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Line bundles with $c_1(L)^2 = 0$

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Topologia. — *Line bundles with* $c_1(L)^2 = 0$. Nota (*) di STEFANO DE MICHELIS, presentata dal Corrisp. E. ARBARELLO.

ABSTRACT. — We prove that on a CW-complex the obstruction for a line bundle L to be the fractional power of a suitable pullback of the Hopf bundle on a 2-dimensional sphere is the vanishing of the square of the first Chern class of L. On the other hand we show that if one looks at integral powers then further secondary obstructions exist.

KEY WORDS: Hopf bundle; Chern classes; Obstructions.

RIASSUNTO. — *Fibrati lineari con* $c_1(L)^2 = 0$. Si dimostra che l'ostruzione per costruire su di un CW-complesso un fibrato lineare L che sia una potenza frazionaria di un opportuno sollevamento del fibrato di Hopf sulla sfera bidimensionale, è dato dall'annullarsi del quadrato della seconda classe di Chern di L, mentre si dimostra che vi sono effettivamente ulteriori ostruzioni se si considerano esclusivamente le potenze intere.

1. INTRODUCTION

Let L be a line bundle over a CW complex M. We consider the problem of finding a map $\varphi: M \to S^2$ such that $\varphi^*(H^k) = L$ where H is the Hopf bundle on S^2. An obvious necessary condition is given by the vanishing of $c_1(L)^2$. We prove that $c_1(L)^2 = 0$ is equivalent to $\varphi^*(H^k) = L^n$ for some $n \neq 0$, so $c_1(L)^2$ is the only obstruction over the rationals.

If we require $n = 1$ and the dimension of $M \geq 5$ there are further secondary obstructions; we will study in particular the first one $\mu \in H^5(X; Z/2)$, and we give a geometric interpretation of it. We show that it vanishes for every 5-dimensional oriented manifold; this is somewhat unexpected because there is a 5-dimensional Poincaré complex on which it is not zero.

We also give an example of a 7-dimensional simply connected manifold for which $\mu \neq 0$. The problem of finding such an example in dimension 6 is still open, as far as the author knows.

PART I: FINITE CW COMPLEXES

Franctional line bundles.

THEOREM 1. Let M be a finite CW complex with a line bundle L over it. Let $c_1(L)^2 = 0$. Then there are a map $\varphi: M \to S^2$ and a line bundle L_0 on S^2 such that $\varphi^*(L_0) = L^n$, where n is a non zero integer.

PROOF. Given L, the classifying map $\psi: M \to CP^\infty$ is defined. L comes from a bundle over S^2 if and only if ψ is homotopic to a $\varphi: M \to S^2 \subset CP^\infty$. It is clear that the ob-

(*) Pervenuta all'Accademia il 19 luglio 1990.

structions lie on the homotopy fibre of the map $S^2 \to CP^\infty$. We study it via the Postnikov tower of S^2: the first layer is

$$\begin{tikzcd}
 & K(Z;3) \arrow[rr] & & X_1 \arrow[d] \\
S^2 \arrow[r] & CP^\infty = K(Z;2) \arrow[urr] \arrow[rr, "i_2 \cup i_2"] & & K(Z;4)
\end{tikzcd}$$

where the map on the right is given by the square of the generator of $M^2(CP^\infty; Z)$. So we see that given $\psi: M \to CP^\infty$ a first obstruction is $\psi^*(i_2^2) = c_1(L)^2$.

Now remember that $\pi_i(S^2) \otimes \mathbf{Q} = 0$ if $i > 3$. It follows that ${}_{\mathbf{Q}}X_1$, the localization of X_1 at $\mathbf{Q}$, is homotopic equivalent to ${}_{\mathbf{Q}}S^2$, the localization of S^2. ${}_{\mathbf{Q}}S^2$ can be described explicitely as the suspension of the infinite telescope on S^1 with multiplications by $n!$ as maps. The canonical map $S^2 \to {}_{\mathbf{Q}}S^2$ defines a homomorphism $\mathbf{Q} \simeq H_2(S^2; \mathbf{Q}) \to H_2({}_{\mathbf{Q}}S^2; \mathbf{Q})$. We can use it to identify $H^2({}_{\mathbf{Q}}S^2; \mathbf{Q})$ with $\mathbf{Q}$ in such a way that the dual class of S^2 corresponds to 1. By the discussion above we have a map

$$M \xrightarrow{\varphi'} {}_{\mathbf{Q}}S^2 \qquad \text{such that} \qquad \varphi'^*(1) = c_1(L).$$

Now it is easy to check that the following facts are true:

1) $\varphi'(M)$ can be pushed into a union of a finite number of segments.

2) Given W, a union of any finite number of segments of ${}_{\mathbf{Q}}S^2$, there is a map $l: W \to S^2$ such that $l^*(\alpha) = n[1]_{|W}$, where $n \neq 0$ and α is a generator for $H^2(S^2; \mathbf{Q})$.

Facts 1 and 2 imply that $l \circ \varphi$: $M \to {}_{\mathbf{Q}}S^2$ is the map we look for.

The integral case, definition of the obstruction μ.

In general the map constructed in the previous section cannot be assumed to give exactly $c_1(L)$ instead of a multiple of it. This becomes evident if we look at the full Postnikov tower of S^2.

$$\begin{tikzcd}
 & \vdots & \\
K(Z/2;4) \arrow[r] & X_2 \arrow[r] \arrow[d] & {} \\
K(Z;3) \arrow[r] & X_1 \arrow[r] \arrow[d] & K(Z_2;5) \\
 & CP^\infty \arrow[r] & K(Z;4)
\end{tikzcd}$$

where the spaces on the left are $K(\pi_i(S^2); i)$: whenever we try to lift a map from X_i to X_{i+1} we are faced with secondary obstructions in $H^i(M; \pi_i(S^2))$, modulo indeterminacies.

To study in detail the first one we need some information on the cell structure of X_1. First we compute the two dimensional cohomology of X_1, with coefficients in $Z/2$. The Serre spectral sequence for the fibration

$$\begin{array}{ccc} K(Z;3) \rightarrow & X_1 \\ & \downarrow \\ & CP^\infty \end{array}$$

has E_2 term:

$$\begin{array}{c|ccccccc}
5 & Sq^2 i_3 & & & & & & \\
4 & 0 & & & & & & \\
3 & i_3 & & i_3 \otimes i_2 & & & & \\
2 & 0 & 0 & 0 & & & & \\
1 & 0 & 0 & 0 & & & & \\
 & Z & 0 & i_2 & 0 & i_2^2 & 0 & i_2^3 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

The $E_2^{0;i}$ term is $H^i(K(Z;3);Z)$.

The first non vanishing differential is $d_4(i_3) = i_2^2$, the class $Sq^2 i_3$ survives for dimensional reasons and gives a generator of $H^5(X_1; Z/2)$: it follows that the five skeleton of X_1 is $S^2 \underset{f}{\cup} e^5$, that is the mapping cone of a map $f\colon S^4 \rightarrow S^3$. We are left with the task of finding $f \in \pi_4(S^2)$ but this is easy: the inclusion $S^2 \hookrightarrow X_1$ is five connected, so f can only be the nontrivial element η^2 of $\pi_4(S^2) = Z/2$.

Now given a map $\psi\colon M \rightarrow CP^\infty$ such that $(\psi^* i_2)^2 = 0$ we lift it to a map $\psi_1\colon M \rightarrow X_1$ and the first obstruction is $\psi_1^*(Sq^2 i_3) \in H^5(M;\ Z/2)$. However this is defined only up to a certain indeterminacy, due to the possible non-uniqueness of the lifting ψ_1. Two different liftings will differ (up to homotopy) by a map into the fiber $K(Z;\ 3)$ of the fibration $X_1 \rightarrow CP^\infty$; the change in the induced cohomology class on M is given by the composition $K(Z;3) \rightarrow X_1 \rightarrow K(Z/2;5)$, which, as we have seen, is classified by $Sq^2 i_3$. It follows that the obstruction is well defined in $H^5(M; Z/2)/«Sq^2»H^3(M;Z)$, where we define $«Sq^2»$ on $H^*(M;Z)$ by first reducing mod2 and then applying Sq^2.

This can be seen in a more explicit way as follows. Consider M^5, the five dimensional skeleton of M; we can assume that $\psi(M^5) \subset CP^2$ by general position. Now collapse $S^2 \subset CP^2$ to a point so that we have a map $\psi\colon M^5 \to CP^2 \to S^4$. ψ is trivial in cohomology if $c_1(L)^2 = 0$ and, in this case, the possible homotopy classes of ψ are in one to one correspondence with $H^5(M^5; Z/2)/Sq^2 H^3(M^5; Z/2)$, by Steenrod's classification theorem. Some further work, left to the reader, allows us to extend this as a cohomology class on the whole of M.

The obstruction μ: geometric definition.

It is also possible to give a simple geometric interpretation of the obstruction. For the sake of clarity we will assume that M is a manifold.

Let L be the line bundle on M and let F and F' be the zero sets of two sections of L. By general position we can assume that F and F' are smooth codimension two submanifolds with $F' \cap F$ a codimension four submanifold G.

Since $c_1(L)^2 = 0$ we can assume that G avoids the 4 skeleton. The intersection of G with the five dimensional skeleton consists of a disjoint union of circles embedded in the five cells e_i^5.

These circles come with a canonical trivialization of their normal bundle, given as follows: using the section, we identify the normal bundles to F and F' with the respective restrictions of L to them, call them $v(F)$ and $v(F')$; this implies that the normal bundle to $F \cap F'$, $v(F \cap F')$, is canonically identified with $L_{|F \cap F'} \oplus L_{|F \cap F'}$. This bundle is always trivial as an $SO(4)$ bundle because the composition of maps of classifying spaces

$$CP^\infty \xrightarrow{\Delta} CP^\infty \times CP^\infty \xrightarrow{\oplus} BSO(4)$$

is trivial as one can see looking at the induced map on the loop spaces

$$SO(2) \xrightarrow{\Delta} SO(2) \times SO(2) \to SO(4).$$

If we restrict to the five dimensional skeleton we can even find a canonical trivialization. In fact L is trivial on $F \cap F'$, which is a disjoint union of circles, so choose any trivialization ∇ of it and consider $\nabla \oplus \nabla$ as a trivialization of $L \oplus L$. This latter does not depend on the choice made because any two trivializations of L differ by a map $F \cap F' \to SO(2)$, and, as before, the composition $\oplus \circ \Delta$, $SO(2) \to SO(4)$ is trivial in homotopy.

In this way we associate to every line bundle L on M a map

$$\{\text{five cells of } M\} \to \{\text{framed circles in } R^5\}.$$

The last set projects onto the stably framed cobordism group of 1-manifolds $\Omega_1^{\text{stab}}(pt)$ which is isomorphic to $\Omega_1^{\text{spin}}(pt) \simeq Z/2$.

We can interpret this as a cochain in $C^5(M; Z/2)$.

It can be proved that this is a cocycle and that it gives a well define element in $H^5(M; Z/2)/Sq^2 H^3(M; Z)$. The proof is left as an entertaining exercise to the reader. We can use the mapping cone $\eta^2 S^4 \to S^2$ to show that the obstruction is realized by a 5-dimensional CW complex.

PART II: LINE BUNDLES OVER COMPACT MANIFOLDS

Now we study the problem of part I in the case in which M is a manifold or a Poincaré complex. In particular we will prove:

THEOREM 2. There exist a 5-dimensional Poincaré complex X and a line bundle L over M such that $c_1(L)^2 = 0$ but there is no map φ: $X \to S^2$ with $L = \varphi^*(L_0)$ with L_0 a line bundle on S^2.

However the M constructed in the proof of Theorem 1 cannot be a topological manifold, indeed more is true, that is:

THEOREM 3. Given any 5-dimensional orientable compact manifold M and a line bundle L over it such that $c_1(L)^2 = 0$; there exist a map q: $X \to S^2$ such that $L = \varphi^*(L_0)$ with L_0 a line bundle on S^2.

However this is a phenomenon typical of dimension 5 as the following shows:

PROPOSITION 4. There exists a 7-dimensional manifold with a line bundle L on it such that $c_1(L)^2 = 0$ but L is not induced by a map into S^2.

PROOF OF THEOREM 2. Consider the 2-complex $M_0 = S^2 \vee S^3$. By the Hilton-Milnor theorem $\pi_4(M_0) = Z \oplus Z/2 \oplus Z/2$, with generators the Whitehead product $[i_2; i_3]$ of infinite order, the suspension of the Hopf map $\sigma\eta$: $S^4 \to S^3$ and the composite $\eta^2 = \eta \cdot \sigma\eta$ $S^4 \to S^2$, these latters have order two.

Let's now take an element $f \in \pi_4(M_0)$ and construct the mapping cone of M_f. We have

$$H_i(M_f; Z) = H^i(M_f; Z) = Z \qquad \text{for } i = 0;2;3;5.$$

$$H_i(M_f; Z) = H^i(M_f; Z) = 0 \qquad \text{for } i = 1;4.$$

Let α be a generator for $H^2(M; Z)$, β one for $H^3(M; Z)$, and γ one for $H^5(M; Z)$. If f is written as $f = n[i_2; i_3] + \varepsilon(\sigma\eta) + \delta(\eta^2)$ with $n \in Z$, $\varepsilon, \delta \in Z/2$, the multiplicative structure is given by $\alpha \cup \beta = n\gamma$, so M_f is a Poincaré complex if and only if $n = \pm 1$. Reversing the orientation of one of the spheres we can assume $n = 1$.

We want to see also the action of the Steenrod algebra; the only possible operations are

$$Sq^1 : H^2(M_f; Z/2) \to H^3(M_f; Z/2); \qquad Sq^2 : H^3(M_f; Z/2) \to H^5(M_f; Z/2).$$

Sq^1 vanishes for every f, because all $H^2(M_f; Z/2)$ comes from reducing integral classes. In order to compute Sq^2, collapse $S^2 \subset M_f$ to a point and identify the result with the mapping cone of g: $S^4 \to S^3$ with $g = \varepsilon(\sigma\eta)$. Since Sq^2 detects the stabilization of the Hopf map, we have: $Sq^2\beta = \varepsilon \cdot \gamma$.

So, in order to have a nonvanishing obstruction group, $H^5(M; Z/2)/Sq^2 H^3(M; Z/2)$, we need $\varepsilon = 0$. The only possibility left is: $f = [i_2; i_3] + \delta\eta^2$ with $\delta \neq 0$, otherwise M_f would be homotopy equivalent to $S^2 \times S^3$ and the projection onto S^2 would give any possible line bundle. Once fixed such an M_f take the line bundle with $c_1(L) = \alpha$ and consider the lifting problem

$$\begin{tikzcd} & S^2 \arrow[d, dotted] & \\ K(Z;3) \arrow[r] & X_1 \arrow[r] \arrow[d] & K(Z/2;5) \\ M_f \arrow[r] \arrow[ur] & CP^\infty \arrow[r] & K(Z;4). \end{tikzcd}$$

The composition of any lifting to X_1 with the map into $K(Z/2;5)$ gives a well defined cohomology class in $H^5(M_f; Z/2)$; the reader will have no problems in proving that this obstruction is not vanishing.

In order to prove Theorem 3 we will need some deeper understanding of the five dimensional obstruction. This is given by the following lemma:

Lemma. There is a commutative diagram:

$$\begin{array}{ccc} \{\mathrm{Ker}\, U \colon H^2(M;Z) \to H^4(M;Z)\} & \xrightarrow{\mu} & H^5(M;Z/2)/Sq^2 H^3(M;Z/2) \\ \downarrow & & \downarrow \\ \{\mathrm{Ker}\,(Sq^1; Sq^2) \subset H^2(M;Z/2)\} & \xrightarrow{\Phi} & H^5(M;Z/2)/Sq^1 H^4(M;Z/2) \oplus Sq^2 H^2(M;Z/2) \end{array}$$

where Φ is the secondary cohomology operation associated to the Adem relation $Sq^3 Sq^1 + Sq^2 Sq^2 = 0$.

Proof. The proof consist in a diagram chase for the map of Serre-spectral-sequences induced by the map of fibrations:

$$\begin{array}{ccc} K(Z;3) \to X_\varepsilon \to K(Z/2;5) & & K(Z/2;2) \oplus K(Z/2;3) \to X_\Phi \to K(Z/2;5) \\ \downarrow & \Rightarrow & \downarrow \\ K(Z;2) \xrightarrow{u} K(Z;4) & & K(Z/2;2) \xrightarrow{(Sq^1;Sq^2)} K(Z/2;3) \oplus K(Z/2;4) \end{array}$$

This exercise in algebraic topology is left to the reader.

In the proof of Theorem 2 we will also need the lemma:

Lemma. If M is a 5-dimensional manifold the map $Sq^2 : H^3(M;Z/2) \to H^5(M;Z/2)$ is given by cup product with w_2, the second Stiefel-Whitney class of the tangent bundle.

Proof. For an elegant proof the reader is refered to the book of Browder [1].

PROOF OF THEOREM 3. The proof is inspired by [2]: assume M to be an orientable 5-manifold; in order to have $H^5(M; Z/2)/Sq^2 H^3(M; Z/2)$ non zero we want Sq^2 to be zero, so $w_2(M) = 0$ by the previous lemma. This implies that M must be spin, and so we should have a classifying map $M(v) \to M_{\text{spin}}$ where $M(v)$ and M_{spin} are the Thom spaces of the stable normal bundle to M and the universal spin bundle, respectively.

Now we compute the homology structure of $M(v)$: remember that it has the same homotopy type of the Spivak normal bundle M, and that the latter can be taken to be the Spanier-Whithead dual of $M \cup p$. (For the proof see [1]). Alexander duality gives for the cohomology of $M(v)$: $H^{N-i}(M(v); Z/2))^* \simeq H^i(M; Z/2)$.

In particular the dual of the orientation class in $H^5(M; Z/2)$ is the Thom class of $M(v)$. Also the dual of the cohomology operation Φ is the operation Φ itself defined on the Thom class u, we have $Sq^1 u = 0$ because of orientability and $Sq^2 u = 0$ because M has to be spin.

But now we have a contradiction: $\Phi(u)$ is non zero by hypothesis, but, if M were a spin manifold, u would be the pull back of the Thom class u' of B spin, and we have $\Phi(u') \in H^{N-2}(B\,\text{spin}, Z/2)$, which can be computed to be zero, see [3].

We now exhibit a 7-dimensional example in which the obstruction is non vanishing; such an example was suggested to the author by Ravenel.

PROPOSITION 5. There is a line bundle on a 7-dimensional manifold which has $c_1(L)^2 = 0$ but L is not induced by a map onto S^2.

PROOF. Consider the non trivial element of $\pi_4(SO(3)) \simeq Z/2$; this gives an S^2-bundle over S^5

$$\begin{array}{ccc} S^2 & \to & X \\ & & \downarrow \\ & & S^5 \end{array}$$

X can be written as $\left(S^2 \underset{f}{\cup} e^5\right) \underset{g}{\cup} e^7$. The attaching map $S^4 \to S^2$ is the composition $S^4 \to SO(3) \to S^2$, but the fibration $SO(3) \to S^2$ has fibre S^1 and so induces the identity of π^4. It follows that f is the only nontrivial attaching map. It has been proved in the previous section that the mapping cone for f supports the non-trivial cohomology operation μ. Since the inclusion of $S^2 \underset{f}{\cup} e^5$ in X is 6-connected, this ends the proof.

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References

[1] W. Browder, *Surgery on simply connected manifolds.* Springer Verlag, Berlin 1972.

[2] I. Madsen - R.J. Milgram, *The universal smooth surgery class.* Comm. Math. Helv., 50, 1975, 281-310.

[3] R. E. Stong, *Notes on cobordism theory.* Princeton University Press, Princeton 1968.

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