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On the cohomological strata of families of vector bundles on algebraic surfaces

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Geometria algebrica. — *On the cohomological strata of families of vector bundles on algebraic surfaces.* Nota di Edoardo Ballico, presentata (*) dal Socio G. Zacher.

Abstract. — In this *Note* we study certain natural subsets of the cohomological stratification of the moduli spaces of rank 2 vector bundles on an algebraic surface. In the last section we consider the following problem: take a bundle E given by an extension, how can one recognize that E is a certain given bundle? The most interesting case considered here is the case $E = T\boldsymbol{P}^3(t)$ since it applies to the study of codimension 1 meromorphic foliations with singularities on $\boldsymbol{P}^3$.

Key words: Vector bundles; Algebraic surface; Moduli scheme; Meromorphic foliations; Meromorphic foliations with singularities.

Riassunto. — *Stratificazioni coomologiche di famiglie di fibrati vettoriali sulle superfici algebriche.* In questa *Nota* si studiano sottoinsiemi opportuni della stratificazione per gruppi di coomologia degli schemi di moduli di fibrati vettoriali di rango 2 sulle superfici algebriche. Inoltre si considera il seguente problema: dato un fibrato E come estensione, come riconoscere che E è isomorfo ad un altro fibrato assegnato? Si considera il caso $E = T\boldsymbol{P}^3(t)$ che viene applicato allo studio delle fogliazioni meromorfe con singolarità su $\boldsymbol{P}^3$.

This paper (except the last section) is devoted to the definition and study of particular families of rank 2 vector bundles on an algebraic surface, X (with the strong restriction that $p_g(X) = 0$). These families (which will be called *sheets*) are defined by cohomological conditions. In favorable cases the sheets give a (rather coarse) stratification of the moduli spaces of bundles on X. Under very strong assumptions on X we will give the birational structure of the sheets (which turned out to be irreducible) (see the end of § 1). In § 2 we consider a few examples ($\boldsymbol{P}^2$ and rational ruled surfaces). In these examples we study the properties of the general bundle in each sheet and the existence in each sheet of bundles with exceptional behavior (a given family of higher order «jumping lines»). In § 3 we introduce (and study in very particular cases) other related notions for a vector bundle, E, on a variety V with rank $(E) = \dim(V)$. In the last section we consider the following *recognition problem*; fix a vector bundle E, given, for instance, as an extension; how to recognize that E is isomorphic to a certain assigned bundle? Here we consider three examples; by far the more interesting one is the case $E \cong T\boldsymbol{P}^3(t)$. This case can be applied (see 4.3 for a beginning) to the study and construction of codimension-1 meromorphic foliations with singularities on $\boldsymbol{P}^3$. The interesting point on such foliations is that, due to the integrability condition, this is a non linear problem (and contrary to most of the other cases considered in [14-18] (and many other papers) with algebro-geometric techniques).

1. Every scheme in this paper will be algebraic over an algebraically closed field $\boldsymbol{K}$. Denote by $\boldsymbol{I}_{Z,T}$ the ideal sheaf of the subscheme Z of the scheme T. Fix a smooth, con-

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nected, complete surface X. Set $K := K_X$, $\mathbf{O} := \mathbf{O}_X$, $\mathbf{I}_Z := \mathbf{I}_{Z,X}$; for a sheaf Q on X, set $H^i(Q) := H^i(X, Q)$. Fix $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbf{Z}$. Let $V(c_1, c_2)$ be the set of (isomorphism classes of) rank two vector bundles on X with Chern classes c_1 and c_2.

Definition 1.1. Take $M \in \mathrm{Pic}(X)$. Set $U(M; c_1, c_2) := \{E \in V(c_1, c_2)$: $E(M)$ has a section, s, with zero dimensional 0-locus $(s)_0$ and $h^0(E(M \otimes K)) = 0\}$; we will write $U(M)$ instead of $U(M; c_1, c_2)$ when there is no danger of misunderstanding. $U(M)$ will be called a sheet of $V(c_1, c_2)$. Let $U'(M; c_1, c_2)$ (or $U'(M)$) the subset of $U(M; c_1, c_2)$ formed by the bundles such that in the definition of $U(M; c_1 c_2)$ we may take $(s)_0$ reduced.

Of course, $U(M) = \emptyset$ for every M if $p_g(X) > 0$. In this case (or if $q(X) > 0$: see below) a reasonable notion is the notion of *good strata* introduced in [5]; the corresponding problem is to see how the good strata fit together (*i.e.* seen as subvarieties of a moduli space, when the closure of a good stratum contains (or intersects) another good stratum), the cohomological properties of a general member of a good stratum and the possible pathologies of particular members of a good stratum.

Fix $M \in \mathrm{Pic}(X)$ and $E \in U(M)$; set $L := M^{\otimes 2} \otimes c_1$ and $F := E(M)$. For line bundles we will shift often between additive and multiplicative notations (even between divisors and associated line bundles, if there is no danger of misunderstanding) and use standard abuses of notations like $|L|$. Choose $s \in H^0(E(M))$ with $\dim((s)_0) = 0$: set $Z := (s)_0$. The choice of s induces the following exact sequence:

$$0 \to \mathbf{O} \to E(M) \to L \otimes \mathbf{I}_Z \to 0. \tag{1}$$

By the general theory of Chern classes, we have length $(Z) = c_2(E(M))$; it is very well known (see for instance [20], beginning of §2) that $c_2(E(M)) = c_2(E) + c_1(E) \cdot M + M^2$.

Lemma 1.2. *Assume $q(X) = p_g(X) = 0$ and $h^0(K \otimes L) \neq 0$. We have $U(M) \neq \emptyset$ if and only if $c_2 + M^2 + c_1 M > h^0(K \otimes L)$; furthermore $U(M) \neq \emptyset$ if and only if $U'(M) \neq \emptyset$. Assume $p_g(X) = h^0(K \otimes L) = 0$; then $U'(M) \neq \emptyset$.*

Proof. First assume $h^0(K \otimes L) \neq 0$. We just remarked that if $E \in U(M)$, the left hand side of the inequality in the statement to be proved is $c_2(E(M))$. Fix $E \in U(M)$ fitting in (1). By definition of sheet and the fact that $q(X) = 0$, in the statement of 1.2 we have at least the non strict inequality. The strict inequality follows from the non strict one, the definition of sheet and the Cayley-Bacharach condition (see [10] for the case in which Z is not reduced). Viceversa, by the other half of the Cayley-Bacharach property (for reduced Z) if the strict inequality in 1.2 holds then for a general Z the general extension $E(M)$ in (1) is a bundle. By the generality of Z and $q(X) = 0$, $E(M) \in U'(M)$. Now assume $h^0(K \otimes L) = 0$; the thesis follows at once from the other implication of the Cayley-Bacharach property (for reduced Z). ♦

Under suitable assumptions on M (or for good X for every M), the notion of sheet and good stratum coincide; in that case we have irreducibility results for $U(M)$, results on the irregularity of any smooth model of a compactification of $U(M)$ and cohomological results on the general member of $U(M)$ (see [4, 6]).

2. In this section we will study in details two cases: X is a Segre-Hirzebruch surface $F_e (e \geq 0)$ and $X = \boldsymbol{P}^2$. Furthermore, we will introduce (and begin to study) other natural problems.

(2.1) First we assume $X := F_e$ with $e \geq 0$. Choose the usual basis h, f of Pic (S) with $h^2 = -e$, $hf = 1$, $f^2 = 0$. Fix $M \in$ Pic (S). Set $L := c_1 + 2M$, say $|L| = |ah + bf|$. Assume $a \geq 0$ and $b \geq ae$. The general splitting type of a rank two bundle, E, is the ordered set of degree of the factors of the restriction of E to a general fiber $F \in |f|$ of S; an element $D \in |f|$ is called a jumping line of order (v, u) (with $v < u - 1$) if $E|D$ has not the general splitting type of E and $E|D \cong \boldsymbol{O}_D(v) \oplus \boldsymbol{O}_D(u)$.

Remark 2.1.1. By 1.2 and the adjunction formula, we have $U(M) \neq \emptyset$ if and only if $c_2 + M^2 + c_1 M > \dim |(a-2)h + (b-e-2)f|$ (and if and only if $U'(M) \neq \emptyset$).

Proposition 2.1.2. *Let (u, ν) with $\nu \leq u$, be the general splitting type of general $E \in U(M)$. Then $\nu = [a/2]$. Furthermore for all integers t with $0 \leq t \leq b - ea + e - 2 + 1$, any t fibers F_i, $1 \leq i \leq t$, of X and sets of integers (ν_i, u_i), $1 \leq i \leq t$, with $0 < \nu_i < [a/2]$, there is $E' \in U(M)$ with each F_i as jumping line of splitting type (ν_i, u_i)*

Proof. Set $u' := [(a+1)/2]$. Take $E(M)$ given by a general extension (1) with Z as general as possible subject to the only restriction that on a fixed fiber, F, we have card $(Z) \cap F = u'$. The restriction of (1) to F gives an inclusion $\boldsymbol{i}$ of $\boldsymbol{O}_F$ into $E(M)|F$ which drops rank exactly on $F \cap Z$. Furthermore, since Z is reduced, a local calculation shows that at each point of $Z \cap F$ the torsion part of coker $(\boldsymbol{i})$ has length 1. Thus $E(M)|F$ has a subbundle of degree u'. Since $\deg(E(M)|F) = a$, $E(M)|F$ must have the splitting type we want. It remains to check that E is a bundle, *i.e.* the Cayley-Bacharach property. It is sufficient to check that for every $P \in Z$, we have $h^0(X, L \otimes K \otimes \boldsymbol{I}_{Z \setminus \{P\}}) = 0$. This is true because $|L + K| = |(a-2)(h + ef) + (b - ea + e - 2)f|$, card $(Z) > h^0(K \otimes L)$ by 2.1.1, and $|h + ef|$ has no base point. This fact explains also the inequality on t appearing on the second part of the statement. For the second part, fix Z as general as possible with the restriction that card $(Z \cap F_i) = u_i$ for every i, and repeat the first part of the proof. ♦

(2.2) Now assume $X = \boldsymbol{P}^2$. In this case sheets and good strata agree, since $K \cong \boldsymbol{O}(-3)$ (hence we have irreducibility and unirationality of the sheets (see [5]). Here we point out only that, in the range of stable bundles, the proof of 2.1.2 gives that a general member of a sheet has balanced general splitting type even if char $(\boldsymbol{K}) > 0$ (hence, when Grauert-Mulich restriction theorem fails) and that there are particular bundles with certain assigned jumping lines of suitable orders. Furthermore, again in the range of stable bundles, we see that if $U(M; c_1, c_2) \neq \emptyset$ the closure of $U(M; c_1, c_2)$ contains $U(M(-1); c_1, c_2)$ (using again that $K \cong \boldsymbol{O}(-3)$).

(2.3) We denote by $\oplus_L$ a direct sum over all $L \in \mathrm{Pic}(X)$. Set $A := \oplus_L H^0(X, L)$ (a commutative ring); for any sheaf G on X, set $M^i(G) := \oplus_L H^i(X; G \otimes L)$; we are interested in $M^i(G)$ as an A-module. About $M^i(G)$ we are interested essentially in three cases:

(*a*) $i = 1$ and $G = \mathbf{O}$;

(*b*) G locally free (essentially if $i \leqslant 1$);

(*c*) $i = 1$ and $G = \mathbf{I}_Z$ with Z 0-dimensional subscheme.

By (1), (*b*) and (*c*) are related, although except for very particular X, with the notations of (1), solving (*c*) does not give at once the solution of (*b*) for $G := E$. Case (*c*) is very easy in the case arising (using (1)) from case (*b*) for general bundles in a good stratum in the sense of [5], since then we may take Z general in its component of Hilb (X). But even if $X = \mathbf{P}^2$ there is an interesting related question (see 2.4) and more should investigated.

(2.3.1) Fix $X = \mathbf{P}^2$, $L = \mathbf{O}(n)$ with $n > 0$, $Z \subset X$ (Z reduced for simplicity), Z of length d, and $z \in H^0(L)$, $z \neq 0$, set $C := (z)_0$.

Claim. For every t the multiplication by z is surjective.

Proof of the claim. Tensor (1) by $\mathbf{O}_C$. If $C \cap Z \neq \emptyset$, note that the torsion part, T, of $\mathbf{I}_Z \otimes \mathbf{O}_C$ has finite support, hence $H^1(X, T) = 0$, and that $(\mathbf{I}_Z \otimes \mathbf{O}_C)/T$ is isomorphic to the ideal sheaf of $Z \cap C$ in C. ♦

Of course, the claim holds for suitable C on other surfaces.
What is not trivial, even in $\mathbf{P}^2$, is the following related question:

(2.4) Question. Fix an integer $h > 0$ and a vector space $W \subseteq H^0(\mathbf{O}(h))$. For every integer t, W induces a map t_W: $H^1(\mathbf{I}_Z(t)) \to W^* \otimes H^1(\mathbf{I}_Z(t+h))$. What can be said about the dimensions of Ker (t_W) and Coker (t_W)?

The more interesting case is the case $h = 1$ (although related very interested bundles appeared for all $h > 1$ in [24, 3]). We will assume $h = 1$; since if dim $(W) = 1$, the question is covered by the claim, we have two cases to consider, according to dim (W).

Case (*i*). Assume $W = H^0(\mathbf{O}(1))$. Tensoring the Euler's sequence of $T\mathbf{P}^2$ by $\mathbf{I}_Z(t)$, we get:

$$0 \to \mathbf{I}_Z(t) \to W^* \otimes \mathbf{I}_Z(t+1) \to T\mathbf{P}^2 \otimes \mathbf{I}_Z(t) \to 0 \tag{2}$$

whose cohomology determines the solution to our question. Assume that Z has good postulation («maximal rank»), *i.e.* assume that for every integer m either $h^0(\mathbf{I}_Z(m)) = 0$ or $h^1(\mathbf{I}_Z(m)) = 0$ (*e.g.* assume Z general). Then Coker (t_W) has dimension $h^0(T\mathbf{P}^2 \otimes \mathbf{I}_Z(t))$ if length $(Z) \leqslant (t+2)(t+1)/2$, dimension $3h^0(\mathbf{I}_Z(t+1)) = 3(\max(0, (t+3)(t+2)/2 - \text{length}(Z)))$ if length $(Z) > (t+2)(t+1)/2$. For every pair

(t, length (Z)) and general Z the cohomology of $TP^2 \otimes I_Z$ was determined for the first time in [13] (in a very different language; see [8] for the translation).

Case (*ii*). Assume $\dim(W) = 2$. Let P the base point of the pencil determined by W. W gives an exact sequence

$$0 \to O \to 2O(1) \to I_{\{P\}}(1) \to 0. \tag{3}$$

If $P \notin Z$ the cohomology of $Z \cup \{P\}$ determines $\dim(\mathrm{Coker}(t_W))$. Now assume $P \in Z$. Let Z' be the union of $Z \setminus \{P\}$ and the first infinitesimal neighborhood of P. Note that the quotient of $I_Z \otimes I_{\{P\}}$ by its torsion is isomorphic to $I_{Z'}$. Hence the cohomology of Z' determines the solution to our question. We have the following result.

PROPOSITION 2.4.1. *Fix integers d, n; set $R := O(n)$. For general $Z \subset P^2$ with* card $(Z) = d$, *and every choice of $P \notin Z$, set $Z''(P) := Z \cup \{P\}$ we have* $h^0(R \otimes O_{Z''(P)}) = \max(0, h^0(R) - (d+1))$, *except in the following two cases:*

(*a*) *$h^0(R) = d+1$ and $P \in C$ with C base locus of $H^0(R \otimes I_Z)$; furthermore C is a smooth curve;*

(*b*) *$h^0(R) = d+2$ and P is one of $n^2 - d$ base points (outside Z) of $H^0(R \otimes I_Z)$.*

PROOF. If $h^0(R) \leq d+1$ the result is clear. Assume $h^0(R) = d+2$. Fix two general curves A, B of degree n; in particular A and B intersects transversally. There is $W \subset A \cap B$ with card $(W) - d$, and $h^0(R \otimes I_W) = 2$. Take as Z any small deformation of W. Now assume $x := h^0(R) > d+2$. It is sufficient to consider the case in which $d = x - 3$. We start again with general curves A, B of degree d and with $W \subset M := A \cap B$ with card $(W) = d+1$. It is sufficient to show that if $Z \subset W$, card $(Z) = d$, the base locus D of $H^0(R \otimes I_Z)$ is disjoint from $M \setminus Z$. By the irreducibility of $|R|$ and $S^d(X)$, if $D \cap (M \setminus W) \neq \emptyset$, then $(M \setminus W) \subseteq D$; however it is easy to find reducible A and B for which this is false. It remains the case in which $W \subseteq D$ for any choice of Z. Since $h^0(R \otimes I_Z) = h^0(R \otimes I_W) + 1$ by construction, this is impossible. ♦

For $X \cong F_e$, the question corresponding to 2.4 is very easy if $W = |f|$ and can be done without too much effort for some very special $W \subseteq |ah + bf|$ (say $a = 1$, $b \geq e$ but $b - e$ small and $\dim(W)$ very small), but it seems to us that nothing general is possible with the available methods.

REMARK 2.4.2. On the postulation of Z' (defined just before 2.4.1) there is very general and very refined conjecture due to Hirschowitz (see [22]). We stress that, if useful for construct examples, it is very easy (as in the geometric proof of 2.4.1) to handle particular cases or prove weaker statements; to do more (but less than the general «abstract» conjecture of [22]), the methods of [21] and [23] are much stronger and gives a lot.

3. In this section we discuss other invariants of bundles. For other tools for the cohomological study of families of rank 2 vector bundles on an algebraic surface, see [6]. Fix a bundle E on a complete variety X with $\dim(X) = \mathrm{rank}(E)$. For simplicity we will

assume $\dim(X) = 2$, stating in 3.2 the definition for the higher dimensional case. Furthermore, for simplicity in the examples we assume that X is smooth. If E is spanned by its global sections, in the introduction of [2] we defined an integer $s(E)$; for such a bundle E, let $c(E)$ be the maximal integer t such that for t general points of X there is $s \in H^0(E)$ with $(s)_0$ containing them and $\dim((s)_0) = 0$; we have $0 < 2c(E) \leqslant s(E)$. Now drop the assumption on the global sections of E. Fix an ample line bundle H on X. Thus for big n $E(nH)$ is spanned. We are interested on the behaviour of $s(E(nH))$ and $c(E(nH))$ as n goes to infinity. Fix a positive real number a. Set $m_{H,a}(E) := = \liminf((s(E(nH)) - 2c(nH))/n^a)$ and $M_{H,a}(E) := \limsup((s(E(nH)) - 2c(nH))/n^a)$. For a sheaf Q on X, set $\gamma(Q) := \{L \in \mathrm{Pic}(X)\colon h^0(Q \otimes L) = 1\}$.

(3.1) Now we will discuss the invariants just introduced under the following assumptions on X:

($a1$) For every $R \in \mathrm{Pic}(X)$ with $h^0(L) > 0$, we have $h^1(L) = 0$.

($a2$) For every $R \in \mathrm{Pic}(X) \setminus \mathbf{O}$ with $h^0(L) > 0$, we have $h^0(L) \geqslant 2$.

Note that ($a2$) means that $\gamma(\mathbf{O}) = \{\mathbf{O}\}$. Note that the assumptions ($a1$) and ($a2$) are satisfied if X is a «generic» smooth surface of degree at least 4 in $\boldsymbol{P}^3$ or if $X = \boldsymbol{P}^2$. Fix X satisfying ($a1$) and ($a2$), E and H. Fix an integer m such that $E(tH)$ is spanned and $h^1(E(tH)) = 0$ for every $t \geqslant m$. If $t \geqslant m$ set $c(t) := c(E(tH))$. Fix a general $T \subset X$ with $\mathrm{card}(T) = c(m)$. Let A' be the subsheaf of E generated by $H^0(E(mH) \otimes \boldsymbol{I}_T)$. By definition of $c(E(mH))$, $A' \neq 0$. First assume $\mathrm{rank}(A') = 2$. This implies $c(m) = = h^0(E(mH)) - 1$ (the maximal possible value); this is essentially a trivial case in the sense that $2c(E(mH)) = s(E(mH))$ and $h^0(E(mH))$ is given by Riemann-Roch. Now assume $\mathrm{rank}(A') = 1$. Set $L := A'^{**} \in \mathrm{Pic}(X)$; since E is locally free, L is a subsheaf of E and there is a non negative divisor D such that $L(D)$ is a saturated subsheaf of $E(mH)$. By assumption ($a2$) we have $D = 0$, hence the following exact sequence:

$$0 \to L \to E(mH) \to R \otimes \boldsymbol{I}_Z \to 0 \tag{4}$$

with $R \in \mathrm{Pic}(X)$. By ($a1$) we have $h^1(L) = 0$; by generality of T and the assumption on $c(m)$, we have $h^1(E(mH) \otimes \boldsymbol{I}_T) = h^1(L \otimes \boldsymbol{I}_T) = 0$. The surjectivity of the map $H^0(E(mH) \otimes \boldsymbol{I}_T) \to H^0(R \otimes \boldsymbol{I}_{T \cup Z})$ and the definition of $c(m)$ implies $h^0(R \otimes \boldsymbol{I}_{Z \cup T}) = 0$. Hence by (4) we have $h^0(E \otimes L^*) = 1$, *i.e.* $L^* \in \gamma(E)$. In special (but interesting) cases this implies that L is uniquely determined by E and the integer m. One such case is the following one:

(b) $\mathrm{Pic}(X) \cong \boldsymbol{Z}$; let H be the ample generator of X; assume $h^1(jH) = 0$ for every integer j and $h^0(H) > 0$.

Again, note that the assumption (b) is satisfied if X is a «generic» smooth surface of degree at least 4 in $\boldsymbol{P}^3$ or if $X = \boldsymbol{P}^2$. Under assumption (b) L gives essentially what is the first integer r with $h^0(E(rH)) \neq 0$ and the fact that $s(E(mH)) > 2c(E(mH))$ means that $H^0(E(rH)) > 1$.

(3.2) Now we assume $n := \dim(X) > 2$ and $\mathrm{rank}(E) = n$. Fix $H \in \mathrm{Pic}(X)$, H ample.

Again one can define a huge number (combining in a different way several lim inf and lim sup) of «asymptotic» invariants of E generalizing $m_{H,a}$ and $M_{H,a}$ if we define the notions equivalent to the notion of $c(E)$ when $n=2$ and E is spanned. Thus now we assume that E is spanned by its global sections. For every integer i with $0 \leqslant i < n$, we will define an integer $d(i) \geqslant 0$ in the following way; $d(0)$ is the maximal integer t such that for t general points of X there is $s \in H^0(E)$ with $(s)_0$ containing them and $\dim((s)_0)=0$; we have $d(0)>0$. If $i>0$ and $d(j)$, $j<i$ are defined (with $d(j) \geqslant 0$ if $1<j<i$) let $d(i)$ be the maximal integer t such that there are subvarietes $Q_{u,v}$, $0 \leqslant u \leqslant i$, with $v \leqslant d(u)$ if $u<i$ and $v \leqslant t$ if $u=i$, $\operatorname{codim}(Q_{u,v})=u$, $Q_{u,v}$ irreducible, and such for general $P_{u,v} \in Q_{u,v}$ there is $s \in H^0(E)$ with $\dim((s)_0)=0$ and s vanishing at each $P_{u,v}$. Variation on the theme: assume $(s)_0$ reduced.

4. In this section we consider the following *recognition problem*. Take a bundle E (given for instance by an extension) on a scheme V. How can one recognize that E is a certain given bundle? The most interesting case is certainly the case of TV (*e.g.* if V is smooth) since it is connected with the notion of holomorphic foliation (when $\boldsymbol{K}=\boldsymbol{C}$). However if V is complete to have a holomorphic foliation (even allowing singularities) is a very restrictive assumption on V. A natural generalization is the notion of meromorphic foliation with singularities; the corresponding bundle is $TV \otimes R$ with $R \in \operatorname{Pic}(V)$. We will consider this problem in a few examples. By far the most interesting case considered here is the case $V=\boldsymbol{P}^3$, $E=T\boldsymbol{P}^3(t)$. Hopefully, this case will be applied (see 4.3 for a beginning) to the study of codimension 1 meromorphic foliations with singularities on $\boldsymbol{P}^3$. This is very interesting case, since (contrary to the case of foliations by curves) it is a non-linear problem, due to the integrability condition. We do not have results for general V and we do not think that there are such a general results: we need to know too much about V to recognize $TV \otimes R$. We will give three examples: (*a*) $\boldsymbol{V}=\boldsymbol{P}^2$ (see 4.1), (*b*) $\boldsymbol{V} \cong F_0=\boldsymbol{P}^1 \times \boldsymbol{P}^1$ (see 4.2), (*c*) $\boldsymbol{V} \times \boldsymbol{P}^3$ (see 4.3). We will set $X:=V$ even if V is not a surface and use the previous conventions.

(4.1) Set $X=V=\boldsymbol{P}^2$ and let E be a rank two bundle given the extension (1); we want to find when (for a fixed integer t) $E \cong TX(t)$. Of course, we need $t \geqslant -1$, $\deg(L)=2t+3$ and $\operatorname{length}(Z)=c_2(TX(t))=t^2+3t+3$ (see *e.g.* [20], beginning of §2). Assume this. Note that by (1) and the cohomology of line bundles on X, the conditions «$h^0(E(-t-2))=0$» and «$h^0(E(-t-1)) \neq 0$» depends only on the cohomology of Z (respectively on $h^0(\boldsymbol{I}_Z(t+1))$ and $h^0(\boldsymbol{I}_Z(t+2))$). The first condition gives the stability of E; we conclude by the classification of stable rank 2 vector bundles on $\boldsymbol{P}^2$ with $c_1=-1$ and $c_2=1$. Alternatively, one can use both conditions to show that $E(-t-2)$ is uniform of splitting type $0,-1$ in the sense of [29] and conclude by [29] and [11].

(4.2) Set $X=F_0=\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and E given by (1) with $L=\boldsymbol{O}(2+2a, 2b+2)$; we want to give cohomological conditions involving only Z, not E, which are equivalent to the existence of an isomorphism between E and $TX(a,b)$, *i.e.* $E \cong \boldsymbol{O}(a+2,b) \oplus \boldsymbol{O}(a,b+2)$.

First assume $a \geq 0$ and $b \geq 0$. The isomorphism implies $h^0(E(-a-2,0)) \neq 0$, $h^0(E(-a,-b-2)) \neq 0$, $h^0(E(-a-2,-b-1)) = h^0(E(-a-3,-b)) = 0$ (and similarly $h^0(E(-a-1,-b-2)) = h^0(E(-a,-b-3)) = 0$ but we do not need the last two equalities). Viceversa, assume that the two inequalities and three equalities just given are satisfied and that E has the Chern classes of $TX(a,b)$. Note that by (1) and the cohomology of the line bundles on X all these conditions involve only the cohomology of Z. Let (u,v) (resp. (m,n)) be the splitting type of $E(-a,-b)$ on a generic line type $(1,0)$ (resp. $(0,1)$). The two inequalities imply that $(u,v) \neq (1,1)$ and $(m,n) \neq (1,1)$. Fix $s \in H^0(E(-2-a,-b))$, $s \neq 0$. Since $h^0(E(-a-3,-b)) = h^0(E(-a-2,-b-1)) = 0$, the image of s does not drop rank on a divisor (*i.e.* we have an extension similar to (1)). Checking the Chern classes (using again [20], beginning of §2), we see that s does not drop rank at all and gives an extension:

$$0 \to \boldsymbol{O} \to E(-a-2,-b) \to \boldsymbol{O}(-2,2) \to 0. \tag{5}$$

Since $(u,v) \neq (1,1)$ and $(m,n) \neq (1,1)$, (5) implies that the restriction of $E(-a,-b)$ to each line of each ruling has splitting type $(2,0)$ (*i.e.* E is weakly uniform of type $(2,2)$ in the sense of [9]). By [9], end of page 211 and beginning of page 212, $E(-a,-b)$ must be decomposable. We conclude easily that $E(-a,-b) \cong TX$. ♦

(4.3) Set $X = \boldsymbol{P}^3$ and take as E a candidate (rank and Chern classes) to be isomorphic to $TX(t)$ with $t \in \boldsymbol{Z}$. Assume that E is given by the following extension:

$$0 \to 2\boldsymbol{O} \to E \to \boldsymbol{I}_C(3t+4) \to 0 \tag{6}$$

in which C is a pure one dimensional locally Cohen-Macaulay and generically complete intersection subscheme of X («a curve»). Note that such an extension is the usual way to give any spanned bundle on any variety; if rank$(E) = r$, then take $(r-1)\boldsymbol{O}$ instead of $2\boldsymbol{O}$; on any variety C has codimension 2 (Vogelaar's extension of Serre construction to the case of rank>2; for more details, see [26]). By the cohomology of X and the general theory (see *e.g.* [26]) the extensions (6) correspond bijectively to the choice of C and of two sections of $\omega_C(-3t)$ which span $\omega_C(-3t)$. Fix a bundle E given by (6). For the existence of an isomorphism $\boldsymbol{i}$: $E \to TX(t)$ we certainly need $h^1(E(m)) = 0$ for every $m \in \boldsymbol{Z}$. By (6) this is equivalent to the fact that C is arithmetically Cohen-Macaulay («a.C.M.» for short). Such a curves are completely described (see [27, 12, 19]), it is very easy to give the equations of all of them («Hilbert-Burch theorem»); hence this part of the *recognition problem* (and the opposite problem: the *construction problem*) can be safely considered done; note also that the Chern classes of E give the degree and arithmetic genus of C. For the existence of the isomorphism $\boldsymbol{i}$ we certainly need $h^0(E(m)) = h^0(TX(t+m))$ for every $m \in \boldsymbol{Z}$; note that by (6) this condition can be read off from C: it is equivalent to known the postulation of C. Again, this is allowable for both the recognition and the construction problems. In our situation, *i.e.* for a.C.M. curves, the postulation of C (with given genus and degree) is equivalent to the choice of a finite (bounded!) set $\boldsymbol{S}$ of integer («the numerical character» of C, see [19] or [7], 3.3) and we know exactly what are the allowable $\boldsymbol{S}$, for what $\boldsymbol{S}$ there are such a curve integral and/or smooth [18,28], how to describe the set of a.C.M. curves with given S (they are parametrized from an irreducible variety); everything is explicit, hence allowable for both the recognition and the construction problems. By the existence of $\boldsymbol{i}$ and (6) we may assume $t \geq -1$. Note that, since $h^0(TX(-2t-4)) = 0$, by (6) we have $H^0(X, \mathbf{O}(t)) \cong H^0(C, \mathbf{O}_C(t))$. Set $T := TX(-1)$ and $G := E(-t-1)$. Consider the following conditions (\$) and (\$'):

Condition (\$): The sections of $\omega_C(-3t)$ corresponding to (6) induce an injection of $H^0(C, \mathbf{O}_C(t))^{\oplus 2}$ into $H^0(C, \omega_C(-2t))$.

Condition (\$'): The sections of $\omega_C(-3t)$ corresponding to (6) induce an injection of $H^0(C, \mathbf{O}_C(t+3))^{\oplus 2}$ into $H^0(C, \omega_C(-2t-3))$.

By the construction of E as extension (6) starting from the two sections ([26]), the remark just made on $H^0(C, \mathbf{O}_C(t))$, and the application of the functor $Hom(, \mathbf{O})$ to (6), the condition (\$) is equivalent to the vanishing of $h^0(G^*(1))$ (which is satisfied if $G \cong T$). Thus we will assume condition (\$); again, this condition is allowable for both the recognition and the construction problems; it is also very easy to find C and sections satisfying (\$) (good for the construction problem); note also that (\$) is an open condition on the set $\boldsymbol{U}$ of pairs (curve, pair of suitable sections). Again, condition (\$') would be allowable; it is *a priori* stronger than (\$). By Serre duality (\$) is equivalent to the vanishing of $h^3(G(-3))$. If there is an isomorphism $\boldsymbol{i}$, then $h^i(G(-i)) = 0$ for every $i \geq 1$; note that if $i = 1$ this condition was used before; if $i = 2$ by (6) and two applications (on X and on C) of Serre duality this condition is equivalent to (\$'). If we assume (\$'), then G is 0-regular in the sense of Castelnuovo-Mumford regularity (see [25], p. 100); hence G is spanned ([25], p. 100); since $c_1(G) = 4$, we see that the restriction of G to every line has splitting type $(1, 0, 0)$; by [11] we see that either $G \cong T$ (as wanted) or $G \cong \mathbf{O}(1) \oplus \mathbf{O} \oplus \mathbf{O}$ (impossible *e.g.* for Chern classes reasons). Now we assume (\$), but not (\$'). Anyway, since (\$') is an open condition, T is isomorphic to itself and the set $\boldsymbol{U}$ of solution forms an irreducible family, we get that for an open dense subset of $\boldsymbol{U}$ the corresponding bundle is isomorphic to T. By semicontinuity (and the irreducibility of U) for our G we have $H^0(Hom(T, G)) \neq 0$. Since $h^0(G(-1)) = h^0(G^*) = 0$, $c_1(G) = 1$ and $\operatorname{rank}(G) = 3$, G is stable. Since $c_1(G) = c_1(T)$, and T is stable, the stability of G implies that every non zero homomorphism $T \to G$ is an isomorphism, as wanted.

We are interested in the case of a meromorphic foliation $\mathcal{F}$ with singularities, *i.e.* the subsheaf of TX must be involutive. This case will be considered in a more general situation elsewhere. Here we remark that in this particular case there is an easy way to satisfy the integrability condition. Fix $P \in X_{\text{reg}}$; a local calculation shows the vanishing at the point P of the commutator of two meromorphic (or rational) vector fields defined around P and vanishing at P; hence it is sufficient to construct $\mathcal{F}$ when $X = \boldsymbol{P}^3$ to find $S \subset X$ with $h^0(TX(t) \otimes I_M) = 0$; for every $t > 0$ such a set M (a finite set) was constructed quite explicitely in [1].

We note explicitely that in (6) we could take $A_1 \oplus A_2$ with $A_i \in \operatorname{Pic}(X)$; the important thing is the trivial higher order cohomology of the bundle $A_1 \oplus A_2$ (and the same should happen in higher dimensions). For any variety V it seems interesting to investigate in general how the cohomological properties of TV and the involutive subsheaf of TV giving a meromorphic foliation $\mathcal{F}$ are reflected in the geometry of $\mathcal{F}$. For the case «V is a geometrically ruled surface», see [17].

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