, the fixed point set will have dimension less than three so that it's topology will be determinated by its homology. In the next sections examples will be produced to show that these results cannot be strengthened in any way, without further information on the action. Let us fix a prime p and let Σ be a rational homology sphere on which Z/p acts. As we will see all the useful information lines in the action of the group on the p-torsion part of the homology, so we will always work with coefficients in Z/p. We first need to study the representation of Z/p.

Proposition 2.1. Every represention of Z/p over Z/p splits as a sum of indecomposable representations V_r, which satisfy the following:

a) V_r has rank r.

b) $\xi \in Z/p$ acts on V_r according to the unimodular matrix

$$\begin{bmatrix} 1 & 1 & 0 & & & & & \\ 0 & 1 & 1 & & & & & \\ 0 & 0 & 1 & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ & & & & & & 1 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \end{bmatrix}$$

PROOF. The proof is left to the reader.

Observe that $r=0$ corresponds to the trivial module and $r=p$ is the free module denoted henceforward by F.

OBSERVATION. We observe that this simple classification does not hold for $Z[Z/p]$-modules. In fact if the ideal class group of $Q[\xi]$ is not trivial, there are other isomorphism types of irreducibile representations (one for each element of the ideal class group) which correspond to non-principal ideal of $Z[\xi]$. Moreover the representations V_r come from representations over Z only for $r=1$, $p-1$ and p. These modules are connected by the exact sequence: $0 \to C_1 \to C_r \to C_{r-1} \to 0$. The Tate cohomology groups are trivial for coefficients in F by definition, those with coefficients in C_r are Z/p for any degree, as the reader can see from an explicit periodic resolution of Z.

Consider now the action of Z/p on $H^1(\Sigma; Z/p)$, this group splits as a sum of irreducible summands isomorphic to F and denote by f, their respective multiplicities, so that $H_1(\Sigma; Z/p) = fF + \sum_{e=1}^{p-1} c_r C_r$, the Universal Coefficient Theorem and Poincare' duality combined give for the two other cohomology groups:

$$H^3(\Sigma; Z/p) = fF + \sum_{r=1}^{p-1} c_r C_r \,; \qquad H^2(\Sigma; Z/p) = 2fF + 2\sum_{r=1}^{p-1} c_r C_r \,.$$

We put together all these informations to obtain the $E_2^{p,q}$ term of the Borel spectral sequence:

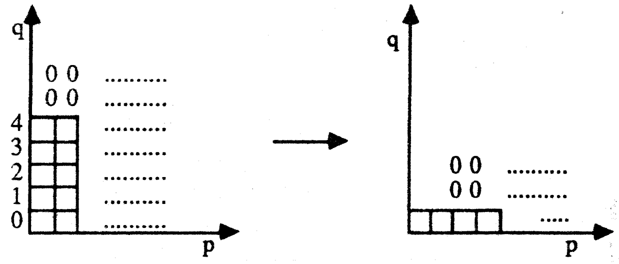
$$E_2^{p,q} = H^p(BZ/p; H^q(\Sigma)) = \begin{cases} (Z/p)^{\Sigma c_r + f} & p=0, \ q=1, \ 3; \\ (Z/p)^{2(\Sigma c_r + f)} & p=0, \ q=2; \\ (Z/p)^{\Sigma c_r} & p>0, \ q=1, \ 3; \\ (Z/p)^{\Sigma 2 c_r} & p>0, \ q=2. \end{cases}$$

To compute the Betti numbers of the fixed point set of the action it is necessary to know $E_\infty^{p,q}$, so one needs some information on the differentials in the spectral sequence. In the appendix a procedure is given to construct them geometrically; moreover one needs only to compute some of them, since the other can be obtained by Bredon duality. Even if the knowledge of the action of the group on the homology is not enough to a complete computation of $E_\infty^{p,q}$, this is possible for the two extreme dimensions, as W. Browder has remarked in [1].

PROPOSITION. If Z/p acts on Σ fixing at least one point, the generators of both $H^p(BZ/p; H^4(\Sigma; Z/p))$ and $H^p(BZ/p; H^0(\Sigma; Z/p))$ are permanent cocycles.

PROOF. For H^0 the proof is standard; the fixed point gives a section of the fibration $\Sigma_G \to BG$ this induces a map of spectral sequences

$E^{p,q}(\Sigma_G)$ $\qquad\qquad$ $E'^{p,q}(BG)$



Let v be a generator of $E^{p,q}$ and let u' be its image in $E'^{p,q}$, since $E^{p,q}$ is degenerated u' is a permanent cocycle, it follows that u is not killed by any differential. But now u has to be a permanent cocycle because it is in the lowest dimension. For $E^{p,4}$ we use a dual construction; if p is the fixed point, there is a map of pairs $\Sigma \to (\Sigma; \Sigma - p)$ which is equivariant; this induces a map of spectral sequences. The map is surjective and the first spectral sequence is degenerated, an argument as before gives the conclusion. Now we interpret all the informations collected: since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$ we have the inequality $r(p, q) \leqslant \operatorname{rank} E_2^{p,q}$, here $r(p, q)$ is the rank of the $E_\infty^{p,q}$ term. The rank of $H^n(\Sigma_G; Z/p)$ is the sum of the $r(p, q)$ for $p+q=n$, and the localization theorem tell us that this is also the rank of $H^r(\Sigma^G \times BG; Z/p)$. Since $H^n(BG; Z/p)$ is Z/p in any dimension, the Kunneth formula gives rank $H(\Sigma^G \times BG; Z/p) =$sum of the Betti numbers of Σ^G. Σ^G is a disjoint union of points and surfaces, let d be the number of points, s the number of connected components of the surfaces F_i and g_i the genus of F_i; then the sum of the Betti numbers of Σ^G is $d + \sum_{i=1}^{s} (2+2g_i)$. Let G be the sum of the g_i; from the computation of the rank of E_∞ and the discussion given above we obtain $d+2s+$ $+2G \leqslant 2 + 4\sum_{r=1}^{p-1} c_r$. The complementary piece of information needed is the Euler characteristic of the fixed point set, which can be computed via the Lefschetz formula and gives $d+2s-2G=2$. Combining this with the last inequality we obtain the result

$$d+2s=2+2G \quad \text{and} \quad G \leqslant \sum_{r=1}^{p-1} c_r .$$

The purpose of the next section is to show that there are cases in which the bounds given are sharp.

3. Counterexamples of Smith's theorem in dimension 4

In this section we will prove the result.

Theorem 3.1. Given any prime, p, and a disjoint union of arbitrary many points and orientable surfaces, F, such that its Euler Characteristic is 2, there exist a manifold Σ with the following properties:

(*a*) Σ^4 is a smooth manifold and its homology with coefficients in $Z[1/p]$ is the same as the one of S^4 (more concisely it is a smooth $Z[1/p]$ homology sphere)

(*b*) Z/p acts on Σ as a group of diffeomorphisms and its fixed point set is a smooth submanifold diffeomorphic to F.

PROOF. For the sake of clarity we will split the construction in several steps.

STEP 1. For any prime, p, there is a smooth action of Z/p on a lens space with foundamental group Z/p such that its fixed point set is the disjoint union of two circles.

PROOF. This fact is probably well known. Let $L(p, q)$ the link of the algebraic singularity $z^p - xy^q = 0$ — in the complex three dimensional space, or, analytically, the set: $L(p, q) = \{x, y, z | z^p - xy^q = 0;\ |x|^2 + |y|^2 + |z|^2 = 1;\ (x, y, z) \in \mathbf{C}^3\}$. It is well known that $L(p, q)$ is a lens space, and that every lens space can be described in this way. Let denote ξ a primitive p root of the unity. The action $Z/p \otimes L(p, q) \to L(p, q)$ is given by $(x, y, z) \to (x, y, \xi \cdot z)$.

By inspection, the fixed point set has equation $z = 0;\ xy = 0;\ |x|^2 + |y|^2 = 0$ it is easily checked that such a set consists of the two unitary circles in the x and y planes.

REMARK. There is another way to describe the same action, which avoids using algebraic singularities. Let R be the Hopf link in S^3, *i.e.* the simplest two component link with linking number 1. Its complement has foundamental group isomorphic to $Z \times Z$. Any surjective map of this group onto Z/p gives a connected covering of the complement of the link of degree p, this covering is regular by definition. If the two canonical generators of the foundamental group, corresponding to the two linking circles, are both mapped to non zero elements of Z/p, a standard theorem in the theory of branched coverings says that the covering can be extended to a regular branched covering of S^3, ramified over the two circles. This covering is a manifold and it is diffeomorphic to a lens space, as can be seen easily from a suitable Heeggart decomposition in solid tori, moreover the covering transformations give the action of Z/p described before.

STEP 2. For any prime p, there is a smooth action of Z/p on a $Z[1/p]$-homology four sphere, such that its fixed point set is the disjoint union of a two dimensional sphere and a two dimensional torus.

PROOF. First construct the manifold $K = L(p, q) \times S^1$, and let Z/p act on it according to the action described in step 1 on first factor and the trivial action on the second.

Künneth formula implies that the homology of K with $Z[1/p]$ coefficients is the same as the one of $S^3 \times S^1$, the fixed point set of the action of Z/p is $(S^1 \amalg S^1) \times S^1$, that is the disjoint union of two copies of the two torus T^2. On one of these two tori choose a simple closed curve to represent a generator for the Z factor of $H_1(K, Z)$, which is isomorphic to $Z \oplus Z/p$. We could take, for instance, the set (point)$\times S^1$. A regular neighborhood of the curve, invariant under the Z/p action, is diffeomorphic to $S^1 \times I \times D^2$. The Z/p action is $\xi \times (p, t, z) \to (p, t, \xi \cdot z)$, where $p \in S^1$, $t \in I$, $z \in D^2$, and the dot indicates the action of Z/p on D^2 in the complex plane given by complex multiplication. Observe also that $S^1 \times I$ is a regular neighborhood of S^1 in T^2.

Perform equivariant surgery on this copy of S^1 in K to obtain a space called Σ'. By inspection one can see that Σ' is a $Z[1/p]$-homology sphere and the fixed point set of the action is the disjoint union of S^2 and T^2.

STEP 3. The general case. With Σ' as above let N be a regular neighborhood of the S^2 component of the fixed point set. N is Z/p-equivariantly diffeomorphic to $S^2 \times D$. The action of Z/p is $\phi: \xi \times (s; z) \to (s; \xi z)$,with $s \in S^2$, $z \in D^2$ and ξ is acting as before. N admits also other simple actions of Z/p such as ϕ' given by $\phi' : \xi \times (s, z) \to (\xi s, \xi z)$, here the action on the first factor is the rotation of S around an axis of $2\pi/p$ radiants. In the case of ϕ' the fixed point set of the action is reduced to the two «poles» of S^2. The key fact is proved in the following proposition:

LEMMA 3.2. The restrictions of ϕ and ϕ' to the boundary of N are smoothly conjugate.

PROOF. The boundary of N is diffeomorphic to $S^2 \times S'$, and we will exhibit the conjugation map as a «twist» along a copy of S^2, more explicitly, in the notation used before: $\psi: (s, z) \to (z \cdot s, z)$, here the dot denotes the natural action of S' on S^2, given by rotation along the same axis as the one of Z/p. It is straightforward to check that $\phi \circ \psi = \psi \circ \phi'$ and this ends the proof of the lemma.

Now we remove N from Σ' and we use the map ψ on the boundary to attach it back again, we call the new manifold Σ. The above lemma gives an action of Z/p which now fixes a torus T^2, which has not been affected by the construction, and two points on the sphere S^2.

To finish the proof we have to check that Σ is a $Z[1/p]$ homology sphere. We shall use the Mayer-Vietoris sequence for the triple $(\Sigma, \overline{N}, \overline{\Sigma - N})$

$$H_i(\partial N) \to H_i(\overline{N}) \oplus H_i(\overline{\Sigma - N}) \to H_i(\Sigma) \to$$

and we will compare it to the one for $(\Sigma', N, \Sigma' - N)$

$$H_i(\partial N) \to H_i(\overline{N}) \oplus H_i(\overline{\Sigma - N}) \to H_i(\Sigma) \to$$

all the terms in the two sequences are the same and the only difference possibly lies in the maps from the homology of the boundary of N into $\Sigma - N$. The maps in the two sequences differ by the isomorphism induced by ψ in $H_i(\partial N)$; but this isomorphism is easily proved to be the identity, hence the homology of Σ and Σ' are the same by the five lemma.

OBSERVATION. ψ cannot be extended as a diffeomorphism to the whole N, to prove this consider the double of N, diffeomorphic to $S^2 \times S^2$, and the connected sum of the complex projective plane with its complex conjugate, one can prove that one is obtained from the other by the twisting construction used above, but they are obviously not diffeomorphic since they have different intersection form. On the other side ψ^2 extends to a diffeomorphism of the whole N, and the latter can be written almost explicitly as: $\psi^2 : (s;\, d) \to (h(d) \cdot s;\, d)$, where $h(d)$ is a map of the disk into $SO(3)$ which, restricted to the boundary S^1, is ψ, such a map exists because the foundamental group of $SO(3)$ is $Z/2$. So, if we had used a different ϕ' and if p were odd, we could have had Σ and Σ' diffeomorphic, but of course not Z/p-diffeomorphic. The general case now follows by taking equivariant connected sums of the Σ and Σ' described above.

Appendix

The task of computing the homology of the fixed point set of an action is made considerably easier if one uses a duality theorem on group actions due to Bredon, see the following discussion for the statement of the result. In this section we will combine this theorem with a method to construct the interesting differentials appearing in the spectral sequence such a method gives geometric representatives for the various cohomology classes involved, and this additional information could be used to gain some specifical control in particular cases. For simplicity we will only discuss Z/p actions for p a prime number. Moreover we will assume the induced action on homology is trivial.

We first state Bredon duality in the setting most convenient to us:

THEOREM. Let Z/p act on a Homology manifold X of dimension n (if p is two assume it preserves the orientation) and let ξ be a generator of $E^2_{0,n}$, which is isomorphic to Z/p. Let the fixed point set be non-empty, then ξ is a permanent cycle and the cap product ξn from $E_k^{p,n-c}$ gives an isomorphism for any k.

OBSERVATION. The fact that ξ is a permanent cycle has been proved by Browder in [1].

In the case of a four dimensional manifold this allows to cut dramatically the number of differentials. In fact both $E_2^{p,0}$ and $E_2^{p,4}$ are generated by permanent cycles and hence all the d_i going or coming to them have to be zero. We are left with homology in dimensions one, two and three; for d_2 there are two maps to check: one from $E^2_{p,1}$ to $E^2_{p,2}$ and the other from $E^2_{p,2}$ to $E^2_{p,3}$. By vector space duality between homology and cohomology with coefficient in a field, the second one corresponds to this same map on cohomology after conjugating by the isomorphism given by Bredon's theorem. Observe that the ring structure over the Tate cohomology of Z/p allows us to treat all the $E^2_{p,l}$ with the same p as isomorphic. It follows that it is necessary to know only one $E^2_{p,2}$ and that the others are deduced from this by cup products with fixed generators of E and Bredon duality, since we use Tate cohomology cup products are defined also for homology. The other map which one needs to compute is d_3 from $E^2_{p,1}$ to $E^2_{p,3}$.

To describe them explicitly let us choose a periodic resolution for Z/p:

$$Z[\xi] \xrightarrow{N} Z[\xi] \xrightarrow{D} Z[\xi]$$

where N is the norm map $a \cdot a\xi + \ldots + a\xi^{p1}$ and D is the difference map $a \to a - \xi a$. If X is a symplicial complex on which Z/p acts we construct the double complex:

$$\begin{array}{ccccc} C_i(X) \otimes Z[\xi] \xrightarrow{N} C_i(X) & \otimes Z[\xi] & \xrightarrow{D} C_i(X) \otimes Z[\xi] \\ \downarrow\partial & \downarrow\partial & \downarrow\partial \\ C_i(X) \otimes Z[\xi] \xrightarrow{N} C_{i-1}(X) & \otimes Z[\xi] & \xrightarrow{D} C_{i-1}(X) \otimes Z[\xi] \end{array}$$

The Borel spectral sequence is the sequence associated to the Altered complex. In the $E_2^{p,l}$ term let h be a homology class and let α be a chain representing it, assume h is in the kernel of N, the case of D is completely analogous, then $N(\alpha)$ is the boundary of some element p in $C_{l+1}(X) \otimes Z[\xi]$, compute D of this chain, the result is a cycle, defined modulo boundaries and images of cycles under the map D, and so is an element of $E_2^{p-2,l+1}$ which is $D_2(h)$. If this is zero it can be changed by an image of D to a boundary of some y in $C_{l+2}(X)$, the norm of y will give a representative for d_3. Once one knows these two maps it is routine to compute the odd and even Betti numbers of the fixed point set; the procedure to change an S^2 component to a couple of points described before proves that it is not possible to refine the information in any case.

References

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