Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Emma Previato

Another algebraic proof of Weil's reciprocity

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **2** (1991), n.2, p. 167–171.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1991_9_2_2_167_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1991.

Rend. Mat. Acc. Lincei
s. 9, v. 2:167-171 (1991)

Geometria algebrica. — *Another algebraic proof of Weil's reciprocity.* Nota di EMMA PREVIATO, presentata (*) dal Socio G. ZACHER.

ABSTRACT. — The Burchnall-Chaundy-Krichever correspondence which converts meromorphic functions on a curve into differential operators is used to interpret Weil's reciprocity as the calculation of a resultant.

KEY WORDS: Riemann surface; Ordinary differential operator; Resultant.

RIASSUNTO. — *Un'altra dimostrazione algebrica della reciprocità di Weil.* Il meccanismo di Burchnall-Chaundy-Krichever che trasforma funzioni meromorfe su una curva in operatori differenziali viene usato per interpretare la reciprocità di Weil come il valore di un risultante.

Weil's reciprocity [9] says that if f, g are two meromorphic functions on a compact Riemann surface S and their divisors (f), (g) are disjoint then

$$\prod_{P \in S} f(P)^{v_P(g)} = \prod_{P \in S} g(P)^{v_P(f)}$$

where v_P signifies the valuation at P, hence is nonzero for a finite number of points only. In the limit, a similar statement for singular curves and/or overlapping divisors could be formulated but is beside the point of this *Note*. The resultant of a pair of monic polynomials in one variable $f(x), g(x) \in \mathbf{C}[x]$ equals

$$\prod_{i=1}^{m} f(\beta_i) = (-1)^{nm} \prod_{i=1}^{n} g(\alpha_i)$$

where

$$f = \prod_{i=1}^{n} (x - \alpha_i), \qquad g = \prod_{i=1}^{m} (x - \beta_i).$$

This gives immediately Weil's reciprocity for $S = \boldsymbol{P}^1$ whereas the traditional proof for higher genus is transcendental, as it involves logarithms [4, 9]. In this *Note* we combine: the Euclidean algorithm for differential operators, which goes back at least to [3]; a remarkable analog of the resultant, which lies in the background of the Burchnall and Chaundy calculations (cf. [7]); and the Krichever dictionary (cf. [6]) and give an algebraic proof of Weil's reciprocity which generalizes the observation we made for $\boldsymbol{P}^1$. We hope that this mechanism may find an extension to the multidimensional case, in the same vein as the adelic construction and the «other» algebraic proof referred to in the title, which introduces the Kac-Peterson representation [1]. The proofs of the basic facts on the resultant are short and elegant and we sketch them for completeness.

(*) Nella seduta del 10 novembre 1990.

1. The algebraic and differential resultant

1. Let $f = a_0 + a_1 x + \ldots + a_n x^n$, $g = b_0 + b_1 x + \ldots + b_m x^m \in \mathbf{C}[x]$. It is well known (cf. for example [5]) that f and g have a common zero if and only if the resultant $R(f, g) = 0$, where

$$R(f,g) = \det \begin{bmatrix} a_0 & a_1 & \ldots & a_n & 0\ldots 0 \\ 0 & a_0 & \ldots & a_n & 0\ldots 0 \\ \ldots & & & & a_n \\ b_0 & \ldots & b_m & 0 & \ldots 0 \\ \ldots & & & & \\ 0 & \ldots & & & b_m \end{bmatrix} = \det \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} (G_2 \text{ is } m \times m,\ G_3 \text{ is } n \times n).$$

This can be seen in many ways, but the spirit of our proof will be reprised in a different context. We consider the vector space $V_f = \mathbf{C}[x]/(f(x))$, with basis $1, x, \ldots, x^{n-1}$, where multiplication by x is given by the companion matrix of f:

$$C_f = \begin{bmatrix} 0 & 0 & \ldots & -a_0/a_n \\ 1 & 0 & & -a_1/a_n \\ \ldots & & & \\ 0 & \ldots & 1 & -a_{n-1}/a_n \end{bmatrix}.$$

As follows from the Euclidean algorithm in $\mathbf{C}[x]$, multiplication by $g(x)$ on V_f is invertible if and only if f and g have no common roots. Now multiplication by g has matrix $(G_3 - G_4 G_2^{-1} G_1)^T$ whose determinant is $R(f, g)/a_n^m$ as can be seen by writing

$$\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} 0 & I \\ G_2^{-1} & -G_2^{-1} G_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ G_4 G_2^{-1} & G_3 - G_4 G_2^{-1} G_1 \end{bmatrix}.$$

2. Let (for simplicity) f and g be monic, $\{\alpha_1, \ldots, \alpha_n\}$ $\{\beta_1, \ldots, \beta_m\}$ be the zeroes of f, g, resp. Then

$$R(f, g) = \prod_{i=1}^{m} f(\beta_i) = (-1)^{mn} \prod_{i=1}^{n} g(\alpha_i).$$

This can be seen by a standard argument of unique factorization by regarding the α and β as indeterminates, but we insist on the point of view taken in 1. $R(f, g)$ as we saw is the determinant of the multiplication by $g(x)$ on V_f; we change basis so as to write C_f as an upper triangular matrix with $\alpha_1, \ldots, \alpha_n$ on the diagonal (Jordan form) and use the fact that it represents multiplication by x; clearly then

$$\det(g(x)) = \prod_{i=1}^{n} g(\alpha_i).$$

3. If $L = \partial^n + u_{n-1}(x)\partial^{n-1} + \ldots + u_0(x)$, $B = \partial^m + v_{m-1}(x)\partial^{m-1} + \ldots + v_0(x)$ are commuting differential operators (here $\partial = d/dx$) whose coefficients are analytic functions in a neighborhood of, say $x = 0$, we define the resultant polynomial $R(L, B) = \det \Lambda(\lambda, \mu)$ where $\Lambda(\lambda, \mu) = [\Lambda_{ji}]$ is the $(n + m) \times (n + m)$ matrix whose first m rows are

given by the coefficients of

$$\partial^j \circ (L-\lambda) = \sum_{i=0}^{n+m-1} \Lambda_{j+1,i+1}\, \partial^i \qquad (j=0,\dots,m-1)$$

and last n rows by the coefficients of $\partial^j \circ (B-\mu)$ similarly arranged $(j=0,\dots,n-1)$. It is a consequence of the commutativity that $R(L,B)$ is independent of x: indeed, it is the characteristic polynomial of the endomorphism obtained by applying B on the vector space $V_\lambda = \mathrm{Ker}(L-\lambda)$. This can be seen by choosing a fundamental system of solutions (at $x=0$) for $L-\lambda$: $y_1(x,\lambda),\dots,y_n(x,\lambda)$; then since B preserves V_λ,

$$\begin{bmatrix} (B-\mu)y_1 & \dots & (B-\mu)y_n \\ ((B-\mu)y_1)' & \dots & ((B-\mu)y_n)' \\ \dots & & \\ ((B-\mu)y_1)^{(n-1)} & \dots & ((B-\mu)y_n)^{(n-1)} \end{bmatrix} =$$

$$= \begin{bmatrix} y_1 & \dots & y_n \\ y_1' & \dots & y_n' \\ & \dots & \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix} M = (G_3 - G_4 G_2^{-1} G_1) \begin{bmatrix} y_1 & \dots & y_n \\ y_1' & \dots & y_n' \\ & \dots & \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}$$

where M is a suitable constant matrix and $G_1,\dots,G_4$ are blocks of the matrix $\Lambda(\lambda,\mu)$ of the same size as above. The required characteristic polynomial is $\det M^T$, which can be seen to coincide with $R(L,B)$ by setting $x=0$ in the formula.

2. Burchnall-Chaundy-Krichever dictionary

Here we only recall how to set up a conversion between meromorphic functions on a Riemann surface, regular outside one fixed point P_∞, say, and differential operators.

1. Construction (Krichever, cf. [6]). Let S be a Riemann surface of genus g with a fixed point P_∞ and a local parameter z^{-1} centered at P_∞; let D be a fixed divisor of degree g on S with $h^0(D)=1$, $D = X_1 + \dots + X_g$; there exists a unique function $\psi(x,P)$, depending on a (small) parameter $x \in \mathbf{C}$ and point $P \in S$ such that ψ is meromorphic on $S \setminus P_\infty$ with poles at most on D, and near P_∞ the expansion $\psi = e^{xz}(1+O(z^{-1}))$ holds. For any meromorphic function f on S with a pole of order n at P_∞ and regular elsewhere, there exists a unique differential operator $L_f = \partial^n + u_{n-2}(x)\partial^{n-2} + \dots + u_0(x)$ such that $L_f \psi = f(P)\psi$.

2. Lemma. *With the notation of the above construction, if f, g are two functions with pole of order n, m at P_∞ resp. and regular elsewhere, and if their expansion in z^{-1} begins with z^n, z^m, resp., then*

$$\prod_{i=1}^{m} f(\beta_i) = (-1)^{mn} \prod_{i=1}^{n} g(\alpha_i)$$

where the α_i's, β_i's are the zeroes of f, g, counted with multiplicity.

PROOF. We set up $R(L_f, L_g)$ as in 1.3, using the operators that correspond to f, g as in 2.1. Viewed as a polynomial in λ, R has leading coefficient $(-1)^{mn}(-1)^m\lambda^m$ and constant term

$$(-1)^m \prod_{(\lambda_j,\mu_j)=\beta_j} \lambda_j;$$

viewed as a polynomial in μ, it has leading coefficient $(-1)^n\mu^n$ and constant term

$$(-1)^n \prod_{(\lambda_i,\mu_i)=\alpha_i} \mu_i.$$

Notice that the normalization assumption comes in when one computes the leading coefficient of the operators L_f, L_g, which in turn appears in the unipotent matrix G_2; a different normalization would cause an explicitly computable constant to appear. Notice also that because of this normalization the poles of λ and μ «cancel out» so that the statement of the lemma is really Weil's reciprocity.

3. WEIL'S RECIPROCITY

1. OBSERVATION. Weil's reciprocity for $\boldsymbol{P}^1$ is the formula for the resultant 1.2.

PROOF. Let f, g be any two functions on $\boldsymbol{P}^1$, with disjoint divisors. We can express them as rational functions in one parameter z and normalize them so that

$$f = \frac{(z-a_1)\dots(z-a_n)}{(z-b_1)\dots(z-b_n)}, \qquad g = \frac{(z-c_1)\dots(z-c_m)}{(z-d_1)\dots(z-d_m)}$$

(after possibly dividing g by a constant); now let f_1, f_2 and g_1, g_2 be the polynomials that appear as numerator, denominator, resp. of f, g, resp. Write the resultant formula 1.2 for the pairs (f_1, g_1) and (f_2, g_2), multiply side by side, write the formula for $(f_1, g_2), (f_2, g_1)$, multiply side by side, divide the results side by side.

2. PROPOSITION. *The Krichever dictionary and the resultant formula prove Weil's reciprocity on any Riemann surface S.*

PROF. Let f, g be meromorphic functions on S with disjoint divisors, and P_∞ a fixed (disjoint) point on S. Let $(f) = (P_1 + \dots + P_r) - (Q_1 + \dots + Q_r)$, P_i being the zeroes of f and Q_i the poles. For n large enough, $nP_\infty - (P_1 + \dots + P_r)$ is linearly equivalent to an effective divisor and if n is the smallest such number, the dimension of $H^0(nP_\infty - (P_1 + \dots + P_r))$ cannot be larger than 1. The dimension of $H^0(nP_\infty - (Q_1 + \dots + Q_r))$ must be the same, for the divisors $\sum P_i$ and $\sum Q_i$ are linearly equivalent. Let f_1, f_2 be the essentially unique functions that have a pole of order n at P_∞ and zeroes on $\sum P_i$, $\sum Q_i$ resp. and notice that $ff_2f_1^{-1}$ must be a constant, for it has neither zeroes nor poles. By choosing a local parameter z^{-1} around P_∞ and normalizing all functions so that their Laurent expansions near P_∞ is monic in z, we can write $f = f_1/f_2$, $g = g_1/g_2$ by the same procedure, apply Lemma 2.2 in the same manner as in the observation 3.1 and conclude.

3. COMMENT. The main advantage I see in this proof is that Weil's reciprocity appears as the consequence of an action by multiplication: indeed, the matrix of the action by L_g say, expressed by using λ and μ as in 1.3, can be viewed as the matrix of multiplication by μ, which is a twisted endomorphism of the vector bundle $\pi_*(\mathcal{L})$, where π is the projection to $\boldsymbol{P}^1$ given by the function $\lambda = f$ and $\mathcal{L}$ is the line bundle of the divisor D of the Baker Akhiezer function:

$$\pi_* \mathcal{L} \xrightarrow{\times g} \pi_* \mathcal{L} \otimes \mathcal{O}([m/n]_+ \infty),$$

where ∞ corresponds to $\lambda = \infty$ on $\boldsymbol{P}^1$ and by $[m/n]_+$ we denote the smallest integer greater than or equal to m/n. This point of view brings the spirit of the formula very close to the multiplication argument in 1.1; it should generalize to two interesting situations: (*a*) π: $S_1 \to S_2$ a morphism of Riemann surfaces where S_2 has genus greater than zero (an elliptic situation is surveyed in [8]) and (*b*) a multipoint Krichever map, where P_∞ is replaced by $P_{\infty 1}, \ldots, P_{\infty d}$ and the Baker function has more than one parameter $z_1, \ldots, z_d$, (cf. [2]); finally, it suggests that letting the divisor D of the Krichever map vary, one may get an interpretation of Weil's reciprocity on the Jacobian of S.

ACKNOWLEDGEMENTS

Boston University seed grant No. 943-MATH is gratefully acknowledged.

REFERENCES

[1] E. ARBARELLO - C. DE CONCINI - V. G. KAC, *The infinite wedge representation and the reciprocity law for algebraic curves*. Preprint 1988.

[2] H. FLASCHKA, *Towards an algebro-geometric interpretation of the Neumann system*. Tôhoku Math. J., 36, 1984, 407-426.

[3] G. FLOQUET, *Sur la théorie des équations différentielles linéaires*. Ann. Sci. de l'Ecole Norm. Supér., 8, 1879, 3-132.

[4] P. A. GRIFFITHS - J. HARRIS, *Principles of Algebraic Geometry*. John Wiley & Sons, New York 1978.

[5] S. LANG, *Algebra*. Addison-Wesley, Reading Mass. 1965.

[6] D. MUMFORD, *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg-de Vries equation and related nonlinear equations*. Proceedings of Intern. Symposium on *Algebraic Geometry* (Kyoto), Kino Kuniya Book Store, 1978, 115-153.

[7] E. PREVIATO, *Generalized Weierstrass $\wp$ functions and KP flows in affine space*. Comment. Math. Helv., 62, 1987, 292-310.

[8] E. PREVIATO, *The Calogero-Moser-Krichever system and elliptic Boussinesq solitons*. In: J. HARNAD - J. E. MARSDEN (eds.), Proceedings CRM Workshop, CRM Press, Montréal 1990, 57-67.

[9] A. WEIL, *Sur les fonctions algébriques à corps de constantes fini*. C.R. Acad. Sci. Paris, 210, 1940, 592-594.

Department of Mathematics
Boston University - BOSTON, Mass. 02215 (U.S.A.)