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Teoria dei numeri. — *A condition for the rationality of certain elliptic modular forms over primes dividing the level.* Nota di ANDREA MORI, presentata (*) dal Corrisp. E. ARBARELLO.

ABSTRACT. — Let f be a weight k holomorphic automorphic form with respect to $\Gamma_0(N)$. We prove a sufficient condition for the integrality of f over primes dividing N. This condition is expressed in terms of the values at particular CM curves of the forms obtained by iterated application of the weight k Maaß operator to f, and extends previous results of the Author.

KEY WORDS: Modular forms; Modular curves; Complex multiplications.

RIASSUNTO. — *Una condizione per la razionalità di certe forme modulari ellittiche su primi che dividono il livello.* Sia f una forma olomorfa automorfa rispetto al gruppo $\Gamma_0(N)$ e di peso k. In questa *Nota* si dimostra una condizione per l'integralità di f rispetto ai primi che dividono N. Tale condizione è in termini dei valori assunti dalle forme ottenute da f mediante l'applicazione degli iterati dell'operatore di peso k di Maaß in punti corrispondenti a particolari curve ellittiche aventi moltiplicazione complessa. Con questa *Nota* si estende un risultato già ottenuto dall'Autore.

1. INTRODUCTION AND STATEMENT OF THE RESULT

Let Γ be a subgroup of finite index of the full modular group $SL_2(\mathbf{Z})$ and assume that Γ contains no elliptic elements. The most natural approach to the arithmetic theory of holomorphic Γ-automorphic forms is via the tensor powers of the line bundle $\mathcal{L}_\Gamma = \pi_{\Gamma,*}(\Omega^1_{\mathcal{E}_\Gamma / Y_\Gamma})$ where $\pi_\Gamma : \mathcal{E}_\Gamma \to Y_\Gamma$ is the «universal» family of elliptic curves obtained as quotient of the natural action of Γ on the tautological family $\pi : \mathcal{E} \to \mathcal{H}$ with $\pi^{-1}(z) = E_z = \mathbf{C}/\mathbf{Z}z \oplus \mathbf{Z}$. More precisely, the *modular curves* Y_Γ have canonical models [10] over finite extensions of $\mathbf{Q}$ and in fact are moduli spaces of elliptic curves with added structures [1, 6] over Dedekind rings, so it is natural to *define* the «Γ-automorphic forms of weight k defined over a ring R» [4] as the R-rational global sections of $\mathcal{L}_\Gamma^{\otimes k}$. In particular we say that a Γ-automorphic form f is *v-integral*, or *integral over v*, where v is a nonarchimedean place in a number field K, if f is defined over the ring of v-integral elements of K.

As automorphic forms defined over a subring of $\mathbf{C}$ give rise to meromorphic automorphic forms (which are just the automorphic forms defined over $\mathbf{C}$), it is natural to ask for criteria of v-integrality for holomorphic forms. The general well-known *q-expansion principle* [4] provides, with some restrictions on v and K, a criterion in terms of the Fourier coefficients of f. In [7, 9] (see also [8] for more detailed motivations) the author proves the following different criterion. Let x be a K-rational point on Y_Γ, E the corresponding elliptic curve (with some extra-structure) defined over K and pick $\tau \in \mathcal{H}$ such that $\phi_\Gamma(\tau) = x$ where $\phi_\Gamma : \mathcal{H} \to Y_\Gamma$ is the quotient map (in particular, the complex tori E_τ and $E \otimes \mathbf{C}$ are isomorphic). Let v be a non-archimedean place of K with residue field k_v of characteristic p, such that the reduction of Y_Γ modulo v is

(*) Nella seduta del 15 dicembre 1990.

smooth (this happens for all but a finite number of v). Also, assume that E has complex multiplications by a quadratic imaginary subfield K_0 of K, with p split in K_0 (*i.e.* E has *ordinary* reduction modulo v). Then

THEOREM 1. *Let f be a holomorphic Γ-automorphic form of weight k. Then if v, K, E and τ are as above, f is v-integral if and only if*

$$(1) \qquad c_n(f) = \frac{(-4\pi)^n}{\Omega_E^{k+2n}} (\delta_k^n f)(\tau) \in \mathcal{O}_v \quad \text{and} \quad v\left(\sum_{j=1}^{n} b_{j,n} c_j(f)\right) \geq v(n!) \quad \text{for all } n \geq 0,$$

where δ_k^n is the n-th iterate of the Maaß operator $\delta_k = (-1/4\pi)\ (2id/dz + k/\mathrm{Im}\,(z))$, Ω_E is the period of E and

$$\sum_{j=1}^{n} b_{j,n} X^j = n! \binom{X}{n}.$$

The goal of this *Note* is to show how the recent results [6] of Katz and Mazur can be used to extend Theorem 1 to places of non-smooth reduction. To fix ideas, we shall limit our considerations to the group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \text{ such that } c \equiv 0 \bmod N \right\}.$$

As we shall explain later in some detail, the modular curve $Y_0(N) = Y_{\Gamma_0(N)}$ can be given a structure of moduli scheme over Spec($\mathbf{Z}$) whose reduction modulo p is smooth if and only if p does not divide N. Our result is:

THEOREM 2. *Let f be a holomorphic $\Gamma_0(p^r M)$-automorphic form of weight k, with $r \geqslant 1$ and $(p, M) = 1$. Let K, E and τ as above, and Ω_E, δ_k and $b_{i,r}$ as in Theorem 1; assume also that $j(\tau) \neq 0$. Let v be a non-archimedean place of K dividing p. For each $n \geqslant 0$ let $c_n(f) = (-4\pi)^n \Omega_E^{-k-2n} (\delta_k^n f)(\tau)$. Then*

a) if $j(\tau)^{p^r} \equiv j(\tau/N) \bmod p$, then f is v-integral if and only if

$$(2) \qquad c_n(f) \in \mathcal{O}_v \quad \text{and} \quad v\left(\sum_{j=1}^{n} b_{j,n} c_j(f)\right) \geq v(n!) \quad \text{for all } n \geq 0;$$

b) if $j(\tau) \equiv j(\tau/N)^{p^r} \bmod p$, then f is v-integral if and only if

$$(3) \qquad c_n(f) \in p^{nr} \mathcal{O}_v \quad \text{and} \quad v\left(\sum_{j=1}^{n} b_{j,n} c_j(f) p^{-jr}\right) \geq v(n!) \quad \text{for all } n \geq 0;$$

c) if $(j(\tau)^{p^{a-1}} - j(\tau/N)^{p^{b-1}})^{p-1} \equiv 0 \bmod p$, for a, b positive integers such that $a + b = r$, then f is v-integral if

$$(4) \qquad c_n(f) \in p^{er} \mathcal{O}_v \quad \text{and} \quad v\left(\sum_{j=1}^{n} b_{j,n} c_j(f) p^{-je}\right) \geq v(n!) \quad \text{for all } n \geq 0;$$

where $e = 1$ if $a \geq b$ and $e = b - a$ if $a \leq b$.

Observe that the different hypothesis on the j-invariant listed in Theorem 2 exhaust all the possibilities and, in particular, Theorem 2 characterizes completely the v-integral forms when $r = 1$.

Notation. If X is a scheme over $\mathrm{Spec}(\mathbf{Z})$, we shall denote X_p the reduction of X modulo p.

2. A brief review of the proof of Theorem 1

Let us review quickly the ideas behind the proof of Theorem 1. This will give us the opportunity to set the notation, which will be slightly different from the one adopted in our main ref. [9].

Let M be an integer which we assume, for the sake of simplicity of exposition, to be at least 4. Let p be a rational prime not dividing M and v a place dividing p in a sufficiently large number field. Let x, E, τ and K_0 be as in the premise of Theorem 1. Recall that a $\Gamma_0(M)$-structure on E is, naively speaking, a cyclic subgroup of E of order M which is identified, via the isomorphism $E \otimes \mathbf{C} \simeq E_\tau$, to the subgroup generated by τ/M. Denote

$$(5) \qquad \phi_M : \mathcal{H} \to \Gamma_0(M) \backslash \mathcal{H} = Y_0 \otimes \mathbf{C}, \qquad \tau \mapsto x_M = x$$

the quotient map, and let $\mathcal{L}_M = \mathcal{L}_{\Gamma_0(M)}$. The curve $Y_0(M)$ has a smooth model [1] over the ring $\mathcal{O}_v$ and its maximal unramified extension $\mathcal{O}_v^{nr}$ (to be thought as the strict henselianization of $\mathcal{O}_v$). Applying the Serre-Tate classification of infinitesimal deformations of abelian varieties to a suitable sequence of deformations of the reduced curve $\widetilde{E} = E \otimes \overline{k_v}$, we can construct an $\mathcal{O}_v^{nr}$-rational element $q_M \in 1 + \hat{m}_x \subset \hat{\mathcal{O}}_x$ such that the (*a priori*) formal uniformizer $T_M = \log(q_M)$ is a K-rational local parameter, and in fact a common eigenvector for the action of $K_0^\times$ on the complete ring $\hat{\mathcal{O}}_x$ induced by the normalized embedding $K_0^\times \hookrightarrow GL_2^+(\mathbf{Q})$, [10], via the local isomorphism [5].

The element q_M depends only on the choice of a generator of the Tate module $T_p(\widetilde{E})$ and provides, because of smoothness of the reduction, a non-canonical identification of the image of the integral jets in $\hat{\mathcal{O}}_x$ and the ring $\mathcal{O}_v[\![q_M - 1]\!]$, [9, Corollary 8]; denote $\tilde{q}_M$ the reduction modulo p of q_M. The chosen identification $\mathbf{Z}_p \simeq T_p(\widetilde{E})$ induces also a local trivialization of the line bundles $\mathcal{L}_M^{\otimes k}$, so we can expand a holomorphic automorphic form f of weight k around x as $f = \sum_{n \geq 0} (b_n(f)/n!)\, T_M^n$. The explicit computation [3] of the Maaß operator δ_k (which can be interpreted as an operator on the k-th symmetric power of the first de Rham bundle relative to the family $\mathcal{E}_M \to Y_0(M)$) together with the explicit computation [5] of the Kodaira-Spencer map in the formal neighborhood of $\widetilde{E}$ shows that $b_n(f) = c_n(f)$. Moreover, we can show that the jet in x of f is v-integral if and only if the form f is itself v-integral. Thus, conditions (1) are nothing but the conditions that characterize the coefficients of the formal power series of the form $\sum_{n \geq 0} (a_n/n!)\, V^n \in K_v[\![V]\!]$ which arise from power series in $\mathcal{O}_v[\![U]\!]$ under the substitution $U = \exp(V) = 1 + V + (1/2)\, V^2 + (1/6)\, V^3 + \ldots$.

It is important to remark that the fact that $Y_0(M)_p$ is smooth is used only to characterize the integral jets, whereas the construction of q_M and the computation of the coefficients $b_n(f)$ in terms of the Maaß operator are independent of it. In fact, it will be clear from the proof of Theorem 2, that in order to obtain a criterion of integrality over a place v dividing M analogous to Theorem 1, it would be enough to characterize numerically the integral jets in terms of the parameter T_M.

3. $Y_0(N)$ over Spec $(\mathbf{Z})$

Now let $N = p^r M$ with $r \geq 1$ and $(p, M) = 1$. In order to give a modular interpretation of the reduction $Y_0(N)_p$ the naive notion of $\Gamma_0(N)$-structure needs to be modified. Elaborating on previous ideas of Drinfeld [2], Katz and Mazur [6] define a $\Gamma_0(N)$-structure on a generalized elliptic curve $E \to S$ as an S-isogeny ψ: $E \to E'$ such that the group scheme $\ker(\psi)$ meets every irreducible component of each geometric fiber of E, and, locally on S, there is a point P of E such that $\ker(\psi) = \sum_{j=1}^{N} [jP]$ as *Cartier divisors*. Any such isogeny is called a *cyclic N-isogeny*. General nonsense machinery [6, Chapter 4] shows that the datum of a $\Gamma_0(p^r M)$-structure on E is equivalent to the datum of a $\Gamma_0(p^r)$ and a $\Gamma_0(M)$-structure on E, so that the problem of studying $Y_0(N)_p$ is essentially reconducted to the analysis of cyclic p^r-isogenies of elliptic curves over $\mathbf{Z}/p\mathbf{Z}$-schemes. The crucial fact is that any p^r isogeny (not necessarily cyclic) ψ: $E \to E_r$ factors uniquely as

$$E \xrightarrow{\mathrm{Fr}^a} E^{(p^a)} \xrightarrow{\sim} E_r^{(p^b)} \xrightarrow{V^b} E_r$$

where Fr is the Frobenius morphism, V the *Verschiebung* and $a + b = r$. Furthermore, ψ is automatically cyclic if either $a = 0$ or $b = 0$, whereas if a and b are non-zero and E is ordinary with $j(E) \neq 0$ or 1728, then ψ is cyclic if and only if $(j(E)^{p^{a-1}} - j(E_r)^{p^{b-1}})^{p-1} = 0$. Hence $Y_0(N)_p$ has exactly $r + 1$ components $Y_{a,b}$. All these components intersect at each supersingular point and the reduction of each is isomorphic to $Y_0(M)_p$. Precisely,

$$Y_{a,b} = \begin{cases} (F^a \times F^b)^{-1}(\Delta), & \text{if either } a = 0 \text{ or } b = 0, \\ (F^{a-1} \times F^{b-1})^{-1}(\Delta_{(p-1)}), & \text{if } a, b \geq 1, \end{cases} \tag{6}$$

where F: $Y_0(M)_p \to Y_0(M)_p$ is the relative Frobenius, and $\Delta_{(p-1)}$ is the infinitesimal neighborhood of the diagonal Δ of order $p - 1$.

4. Proof of Theorem 2

The inclusion $\Gamma_0(N) \subset \Gamma_0(M)$ induces an étale map of non-compact Riemann surfaces $\phi_{N,M}$: $Y_0(N) \otimes \mathbf{C} \to Y_0(M) \otimes \mathbf{C}$, $\phi_{N,M}(x_N) = x_M$. This map is clearly the map between $\mathbf{C}$-rational points of $Y_0(N)$ and $Y_0(M)$ induced by the natural «forget the $\Gamma_0(p^r)$-structure» map of moduli schemes over Spec $(\mathbf{Z})$ $f_{N,M}$: $Y_0(N) \to Y_0(M)$. Denote $\tilde{x}_M$ (respectively $\tilde{x}_N$) the reduction modulo p of x_M (respectively x_N) and fix a and b such that $\tilde{x}_N \in Y_{a,b}$. As the actual Γ_0-structures do not enter in the construction of the Serre-Tate parameter, we must have $T_N = \phi_{N,M}^\natural(T_M)$ and $\tilde{q}_N = f_{N,M}^\natural(\tilde{q}_M)$. On the other hand, the explicit description of the component $Y_{a,b}$ given in the previous section shows that on the reduction of $Y_{a,b}$ the map $f_{N,M}$ is just the restriction of the projection on the first factor:

$$\begin{array}{ccc} (Y_{a,b})_{\mathrm{red}} = \Phi^{-1}(\Delta) \longrightarrow & Y_0(M)_p \times Y_0(M)_p \xrightarrow{\Phi} Y_0(M)_p \times Y_0(M)_p \\ & \Big\downarrow p_1 & \\ & Y_0(M)_p & \end{array}$$

where Φ is the map that defines $Y_{a,b}$ as in (6). Since $Y_{a,b}$ is, as set, just the diagonal in $Y_0(M)_p \times Y_0(M)_p$, a straightforward computation shows that the degree of the composition $(Y_{a,b})_{\mathrm{red}} \to Y_{a,b} \to Y_0(M)$ is

$$d = \begin{cases} 1 & \text{if } a \geq b, \\ p^{b-a} & \text{if } a \leq b. \end{cases}$$

Look at the diagram of complete local rings

$$\begin{array}{lcl} \hat{\mathcal{O}}_{\tilde{x}_M} \simeq k_v[[\tilde{q}_M - 1]] & & \\ \Big\downarrow f^{\natural}_{N,M} & & \\ \hat{\mathcal{O}}_{\tilde{x}_N, Y_{a,b}} \longrightarrow \hat{\mathcal{O}}_{\tilde{x}_N, (Y_{a,b})_{\mathrm{red}}} \xrightarrow{\sim} \hat{\mathcal{O}}_{\tilde{x}_M} \simeq k_v[[\tilde{q}_M - 1]]. & & \end{array} \tag{7}$$

By the above considerations, the element $\tilde{q}_N$ corresponds, under the bottom row identification in (7), to the element $\tilde{q}_M^d$. Therefore, the K-rational local parameter $T'_N = d^{-1} T_N$ is such that $q'_N = \exp(T'_N) = q_N^{1/d}$ is v-integral and $q'_N - 1 \in \hat{m}_{\tilde{x}_N} - \hat{m}^2_{\tilde{x}_N}$. The same formal manipulations used to deduce the conditions (1) in Theorem 1 show that the image of the v-integral jets in $\hat{\mathcal{O}}_{x_N} \simeq K[[T_N]]$ contains the elements $\sum_{n\geq 0}(b_n/n!)\, T_N^n$ with $b_n \in d^n \mathcal{O}_v$ and

$$v\left(\sum_{j=1}^{n} b_{j,n} b_j d^{-j}\right) \geq v(n!).$$

Moreover, these conditions characterize completely the integral jets at x_N if $\tilde{x}_N \in Y_{r,0}$ (and $d = 1$) or $\tilde{x}_N \in Y_{0,r}$ (and $d = p^r$) because the $(r, 0)$ and $(0, r)$-components are always reduced, smooth and intersect the others away from $\tilde{x}_N$.

Let f be a holomorphic $\Gamma_0(N)$-automorphic form and $\sum_{n\geq 0}(c_n(f)/n!)\, T_N^n$ its T_N-expansion. If f is v-integral, then its jet in x_N is v-integral, so if $\tilde{x}_N \in Y_{r,0}$ (respectively $Y_{0,r}$) then the numbers $c_n(f)$ must satisfy conditions (2) (respectively, conditions (3)).

Conversely, if the numbers $c_n(f)$ satisfy one of the conditions (2), (3) or (4) (according to which component $\tilde{x}_N$ belongs to) then jet of f at x_N is v-integral. Thus, we are finally reduced to the following

LEMMA. *Let R be a DVR with field of quotients K. Let X be an irreducible, locally Cohen-Macaulay scheme of relative dimension $\geqslant 1$. Let $\mathcal{L}$ be an invertible sheaf on X and f a global section of the pull-back of $\mathcal{L}$ to $X \otimes \mathbf{C}$. If the jet of f at a K-rational point is R-rational, then f lifts to a global section of $\mathcal{L}$ on X*

PROOF. Let x: $\operatorname{Spec}(R) \to X$ be an R-rational point of X and set $x_0 = x((0))$, $x_\pi = x(\pi R)$ where π is a uniformizer in R. There are natural embeddings $\operatorname{jet}^{(n)}_{x_\pi, X} \to \operatorname{jet}^{(n)}_{x_0, X \otimes K} \to \operatorname{jet}^{(n)}_{x_0, X \otimes C}$ for all n. Let us prove first that f lifts to a K-rational section.

On a sufficiently small open neighborhood of x_0 the section f can be identified to a section of $\mathcal{O}_X$. Since the stalk $\operatorname{jet}^{(n)}_{x_0, X \otimes K}$ is generated, as an $\mathcal{O}_{x_0}$-module, by $\operatorname{jet}^{(n)}(\mathcal{O}_{x_0})$, we can find elements $f_1, \ldots, f_t, g_1, \ldots, g_t \in \mathcal{O}_{x_0}$ such that

$$\operatorname{jet}^{(n)}(f) = \sum_{i=1}^{t} f_i \operatorname{jet}^{(n)}(g_i). \tag{8}$$

Any $h \in \mathcal{O}_{x_0}$ acts on the fiber at x_0 of $\operatorname{jet}^{(n)}$, which is canonically isomorphic to $\mathcal{O}_{x_0} / m_{x_0}^{n+1}$, simply as multiplication, so that (7) can be read, in $\mathcal{O}_{x_0}$, as a congruence

$$f \equiv \sum_{i=1}^{t} f_i g_1 \bmod m_{x_0}^{n+1}.$$

By Krull's theorem $f \in \mathcal{O}_{x_0}$. Hence, f is the extension of the pull-back of a K-rational section defined over an open dense subscheme of $X \otimes K$ and so it is itself K-rational.

To prove R-rationality, argue as above with $f_1, \ldots, f_t, g_1, \ldots, g_t \in \mathcal{O}_{x_\pi}$ in (8). Then f is extended to a neighborhood of x_π, and in fact to an open subscheme $U \subset X$ containing all K-rational points. Then $X - U$ is a finite union of closed points whose ideals have depth $\geqslant 2$. Therefore f extends to a global section of $\mathcal{L}$. QED

This ends the proof of Theorem 2.

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