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Betti's reciprocal theorem for Cosserat elastic shells

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Meccanica dei solidi. — *Betti's reciprocal theorem for Cosserat elastic shells.* Nota di FRANCO PASTRONE, presentata (*) dal Socio C. A. TRUESDELL.

ABSTRACT. — It is proved that, as in three-dimensional elasticity, Betti's theorem represents a criterion for the existence of a stored-energy function for a Cosserat elastic shell.

KEY WORDS: Reciprocity; Elasticity; Cosserat shells.

RIASSUNTO. — *Il teorema di reciprocità di Betti per gusci elastici di Cosserat.* Viene dimostrato che il teorema di reciprocità di Betti per gusci elastici alla Cosserat, nel caso di piccole deformazioni sovrapposte ad uno stato comunque deformato, è una condizione necessaria e sufficiente per l'esistenza del potenziale elastico.

1. It is well known that any reciprocity theorem such as Betti's in three-dimensional elasticity is a criterion for the existence of a potential function (see, for instance, Truesdell [5]). We prove that Betti's theorem remains valid for Cosserat elastic shells, in the case of infinite small deformations superimposed upon an arbitrary strained state, and it provides a necessary and sufficient condition for the elastic shell to be hyperelastic.

A general treatment of Cosserat shell theory has been given by Naghdi [1, 2, 3]; in particular, in [2] Betti's theorem is proved in the linear theory, and historical remarks relevant to this subject are given in [1, 2].

We follow the intrinsic notation introduced in [3], sect. 8, and refer also to Naghdi and Trapp [4], who deal with a uniqueness theorem for shells undergoing small motion superposed on a large deformation.

2. A Cosserat elastic shell S is a smooth surface given by parametric equations $\boldsymbol{r} = \boldsymbol{r}(\vartheta^\alpha, t)$, $\alpha = 1, 2$, and equipped with a vector field (the «director» field) $\boldsymbol{d} = \boldsymbol{d}(\vartheta^\alpha, t)$, such that $\boldsymbol{d}_1 \times \boldsymbol{d}_2 \cdot \boldsymbol{d}_3 \neq \boldsymbol{0}$, $\boldsymbol{d}_\alpha = \boldsymbol{r}_{,\alpha}$, $\boldsymbol{d}_3 = \boldsymbol{d}$, $\boldsymbol{r}_{,\alpha} = \partial \boldsymbol{r}/\partial\vartheta^\alpha$, where ϑ^α are material coordinates on S, and t is time.

Greek indices take the values 1, 2; Latin indices take the values 1, 2, 3; the summation convention holds.

If S_0 is a reference configuration of the shell, $\boldsymbol{D}_i$ are the referential base vectors corresponding to the base vectors $\boldsymbol{d}_i$; $\boldsymbol{d}^i$ and $\boldsymbol{D}^i$ are the reciprocal vectors, *i.e.* $\boldsymbol{d}_i \cdot \boldsymbol{d}^j = \delta^j_i$, $\boldsymbol{D}_i \cdot \boldsymbol{D}^j = \delta^j_i$.

We make use of the deformation measures introduced in [3]:

$$\boldsymbol{F} = \boldsymbol{d}_i \otimes \boldsymbol{D}^i = \boldsymbol{r}_{,\alpha} \otimes \boldsymbol{D}^\alpha + \boldsymbol{d} \otimes \boldsymbol{D}^3\,, \qquad \boldsymbol{G} = \boldsymbol{d}_{3,\alpha} \otimes \boldsymbol{D}^\alpha\,, \tag{1}$$

where $\otimes$ denotes the tensor product.

(*) Nella seduta del 14 giugno 1990.

For our purposes, the equations of motion can be written as

(2) $$\mathrm{Div}_S\, \boldsymbol{N} + \rho_0 \boldsymbol{f} = \boldsymbol{0}, \qquad \mathrm{Div}_S\, \boldsymbol{M} - \boldsymbol{K}\boldsymbol{d}^3 + \rho_0 \boldsymbol{l} = \boldsymbol{0}.$$

The second-order tensors $\boldsymbol{N}$, $\boldsymbol{M}$, $\boldsymbol{K}$ are Piola-Kirchhoff stress tensors corresponding to the contact force, the contact director force and to the intrinsic director force (see, for instance [3], sect. 8); ρ_0 is the mass density on S_0. Let us remark that we use a material (Lagrangean) formulation. The operator Div_s is the material surface divergence operator defined in [3].

The terms $\boldsymbol{f}$ and $\boldsymbol{l}$ are the inertial forces, *i.e.* the differences

(3) $$\boldsymbol{f} = \boldsymbol{f}^{(m)} - (\ddot{\boldsymbol{r}} + y^1 \ddot{\boldsymbol{d}}), \qquad \boldsymbol{l} = \boldsymbol{l}^{(m)} - (y^1 \ddot{\boldsymbol{r}} + y^2 \ddot{\boldsymbol{d}})$$

of the assigned force $\boldsymbol{f}^{(m)}$ and the acceleration term $\ddot{\boldsymbol{r}} + y^1 \ddot{\boldsymbol{d}}$ and of the assigned director force $\boldsymbol{l}^{(m)}$ and the director acceleration term $y^1 \ddot{\boldsymbol{r}} + y^2 \ddot{\boldsymbol{d}}$, respectively; y^α are inertial coefficients (indipendent of time), the dot means differentiation with respect to t.

Traction boundary conditions

(4) $$\boldsymbol{N}\boldsymbol{n} = \boldsymbol{N}_0, \qquad \boldsymbol{M}\boldsymbol{n} = \boldsymbol{M}_0,$$

are prescribed on an appropriate portion of the boundary ∂S_0, with outward unit normal $\boldsymbol{n} = n^\alpha \boldsymbol{D}_\alpha$. Constitutive relations for elastic material are added:

(5) $$\begin{aligned} &\boldsymbol{N} + \boldsymbol{K} = \boldsymbol{N}^*(\boldsymbol{F}, \boldsymbol{G}, {}_R\boldsymbol{G}) = \boldsymbol{N}^{*i} \otimes \boldsymbol{d}^i = \boldsymbol{N}^\alpha \otimes \boldsymbol{r}_{,\alpha} + \boldsymbol{K} \otimes \boldsymbol{d}, \\ &\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{F}, \boldsymbol{G}, {}_R\boldsymbol{G}) = \boldsymbol{M}^\alpha \otimes \boldsymbol{d}, \end{aligned}$$

where ${}_R\boldsymbol{G} = \boldsymbol{D}_{3,\alpha} \otimes \boldsymbol{D}^\alpha$ and the vectors $\boldsymbol{N}^\alpha$ and $\boldsymbol{M}^\alpha$ are the classical stresses (see [1-3]).

The set $\varepsilon = \{\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{N}, \boldsymbol{M}, \boldsymbol{K}, \boldsymbol{f}, \boldsymbol{l}\}$ is called an «elastic state» corresponding to the assigned forces $(\boldsymbol{f}^{(m)}, \boldsymbol{l}^{(m)})$ if it satisfies (1)-(5).

3. Let us consider two different states ε and $\hat{\varepsilon}$ and the differences:

(6) $$\lambda\delta = \boldsymbol{F} - \hat{\boldsymbol{F}}, \qquad \lambda\eta = \boldsymbol{G} - \hat{\boldsymbol{G}}, \qquad \lambda \in R.$$

If the constitutive functionals (5) are Fréchet-differentiable in a neighbourhood of $(\hat{\boldsymbol{F}}, \hat{\boldsymbol{G}})$ we can write

(7) $$\boldsymbol{N}^* - \hat{\boldsymbol{N}}^* = \lambda\nu + o(\lambda^2), \qquad \boldsymbol{M} - \hat{\boldsymbol{M}} = \lambda\mu + o(\lambda^2),$$

where

(8) $$\nu = \boldsymbol{N}^*_F \delta + \boldsymbol{N}^*_G \eta, \qquad \mu = \boldsymbol{M}_F \delta + \boldsymbol{M}_G \eta,$$

and $\boldsymbol{N}^*_F$, $\boldsymbol{N}^*_G$, $\boldsymbol{M}_F$, $\boldsymbol{M}_G$, are fourth-order tensors given by

(9) $$\boldsymbol{N}^*_F = (\partial \boldsymbol{N}^*/\partial \boldsymbol{F})^\wedge, \quad \boldsymbol{N}^*_G = (\partial \boldsymbol{N}^*/\partial \boldsymbol{G})^\wedge, \quad \boldsymbol{M}_F = (\partial \boldsymbol{M}/\partial \boldsymbol{F})^\wedge, \quad \boldsymbol{M}_G = (\partial \boldsymbol{M}/\partial \boldsymbol{G})^\wedge.$$

Since both ε and $\hat{\varepsilon}$ satisfy (2) and (5), ν and μ satisfy the variational system

(10) $$\mathrm{Div}_s \nu + \rho_0 \phi = \boldsymbol{0}, \qquad \mathrm{Div}_s \mu - \nu^3 + \rho_0 \sigma = \boldsymbol{0},$$

where ϕ and σ are the excesses of the inertial forces:

$$\lambda\phi = f - \hat{f}, \qquad \lambda\sigma = l - \hat{l},$$

and $\nu^3 = \nu d^3$, since $\nu = \nu^i \otimes d_i$.

The boundary conditions (4) become $\nu n = t$, $\mu n = m$, t and m being the excesses of surface tractions and moments.

The fields (δ, η) can be seen as the gradients of an infinitesimal displacements (τ, ω) superimposed on the given strained state ε, as soon as we neglect the terms denoted by $o(\lambda^2)$ in (7):

$$\lambda\tau = r - \hat{r}, \qquad \lambda\omega = d - \hat{d}\,, \tag{11}$$

$$\delta = \tau_\alpha \otimes D^\alpha + \omega \otimes D^3, \qquad \eta = \omega_{,\alpha} \otimes D^\alpha\,. \tag{12}$$

4. Let φ^* and σ^* correspond to another infinitesimal displacement τ^*, ω^*, with deformation gradients δ^*, η^*, superimposed upon the same given, underlying strained state.

Scalar product equation $(10)_1$ with τ^*, and $(10)_2$ with ω^*, followed by integration of the sum of such products, over an arbitrary portion P of S_0, and the use of the divergence theorem yields the *Betti's identity* for Cosserat shells:

$$\int_{\partial P} (t \cdot \tau^* + m \cdot \omega^*)\, ds + \int_P \rho_0 (\varphi \cdot \tau^* + \sigma \cdot \omega^*)\, d\Sigma = \int_P (\nu\delta^* + \mu\eta^*)\, d\Sigma\,. \tag{13}$$

The left-hand side of (13) gives the «work» done by the field (t, m, φ, σ) over the displacement field (τ^*, ω^*) and Betti's reciprocal theorem claims that this work is equal to the work done by the field $(t^*, m^*, \varphi^*, \sigma^*)$ over the displacement (τ, ω), for any portion P of S_0.

In order that this be possible, for all displacement fields and for all portions P, it is necessary and sufficient that

$$N_F^* = N_F^{*T}, \qquad M_G = M_G^T, \qquad N_G^* = M_F^T\,, \tag{14}$$

where T means transpose, since, from (13), upon interchange of starred and unstarred quantities, it follows that

$$\int_P (\nu\delta^* + \mu\eta^*)\, d\Sigma = \int_P (\nu^* \delta + \mu^* \eta)\, d\Sigma\,. \tag{15}$$

Thus, by means of (8), we obtain (14).

If we require that Betti's theorem shall hold for infinitesimal deformations from an arbitrary underlying state of strain, conditions (14) must hold as identities in their arguments F and G (because the undeformed state is arbitrary, we can drop the hat).

If the shell is hyperelastic, there is a strain-energy density $W = W(F, G, {}_R G)$ such that

$$N^* = \rho_0 \partial W / \partial F, \quad M = \rho_0 \partial W / \partial G \tag{16}$$

and conditions (14) are obviously satisfied (if W is smooth enough to satisfy the Schwartz theorem).

Viceversa, if (14) hold as identities, they are sufficient conditions for the existence of a function $W = W(\boldsymbol{F}, \boldsymbol{G}, {}_R\boldsymbol{G})$ such that (16) hold, for any $(\boldsymbol{F}, \boldsymbol{G})$, as a consequence of the general statement of Kerner's theorem for potential operators.

Finally, we can claim that Betti's reciprocal theorem, as stated above for infinitesimal deformations from an arbitrary state of strain, provides a necessary and sufficient condition for an elastic Cosserat shell to be hyperelastic.

REMARK. The relation between Betti's theorem and potentiality is due to V. Volterra, [6], pp. 155-161, as a pure analytical statement. Hence, the results obtained in [5] and here are applications of Volterra's general theorem to elasticity.

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