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Nonvariational basic parabolic systems of second order

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Analisi matematica. — *Non variational basic parabolic systems of second order.*
Nota (*) del Corrisp. SERGIO CAMPANATO.

ABSTRACT. — Ω is a bounded open set of $\mathbf{R}^n$, of class C^2 and $T>0$. In the cylinder $Q=\Omega\times(0,T)$ we consider non variational basic operator $a(H(u))-\partial u/\partial t$ where $a(\xi)$ is a vector in $\mathbf{R}^N$, $N\geq 1$, which is continuous in ξ and satisfies the condition (A). It is shown that $\forall f\in L^2(Q)$ the Cauchy-Dirichlet problem $u\in W_0^{2,1}(Q)$, $a(H(u))-\partial u/\partial t=f$ in Q, has a unique solution. It is further shown that if $u\in W^{2,1}(Q)$ is a solution of the basic system $a(H(u))-\partial u/\partial t=0$ in Q, then $H(u)$ and $\partial u/\partial t$ belong to $H^1_{\mathrm{loc}}(Q)$. From this the Hölder continuity in Q of the vectors u and Du are deduced respectively when $n\leq 4$ and $n=2$.

KEY WORDS: Nonlinear non variational systems; (A) condition; Existence theorem.

RIASSUNTO. — *Sistemi parabolici base non variazionali del 2° ordine.* Ω è un aperto limitato di $\mathbf{R}^n$ di classe C^2 e $T>0$. Nel cilindro $Q=\Omega\times(0,T)$ si considera l'operatore non variazionale base $a(H(u))-\partial u/\partial t$ dove $a(\xi)$ è un vettore di $\mathbf{R}^N$, $N\geq 1$, continuo in ξ il quale verifica la condizione (A). Si dimostra che $\forall f\in L^2(Q)$ il problema di Cauchy-Dirichlet $u\in W_0^{2,1}(Q)$, $a(H(u))-\partial u/\partial t=f$ in Q, ha una e una sola soluzione. Si dimostra inoltre che se $u\in W^{2,1}(Q)$ è una soluzione del sistema base $a(H(u))-\partial u/\partial t=0$ in Q, allora $H(u)$ e $\partial u/\partial t$ appartengono ad $H^1_{\mathrm{loc}}(Q)$. Se ne deduce l'holderianità in Q dei vettori u e Du rispettivamente quando $n\leqslant 4$ e $n=2$.

1. – INTRODUCTION

Let Ω be a bounded open set in $\mathbf{R}^n$, $n\geq 1$, of class C^2 and let x be a generic point in it. N is an integer ≥ 1 and Q is the cylinder $\Omega\times(0,T)$ with $T>0$, $X=(x,t)$ a point of $\mathbf{R}^n_x\times\mathbf{R}_t$ and $u(X)$ is a vector $Q\to\mathbf{R}^N$. We shall set $Du=(D_1u,\dots,D_nu)$, $H(u)=\{D_{ij}u\}$, $i,j=1,\dots,n$. Du is a vector in $\mathbf{R}^{nN}$ and $H(u)$ is an element of $\mathbf{R}^{n^2N}$, that is, it is an $n\times n$ matrix of vectors in $\mathbf{R}^N$. If $\tau\in\mathbf{R}^{n^2N}$ we set as usual

$$\mathrm{Tr.}\,\tau=\sum_{i=1}^{n}\tau_{ii}.$$

It is well known that $H^2\cap H^1_0(\Omega)$ is a Hilbert space with the norm $\|H(u)\|_{L^2(\Omega)}$.

We shall denote by $W^{2,1}(Q)$ and $W_0^{2,1}(Q)$, respectively, the Hilbert spaces of vectors u: $Q\to\mathbf{R}^N$ such that

$$u\in L^2(0,T,H^2(\Omega)),\qquad \partial u/\partial t\in L^2(Q) \tag{1.1}$$

and

$$u\in L^2(0,T,H^2\cap H^1_0(\Omega)),\qquad \partial u/\partial t\in L^2(Q),\quad u(x,0)=0\ \text{ in }\Omega. \tag{1.2}$$

We shall provide $W_0^{2,1}(Q)$ with the norm

$$\|u\|^2_{(\alpha)}=\int_Q\left[\|H(u)\|^2+\alpha^2\left\|\frac{\partial u}{\partial t}\right\|^2\right]dx\,dt \tag{1.3}$$

where α is a positive constant.

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Let $a(\xi)$ be a vector in $\mathbf{R}^N$, continuous onto $\mathbf{R}^{n^2N}$ such that $a(0)=0$. Suppose that the vector $a(\xi)$ satisfies the following condition

(A) *There exist three positive constants* α, γ *and* δ *with* $(\gamma+\delta)<1$ *such that,* $\forall \xi, \tau, \in \mathbf{R}^{n^2N}$ *we have*

$$\|\mathrm{Tr.}\ \tau - \alpha[a(\tau+\xi)-a(\xi)]\|_N \le \gamma\|\tau\| + \delta\|\mathrm{Tr.}\ \tau\|_N. \tag{1.4}$$

One shows that if the vector $a(\xi)$ is of class C^1 with its derivatives

$$\partial a(\xi)/\partial\xi_{ij} = \{\partial a^h(\xi)/\partial \xi_{ij}^k\} \qquad h,k=1,\dots,N,$$

bounded, then the fact that $a(\xi)$ satisfies the condition (A) implies that $a(\xi)$ is elliptic (see [5]).

It follows, in particular, from the condition (1.4) that $\forall \tau \in \mathbf{R}^{n^2N}$ we have

$$\|a(\tau)\| \le c(n)\|\tau\|/\alpha. \tag{1.5}$$

We shall consider the basic operator

$$a(H(u)) - \partial u/\partial t \tag{1.6}$$

and consider the Cauchy-Dirichlet problem:

Given $f\in L^2(Q)$ *to find* $u \in W_0^{2,1}(Q)$ *such that*

$$a(H(u)) - \partial u/\partial t = f \quad \text{in } Q. \tag{1.7}$$

We shall prove the following

THEOREM 1.1. *If* Ω *is of class* C^2 *and is convex and the vector* a(ξ) *satisfies the condition* (A), $\forall f \in L^2(Q)$ *the Cauchy-Dirichlet problem* (1.7) *has a unique solution.*

If $X_0=(x^0,t_0)$ and $\sigma>0$ we set $B(x^0,\sigma)=\{x\in\mathbf{R}^n\colon \|x-x^0\|<\sigma\}$, $Q(X_0,\sigma)=$ $=B(x^0,\sigma)\times(t_0-\sigma^2,t_0)$.

We say that $Q(X_0,\sigma)\subset\subset Q$ if $B(x^0,\sigma)\subset\subset\Omega$ and $\sigma^2<t_0\le T$.

Let $u\in W^{2,1}(Q)$ be a solution of the basic system

$$a(H(u)) - \partial u/\partial t = 0 \quad \text{in } Q. \tag{1.8}$$

We shall prove the following

THEOREM 1.2. *If the vector* $a(\xi)$ *satisfies the condition* (A) *then* $H(u)\in H^1_{\mathrm{loc}}(Q)$, $\partial u/\partial t \in H^1_{\mathrm{loc}}(Q)$ *and* $\forall Q(2\sigma)\subset\subset Q$ *we have the following estimates*

$$\int_{Q(\sigma)}\left[\|DH(u)\|^2 + \left\|D\frac{\partial u}{\partial t}\right\|^2\right]dx\,dt \le c(\sigma)\int_{Q(2\sigma)}[\|Du\|^2+\|H(u)\|^2]\,dx\,dt; \tag{1.9}$$

$$\int_{Q(\sigma)}\left[\left\|\frac{\partial}{\partial t}H(u)\right\|^2 + \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2\right]dx\,dt \le c(\sigma)\int_{Q(2\sigma)}\left[\left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial Du}{\partial t}\right\|^2\right]dx\,dt. \tag{1.10}$$

In view of the Sobolev imbedding theorem it follows from the implication of the Theorem 1.2 that, if u is the solution of the system (1.8), then the vector Du is Hölder continuous in Q if $n=2$, the vector u is Hölder continuous in Q if $n\le 4$.

2. – Preliminaries

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two real Banach spaces, eventually two finite dimensional Hilbert spaces. Let A and B be two mappings $\mathcal{B}_1 \to \mathcal{B}_2$.

Definition 2.1. We shall say that A is near B if there exist two positive constants α and K, with $0 < K < 1$, such that $\forall u, v \in \mathcal{B}_1$ we have

$$\|B(u) - B(v) - \alpha [A(u) - A(v)]\|_{\mathcal{B}_2} \leq K \|B(u) - B(v)\|_{\mathcal{B}_2} . \tag{2.1}$$

We have the following

Theorem 2.1. *If B: $\mathcal{B}_1 \to \mathcal{B}_2$ is a bijection and A is near B with constants α and K then A is also a bijection and $\forall u \in \mathcal{B}_1$ we have the estimate*

$$\|B(u) - B(0)\|_{\mathcal{B}_2} \leq \alpha \|A(u) - A(0)\|_{\mathcal{B}_2} / (1 - K) \tag{2.2}$$

([3] *Theorem* 2.1).

Since Ω is of class C^2 and is convex we have the following estimate due to C. Miranda and G. Talenti: $\forall u \in H^2 \cap H_0^1(\Omega)$

$$\int_\Omega \|H(u)\|^2 \, dx \leq \int_\Omega \|\Delta u\|^2 \, dx .$$

As a consequence, we have, if Ω is of class C^2 and is convex and if $Q = \Omega \times (0, T)$, the following

Lemma 2.1. *For each $\alpha > 0$ and $\forall u \in W_0^{2,1}(Q)$*

$$\|u\|_{(\alpha)}^2 \leq \int_Q \left[\|\Delta u\|^2 + \alpha^2 \left\| \frac{\partial u}{\partial t} \right\|^2 \right] dx \, dt . \tag{2.3}$$

We have the following Lemma ([6] Lemma 1.I)

Lemma 2.2. *For each $u \in W_0^{2,1}(Q)$ the following estimate holds*

$$\int_Q \left(\Delta u \;\middle|\; \frac{\partial u}{\partial t} \right)_N dx \, dt \leq 0 . \tag{2.4}$$

As a consequence we obtain

Lemma 2.3. *If Ω is of class C^2 and is convex then $\forall u \in W_0^{2,1}(Q)$ and $\forall \alpha > 0$ we have*

$$\|u\|_{(\alpha)}^2 \leq \int_Q \left\| \Delta u - \alpha \frac{\partial u}{\partial t} \right\|^2 dx \, dt . \tag{2.5}$$

This is a trivial consequence of (2.3) and (2.4).

Lemma 2.4. *If Ω is of class C^2 and is convex, $\forall u \in W_0^{2,1}(Q)$ and $\forall \alpha > 0$ we have*

$$\int_Q \|\Delta u\|_N^2 \, dx\, dt \leq \int_Q \left\| \Delta u - \alpha \frac{\partial u}{\partial t} \right\|_N^2 dx\, dt. \tag{2.6}$$

This is a trivial consequence of the estimate (2.4).

3. – Proof of the Theorem 1.1

In view of the estimate (1.5) the operator

$$a(H(u)) - \partial u/\partial t \tag{3.1}$$

maps $W_0^{2,1}(Q)$ into $L^2(Q)$. On the other hand it is well known that the linear operator

$$\Delta u - \alpha \partial u/\partial t \tag{3.2}$$

is an isomorphism of $W_0^{2,1}(Q)$ into $L^2(Q)$. We choose as α in (3.2) to be exactly the positive constant that appears in the condition (A) on the vector $a(\xi)$. In virtue of Theorem 2.1, in order to show that, $\forall f \in L^2(Q)$, the Cauchy-Dirichlet problem (1.7) has a unique solution one is reduced to show that the operator (3.1) is near the operator (3.2), which means that we should show that there exists a constant $K \in (0, 1)$ such that, $\forall u,\ v \in W_0^{2,1}(Q)$, we have

$$\int_Q \|\Delta(u-v) - \alpha[a(H(u)) - a(H(v))]\|_N^2 \, dx\, dt \leq \tag{3.3}$$

$$\leq K^2 \int_Q \left\| \Delta(u-v) - \alpha \frac{\partial(u-v)}{\partial t} \right\|_N^2 dx\, dt.$$

On the other hand, it follows from the condition (A) that, $\forall \varepsilon > 0$ and $\forall \xi,\ \tau \in \mathbf{R}^{n^2 N}$, we have $\|\text{Tr.}\ \tau - \alpha[a(\tau+\xi) - a(\xi)]\|_N^2 \leq (1+\varepsilon)\gamma^2\|\tau\|^2 + (1+1/\varepsilon)\delta^2\|\text{Tr.}\ \tau\|_N^2$. Assuming $\varepsilon = \delta/\gamma$ we find that

$$\|\text{Tr.}\ \tau - \alpha[a(\tau+\xi) - a(\xi)]\|_N^2 \leq \gamma(\gamma+\delta)\|\tau\|^2 + \delta(\gamma+\delta)\|\text{Tr.}\ \tau\|_N^2 \tag{3.4}$$

We observe that

$$\gamma(\gamma+\delta) + \delta(\gamma+\delta) = (\gamma+\delta)^2 < 1. \tag{3.5}$$

It now follows, from the estimate (3.4) and the Lemmas 2.3 and 2.4, that,

$\forall u, v \in W_0^{2,1}(Q)$, we have

$$\int_Q \|\Delta(u-v) - \alpha[a(H(u)) - a(H(v))]\|_N^2 \, dx\, dt \leq$$

$$\leq \gamma(\gamma+\delta) \int_Q \|H(u-v)\|^2 \, dx\, dt + \delta(\gamma+\delta) \int_Q \|\Delta(u-v)\|_N^2 \, dx\, dt \leq$$

$$\leq \gamma(\gamma+\delta) \|(u-v)\|_{(\alpha)}^2 + \delta(\gamma+\delta) \int_Q \|\Delta(u-v)\|_N^2 \, dx\, dt \leq$$

$$\leq (\gamma+\delta)^2 \int_Q \left\| \Delta(u-v) - \alpha \frac{\partial(u-v)}{\partial t} \right\|_N^2 dx\, dt.$$

And we thus have (3.3) with $K = (\gamma+\delta) < 1$. Hence the Theorem (1.1) is proved.

We observe that, in view of the estimate (2.2) and the Lemma 2.3, the solution u of the Cauchy-Dirichlet problem (1.7) satisfies the following estimate

$$\|u\|_{(\alpha)} \leq \|\Delta u - \alpha\, \partial u/\partial t\|_{L^2(Q)} \leq \alpha \|f\|_{L^2(Q)}/[1-(\gamma+\delta)]. \tag{3.6}$$

4. – Proof of Theorem 1.2

Let $u \in W^{2,1}(Q)$ be a solution of the basic system

$$a(H(u)) - \partial u/\partial t = 0 \quad \text{in } Q \tag{4.1}$$

where $a(\xi)$ is a vector of $\mathbf{R}^N$, continuous into $\mathbf{R}^{n^2N}$, $a(0)=0$ and $a(\xi)$ satisfies the condition (A).

Let us fix a cylinder $Q(2\sigma) = Q(X_0, 2\sigma) \subset\subset Q$.

Let $\theta(x)$ and $g(t)$ be two C^∞ functions, respectively in $\mathbf{R}^n$ and $\mathbf{R}$, with the following properties: $0 \leq \theta \leq 1$, $\theta = 1$ on $B(x^0, \sigma)$, $\theta = 0$ in $\mathbf{R}^n \setminus B(3\sigma/2)$, $|D^\alpha \theta| \leq c\sigma^{-|\alpha|}$ $\forall$ multi-indices α, and $0 \leq g \leq 1$, $g = 1$ for $t \geq t^0$, $g = 0$ for $t \leq t^0 - 3\sigma^2$, $|g'(t)| \leq c\sigma^{-2}$.

We set $\rho_{s,h} u(X) = u(x + he^s, t) - u(X)$, $s = 1, \dots, n$ and $|h| < \sigma/4$; $\rho_{t,h} u(X) = u(x, t + h) - u(X)$, $|h| < \sigma/4$ and let us first consider the increments with respect to the variables x_s, $s = 1, \dots, n$.

Let

$$\mathcal{U}(X) = \theta(x)\, g(t)\, \rho_{sh} u. \tag{4.2}$$

Obviously $\mathcal{U} \in W_0^{2,1}(Q(3\sigma/2))$. From the system (4.1) we have $\rho_{sh}\, a(H(u)) - \rho_{sh}\, \partial u/\partial t = 0$ in $Q(3\sigma/2)$.

On the other hand $\rho_{sh}\, a(H(u)) = a(H(\rho_{sh} u) + H(u)) - a(H(u))$ and hence also $\Delta(\rho_{sh} u) - \alpha \rho_{sh}\, \partial u/\partial t = \Delta(\rho_{sh} u) - \alpha[a(H(\rho_{sh} u) + H(u)) - a(H(u))]$, where α is the positive constant which appears in the condition (A).

In view of the condition (A), we get from this that

$$\|\theta g[\Delta(\rho_{sh} u) - \alpha\rho_{sh}\, \partial u/\partial t]\|_N \leq \theta g[\gamma\|H(\rho_{sh} u)\| + \delta\|\Delta(\rho_{sh} u)\|_N . \tag{4.3}$$

On the other hand

$$\begin{aligned} \Delta\mathcal{U} &= \theta g \Delta(\rho_{sh} u) + A(u), \\ H(\mathcal{U}) &= \theta g H(\rho_{sh} u) + B(u), \\ \partial\mathcal{U}/\partial t &= \theta g \rho_{sh}\, \partial u/\partial t + \theta g' \rho_{sh} u, \end{aligned} \tag{4.4}$$

where

$$A(u) = g\Delta\theta\cdot\rho_{sh} u + 2g\sum_i D_i\theta\cdot\rho_{sh}\, D_i u,\quad B(u) = g\sum_{ij} D_{ij}\theta\cdot\rho_{sh} u + 2g\sum_{ij} D_i\theta\cdot D_j(\rho_{sh} u). \tag{4.5}$$

Hence, $\forall\varepsilon > 0$, we have

$$\begin{aligned} \|\Delta\mathcal{U} - \alpha\,\partial\mathcal{U}/\partial t\|_N \leq (1+\varepsilon)^2 \{\gamma(\gamma+\delta)\,\|H(\mathcal{U})\|^2 + \delta(\gamma+\delta)\,\|\Delta\mathcal{U}\|^2\} + \\ + c(\varepsilon)\,\{\|A(u)\|^2 + \|B(u)\|^2 + \theta^2 g'^2 \|\rho_{sh} u\|^2\}. \end{aligned} \tag{4.6}$$

On integrating (4.6) on $Q(3\sigma/2)$, taking into account that $\mathcal{U} \in W_0^{2,1}(Q(3\sigma/2))$ and taking into consideration the Lemmas 2.3 and 2.4, we obtain, for ε sufficiently small, that

$$\begin{aligned} \int_{Q(3\sigma/2)} \left[\|(H(\mathcal{U}))\|^2 + \alpha^2\left\|\frac{\partial\mathcal{U}}{\partial t}\right\|^2\right] dx\,dt \leq \int_{Q(3\sigma/2)} \left\|\Delta\mathcal{U} - \alpha\frac{\partial\mathcal{U}}{\partial t}\right\|^2 dx\,dt \leq \\ \leq c \int_{Q(3\sigma/2)} [\|A(u)\|^2 + \|B(u)\|^2 + g'^2\, \|\rho_{sh} u\|^2]\, dx\,dt \end{aligned}$$

and also

$$\int_{Q(\sigma)} \left\|\rho_{sh}\left[H(u) + \alpha\frac{\partial u}{\partial t}\right]\right\|^2 dx\,dt \leq c \int_{Q(3\sigma/2)} [\|A(u)\|^2 + \|B(u)\|^2 + g'^2\, \|\rho_{sh} u\|^2]\, dx\,dt\,. \tag{4.7}$$

We evaluate the right hand side of (4.7) using the Lemmas of Nirenberg (see for example [4]).

$$\begin{aligned} \int_{Q(3\sigma/2)} \|A(u)\|^2\, dxdt \leq c\sigma^{-4} \int_{Q(3\sigma/2)} \|\rho_{sh} u\|^2\, dx\,dt + c\sigma^{-2} \int_{Q(3\sigma/2)} \|\rho_{sh} Du\|^2\, dx\,dt \leq \\ \leq c|h|^2 \left\{\sigma^{-4} \int_{Q(2\sigma)} \|Du\|^2\, dx\,dt + \sigma^{-2} \int_{Q(2\sigma)} \|H(u)\|^2 dx\,dt\right\}. \end{aligned} \tag{4.8}$$

One estimates

$$\int_{Q(3\sigma/2)} \|B(u)\|^2\, dx\,dt$$

in a similar way. Finally

$$\int_{Q(3\sigma/2)} \theta^2 g'^2\, \|\rho_{sh} u\|^2\, dx\,dt \leq c\sigma^{-4}|h|^2 \int_{Q(2\sigma)} \|Du\|^2\, dx\,dt. \tag{4.9}$$

From the estimates (4.7), (4.8) and (4.9) we conclude that $DH(u)$ and $D\,\partial u/\partial t$ exist in $L^2(Q(\sigma))$ and we have the following estimate

$$\int_{Q(\sigma)} \left[\|H(Du)\|^2 + \alpha^2 \left\| \frac{\partial Du}{\partial t} \right\|^2\right] dx\,dt \le c(\sigma) \int_{Q(2\sigma)} [\|Du\|^2 + \|H(u)\|^2]\, dx\,dt. \tag{4.10}$$

The same procedure can be repeated to estimate the increment with respect to the variable t and one obtains

$$\int_{Q(\sigma)} \left\| \rho_{th}\left[H(u) + \alpha \frac{\partial u}{\partial t}\right] \right\|^2 dx\,dt \le c(\sigma)\,|h|^2 \int_{Q(3\sigma/2)} \left[\left\| \frac{\partial u}{\partial t} \right\|^2 + \left\| \frac{\partial Du}{\partial t} \right\|^2\right] dx\,dt. \tag{4.11}$$

From this, in view of the Lemmas of Nirenberg, we conclude that $\partial H(u)/\partial t$ and $\partial^2 u/\partial t^2$ also exist and belong to $L^2(Q(\sigma))$ and we have the following estimate

$$\int_{Q(\sigma)} \left[\left\| \frac{\partial}{\partial t} H(u) \right\|^2 + \alpha^2 \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2\right] dx\,dt \le c(\sigma) \int_{Q(3\sigma/2)} \left[\left\| \frac{\partial u}{\partial t} \right\|^2 + \left\| \frac{\partial Du}{\partial t} \right\|^2\right] dx\,dt. \tag{4.12}$$

5. – A result on Hölder continuity

A result on Hölder continuity of the vector Du follows from the Theorem 1.2.

THEOREM 5.1. *If $u \in W^{2,1}(Q)$ is a solution of the basic system*

$$a(H(u)) - \partial u/\partial t = 0 \quad \textit{in } Q \tag{5.1}$$

and if $n = 2$ *then the vector* Du *is Hölder continuous in* Q.

In fact, $\forall Q(\sigma) \subset Q$ we have the Poincaré inequality

$$\int_{Q(\sigma)} \|Du - (Du)_{Q(\sigma)}\|^2\, dx\,dt \le C(Q)\,\sigma^2 \int_{Q(\sigma)} \left[\|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2\right] dx\,dt \tag{5.2}$$

(see for example, Lemma 2.II in [7]).

On the other hand $H(u) \in H^1_{\mathrm{loc}}(Q)$ and $\partial u/\partial t \in H^1_{\mathrm{loc}}(Q)$ and hence, by the Sobolev imbedding Theorem, we have $H(u) \in L^p_{\mathrm{loc}}(Q)$ and $\partial u/\partial t \in L^p_{\mathrm{loc}}(Q)$ where $1/p = (n-1)/[2(n+1)]$. We also have the estimate

$$\int_{Q(\sigma)} \|H(u)\|^2\, dX \le c \left(\int_{Q(\sigma)} \|H(u)\|^p\, dX \right)^{2/p} \sigma^{(n+2)(1-2/p)}, \tag{5.3}$$

$$\int_{Q(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 dX \le c \left(\int_{Q(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^p dX \right)^{2/p} \sigma^{(n+2)(1-2/p)}. \tag{5.4}$$

It follows from (5.2), (5.3) and (5.4) that

$$Du \in \mathcal{L}^{2,2+(n+2)(1-2/p)}_{\mathrm{loc}}(Q) \tag{5.5}$$

and hence Du is Hölder continuous in Q if $n = 2$.

For each $Q(\sigma) \subset Q$ we also have the following Poincaré inequality

$$\int_{Q(\sigma)} \|u - (u)_{Q(\sigma)}\|^2 \, dX \leq c\sigma^2 \int_{Q(\sigma)} \|D(u)\|^2 \, dX + c\sigma^4 \int_{Q(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 dX \tag{5.6}$$

(see for instance Lemma 2.I in [7]).

Using (5.4) and (5.5), it follows from this that

$$u \in \mathcal{L}^{2,4+(n+2)(1-2/p)}_{\mathrm{loc}}(Q) \tag{5.7}$$

and hence u is Hölder continuous in Q if $n \leqslant 4$.

Remembering what happens for solutions of a basic non variational elliptic system [5], we think that the Hölder continuity result of this last section is not optimal.

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