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Teoria dei gruppi. — *Prime divisors of conjugacy class lengths in finite groups.* Nota di Carlo Casolo, presentata (*) dal Socio G. Zacher.

Abstract. — We show that in a finite group G which is p-nilpotent for at most one prime dividing its order, there exists an element whose conjugacy class length is divisible by more than half of the primes dividing $|G/Z(G)|$.

Key words: Finite groups; Conjugacy classes; Lengths.

Riassunto. — *Divisori primi della lunghezza delle classi di coniugio in gruppi finiti.* Dimostriamo che in un gruppo finito G che è p-nilpotente per al più un primo che divide il suo ordine, esiste un elemento la cui classe di coniugio ha ordine divisibile da più della metà dei primi che dividono $|G/Z(G)|$.

Introduction

If G is a finite group and $g \in G$, we denote by $\sigma_G(g)$ the set of all prime divisors of $|G: C_G(g)|$, the length of the conjugacy class of g. Also we put:

$$\rho'(G) = \bigcup_{g \in G} \sigma_G(g) \qquad \text{and} \qquad \sigma'(G) = \max_{g \in G} |\sigma_G(g)|.$$

It is easy to show that $\rho'(G)$ is the set of all prime divisors of $|G/Z(G)|$. In this *Note* we prove (Corollary 2), by an elementary method, that if G is a finite group which is not p-nilpotent for two or more primes dividing its order, then

$$(*) \qquad |\rho'(G)| \leqq 2\sigma'(G).$$

At the International Group Theory Conference in Bressanone 1989, Prof. Huppert asked whether such a bound holds for every finite soluble group. The motivation for this question comes from the observation of a not yet well understood parallelism between results on characters and results on conjugacy classes.

There is for example the conjecture that for every finite soluble group G, $|\rho(G)| \leqq 2\sigma(G)$, where $\rho(G)$ is the set of all primes dividing the degree of some irreducible complex character of G, and $\sigma(G)$ is the maximum number of distinct primes dividing the degree of a single irreducible character of G. The best result published so far in this direction, due to D. Gluck and O. Manz [3], states that for every finite soluble group G, $|\rho(G)| \leqq 3\sigma(G) + 32$.

Our result, which gives an affirmative answer to Huppert's question on conjugacy classes for a large class of finite groups, seems also to indicate that restricting to soluble groups might not be unavoidable in this contest: an immediate corollary of our Theorem is that the bound (*) holds in every finite perfect group G.

We recall that it follows from results of D. Chillag and M. Herzog [1], that $|\rho'(G)| \leqq 2$ for all finite groups G with $\sigma'(G) = 1$ (such a group is necessarily soluble); also P. Ferguson [2] has shown that $|\rho'(G)| \leqq 4$ for every finite soluble group G with

(*) Nella seduta del 15 dicembre 1990.

$\sigma'(G)=2$. In a subsequent work we will show that the inequality $(*)$ holds for every finite group G such that $\sigma'(G)=2$, and every finite soluble group with $\sigma'(G)=3$.

All groups considered in this *Note* are finite.

PROOFS. We start by fixing some more notations. If G is a group, we denote by $\pi(G)$ the set of all prime divisors of $|G|$, and by $\Delta(G)$ the set of all primes $p \in \pi(G)$ such that G is not p-nilpotent with abelian Sylow p-subgroups. If $p \in \pi(G)$ we let G_p be a Sylow p-subgroup of G and put $n_p(G) = |N_G(G_p): C_G(G_p)|$ (we simply write n_p when it will be obvious to which group we refer). Now, $n_p(G)=1$ if and only if $G_p \leqq Z(N_G(G_p))$; thus Burnside's criterion for p-nilpotency implies that $n_p(G)=1$ *if and only if G is p-nilpotent with abelian Sylow p-subgroups.*

In particular $\Delta(G) = \{p \in \pi(G);\ n_p(G) \neq 1\}$.

THEOREM. *Let G be a non-abelian group. Then:*

$$\sigma'(G) > \sum_{p \in \Delta(G)} \frac{n_p - 1}{n_p}$$

PROOF. For every $p \in \pi(G)$, we put $\mathcal{L}_p = \{g \in G;\ p \in \sigma_G(g)\} = \{g \in G;\ p \text{ divides } |G: C_G(g)|\}$.

Let $x \in G$; then $x \notin \mathcal{L}_p$ if and only if p does not divide $|G: C_G(x)|$, if and only if there exists a Sylow p-subgroup G_p of G such that $G_p \leqq C_G(x)$. Thus:

$$G \setminus \mathcal{L}_p = \bigcup_{g \in G} C_G(G_p^g).$$

Hence

$$|G| - |\mathcal{L}_p| \leqq |G|(|C_G(G_p)| - 1)/|N_G(G_p)| + 1 \leqq |G|/|N_G(G_p): C_G(G_p)| = |G|/n_p$$

and so:

$$|\mathcal{L}_p| \geqq |G| - \frac{|G|}{n_p} = |G|\left(\frac{n_p - 1}{n_p}\right). \tag{1}$$

In particular, we have that if $p \in \Delta(G)$, then $|\mathcal{L}_p| \geqq |G|/2$. We now consider the following subset of $\pi(G) \times G$: $\mathcal{S} = \{(p,x) \in \pi(G) \times G;\ p \in \sigma_G(x)\}$.

Observing that $(p,x) \in \mathcal{S}$ if and only if $x \in \mathcal{L}_p$, and by counting the number of elements of $\mathcal{S}$ in two ways, we get:

$$\sum_{p \in \pi(G)} |\mathcal{L}_p| = |\mathcal{S}| = \sum_{x \in G} |\sigma_G(x)|.$$

Since $\sigma_G(1) = \emptyset$, we may write:

$$\sum_{x \in G^*} |\sigma_G(x)| = \sum_{p \in \pi(G)} |\mathcal{L}_p|. \tag{2}$$

Hence, by formula (1):

$$\sigma'(G)(|G| - 1) \geqq \sum_{p \in \pi(G)} \left(|G| \frac{n_p - 1}{n_p}\right).$$

Now, for $p \in \pi(G) \setminus \Delta(G)$, it is $n_p - 1 = 0$, so we have:

$$|G|\,\sigma'(G) > |G| \sum_{p \in \Delta(G)} \frac{n_p - 1}{n_p},$$

and the desired result.

COROLLARY 1. *Let G be a non-abelian group. Then*: $|\Delta(G)| < 2\sigma'(G)$.

PROOF. By the Theorem:

$$\sigma'(G) > \sum_{p \in \Delta(G)} \frac{n_p - 1}{n_p} \geqq |\Delta(G)| \frac{1}{2}.$$

COROLLARY 2. *Let G be a finite group and assume that the quotient group of G by the derived subgroup contains a subgroup isomorphic to a Sylow p-subgroup of G for at most one prime* $p \in \pi(G)$, *then*: $|\rho'(G)| \leqq 2\sigma'(G)$.

PROOF. The hypothesis on G implies $|\Delta(G)| \geqq |\rho'(G)| - 1$ hence, by Corollary 1: $2\sigma'(G) > |\rho'(G)| - 1$ and so $2\sigma'(G) \geqq |\rho'(G)|$.

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