Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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A variationally consistent generalized variable formulation of the elastoplastic rate problem

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **2** (1991), n.2, p. 177–190.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1991_9_2_2_177_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1991.

Rend. Mat. Acc. Lincei
s. 9, v. 2:177-190 (1991)

Meccanica dei solidi. — *A variationally consistent generalized variable formulation of the elastoplastic rate problem.* Nota di Claudia Comi e Umberto Perego, presentata (*) dal Corrisp. G. Maier.

Abstract. — The elastoplastic rate problem is formulated as an unconstrained saddle point problem which, in turn, is obtained by the Lagrange multiplier method from a kinematic minimum principle. The finite element discretization and the enforcement of the *min-max* conditions for the Lagrangean function lead to a set of algebraic governing relations (equilibrium, compatibility and constitutive law). It is shown how important properties of the continuum problem (like, *e.g.*, symmetry, convexity, normality) carry over to the discrete problem if «generalized variables» are used in the discretization. A couple of dual kinematic and static minimum properties in generalized variables are finally derived.

Key words: Plasticity; Finite elements; Generalized variables; Extremum properties.

Riassunto. — *Formulazione variazionalmente coerente in variabili generalizzate del problema elastoplastico incrementale.* Il problema elastoplastico per il continuo, in termini di velocità di variazione, viene formulato come un problema di punto sella non vincolato partendo da un principio di minimo cinematico e utilizzando il metodo dei moltiplicatori di Lagrange. L'imposizione delle condizioni di *min-max* per la funzione lagrangiana, discretizzata ad elementi finiti, porta ad un sistema algebrico di equazioni governanti (equilibrio, congruenza e legge costitutiva). Si dimostra come importanti proprietà del problema continuo (quali ad es. simmetria, convessità, normalità) si trasferiscano al problema discreto qualora si utilizzino variabili generalizzate per la discretizzazione. Infine, si formula una coppia di proprietà duali di minimo cinematico e statico.

1. Introduction

In the formulation of elastic-plastic problems, it is customary to express the plasticity condition in the space of stresses and static internal variables. In displacement based finite element methods, this fact implies the evaluation of static variables at a discrete number of points and the enforcement of the constitutive law at these points only [1]. Corradi [2, 3] showed how the discrete enforcement of the constitutive law implicitly amounted to the introduction of an arbitrary interpolation for the static variables. In order to provide a more consistent formulation, he proposed to discretize all fields in terms of conjugate static and kinematic variables, «generalized» in Prager's sense [4]. The generalized variables are interpolation parameters such that the scalar product of vectors containing conjugate generalized variables is equal to the integral over the domain of the scalar product of the relevant fields. The adoption of such a discretization for the constitutive law results in a set of relations in terms of generalized variables which can be interpreted as the «constitutive law» of a finite portion of material (typically a finite element). The same concepts have been applied for the modelling of fields over the internal cells in boundary element plastic analysis. In this case, the use of generalized variables allows for a consistently symmetric formulation with noteworthy theoretical and computational advantages [5-7].

(*) Nella seduta del 15 dicembre 1990.

A variational motivation for the constitutive law in generalized variables, resting on the principle of maximum plastic work, has been proposed in this context by Polizzotto [8]. The independent modelling of displacements, strains and stresses has also been given a rigorous treatment in [9, 10]. In [9], conditions on the stress and strain modellings were derived starting from a Hu-Washizu functional, while in [10] the equations governing the boundary value problem were obtained by independent modelling of displacements and stresses on the basis of a Hellinger-Reissner-type functional. In [11], the generalized variable approach was investigated from both the theoretical and computational point of view. As in [9], a Hu-Washizu-type functional was used to derive the compatibility equations between nodal displacements and generalized strains.

In the present paper use is made of an extension of a rate kinematic theorem [12] to a more general internal variable material model. According to this theorem, the solution of the elastoplastic boundary value rate problem can be obtained by minimizing a constrained functional with respect to displacements, elastic strains, kinematic internal variables and plastic multipliers. By using the Lagrange multiplier method, this variational principle is transformed into an unconstrained *min-max* problem. Since all fields (static and kinematic) are considered as independent, the saddle-point problem can also be regarded as a generalization to the present elastoplastic context of Hu-Washizu principle. The finite element discretization of the Lagrangean functional and the enforcement of the saddle-point conditions straightforwardly generate the discretized version of equilibrium, compatibility and constitutive equations. If, in addition, the unknown fields are modelled in terms of generalized variables, the discretized constitutive relations are formally identical to the local ones and preserve essential properties like symmetry, normality, convexity.

Finally, a couple of dual constrained minimization theorems are obtained by eliminating some variables from the Lagrangean function and by adding suitable constraints. The extremum properties proved herein represent the counterparts in rates (for $\Delta t \rightarrow 0$) to those established in [11] for a finite-step backward-difference time integration.

2. Elastoplastic rate problem for continua

Reference is made to a solid of volume Ω and boundary $\Gamma = \Gamma_u \cup \Gamma_t$, Γ_u and Γ_t being the constrained and free part of the boundary, respectively. We are concerned with the evaluation of the quasi-static, small-displacement response to given external actions, namely: body forces $\boldsymbol{F}(\boldsymbol{x})$ $(\boldsymbol{x} \in \Omega)$ and surface tractions $\boldsymbol{t}(\boldsymbol{x})$ $(\boldsymbol{x} \in \Gamma_t)$. Let the deformation process be described by the vector $\boldsymbol{u}(\boldsymbol{x})$ of displacement components and by the second order tensors of total, elastic and plastic strains, represented in vector notation by $\boldsymbol{\varepsilon}(\boldsymbol{x})$, $\boldsymbol{e}(\boldsymbol{x})$ and $\boldsymbol{p}(\boldsymbol{x})$, respectively. The material is assumed to obey an elastic-plastic, time-independent constitutive law described by an internal variable model in terms of strains, stresses $\boldsymbol{\sigma}(\boldsymbol{x})$ and static and kinematic internal variables $\boldsymbol{\chi}(\boldsymbol{x})$ and $\boldsymbol{\eta}(\boldsymbol{x})$ [13]. We postulate the existence of two convex potential functions: $U[\boldsymbol{e}(\boldsymbol{x})]$, the elastic strain energy potential and $V[\boldsymbol{\eta}(x)]$, the stored strain energy due to structural rearrangements at the microscale. The latter assumption rules out unstable plastic behaviour («softening»).

With the above symbology, considering the system in a known configuration and state at a given instant, the governing relations in terms of rates (dotted symbols) can be expressed as follows:

$$\text{(1)} \qquad \boldsymbol{C}^T \dot{\boldsymbol{\sigma}}(\boldsymbol{x}) + \dot{\boldsymbol{F}}(\boldsymbol{x}) = \boldsymbol{0}, \ \ \boldsymbol{x} \in \Omega; \qquad \boldsymbol{n}(\boldsymbol{x}) \dot{\boldsymbol{\sigma}}(\boldsymbol{x}) = \dot{\boldsymbol{t}}(\boldsymbol{x}), \ \ \boldsymbol{x} \in \Gamma_t ;$$

$$\text{(2)} \qquad \boldsymbol{C} \dot{\boldsymbol{u}}(\boldsymbol{x}) = \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) = \dot{\boldsymbol{e}}(\boldsymbol{x}) + \dot{\boldsymbol{p}}(\boldsymbol{x}), \ \ \boldsymbol{x} \in \Omega; \qquad \dot{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{0}, \ \ \boldsymbol{x} \in \Gamma_u ;$$

$$\text{(3)} \qquad \dot{\boldsymbol{\sigma}}(\boldsymbol{x}) = \frac{\partial^2 U}{\partial \boldsymbol{e}\, \partial \boldsymbol{e}^T} \dot{\boldsymbol{e}}(\boldsymbol{x}), \qquad \dot{\boldsymbol{\chi}}(\boldsymbol{x}) = \frac{\partial^2 V}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}^T} \dot{\boldsymbol{\eta}}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega;$$

$$\text{(4)} \qquad \dot{\phi}(\boldsymbol{\sigma}, \boldsymbol{\chi}) = \frac{\partial \phi}{\partial \boldsymbol{\sigma}^T} \dot{\boldsymbol{\sigma}}(\boldsymbol{x}) + \frac{\partial \phi}{\partial \boldsymbol{\chi}^T} \dot{\boldsymbol{\chi}}(\boldsymbol{x}) \leqslant 0, \qquad \boldsymbol{x} \in \Omega_p ;$$

$$\text{(5)} \qquad \dot{\lambda}(\boldsymbol{x}) \geqslant 0, \ \ \boldsymbol{x} \in \Omega_p ; \qquad \dot{\lambda}(\boldsymbol{x}) = 0, \ \ \boldsymbol{x} \in \Omega_e ; \ \ \Omega = \Omega_e \cup \Omega_p ;$$

$$\text{(6)} \qquad \dot{\phi}(\boldsymbol{x})\, \dot{\lambda}(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega ;$$

$$\text{(7)} \qquad \dot{\boldsymbol{p}}(\boldsymbol{x}) = \frac{\partial \phi}{\partial \boldsymbol{\sigma}} (\boldsymbol{\sigma}, \boldsymbol{\chi})\, \dot{\lambda}(\boldsymbol{x}); \qquad \dot{\boldsymbol{\eta}}(\boldsymbol{x}) = -\frac{\partial \phi}{\partial \boldsymbol{\chi}} (\boldsymbol{\sigma}, \boldsymbol{\chi})\, \dot{\lambda}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega_p .$$

Equations (1) and (2) express equilibrium and compatibility, $\boldsymbol{C}$ being the compatibility linear differential operator and $\boldsymbol{n}(\boldsymbol{x})$ being a suitable matrix containing the components of the outward normal to the (smooth) boundary. Equations (3) relate the static variables $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\chi}}$ to the corresponding kinematic variables $\dot{\boldsymbol{e}}$ and $\dot{\boldsymbol{\eta}}$. In (4) $\phi(\boldsymbol{\sigma}, \boldsymbol{\chi})$ denotes the continuously differentiable yield function. The elastic domain is therefore defined by the condition $\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0$. Ω_p is that part of the volume where, at the considered instant, $\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$, *i.e.*, the stress point $\boldsymbol{\sigma}$ belongs to the boundary of the instantaneous yield surface of the material. On the other hand, Ω_e is the complementary part of the volume where $\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) < 0$. Plastic strain rates can occur only at material points belonging to Ω_p and eqs. (4)-(6) express the loading-unloading conditions, $\dot{\lambda}$ being a nonnegative plastic multiplier.

The elastoplastic rate problem can equivalently be formulated as a constrained minimization problem, *e.g.*:

$$\text{(8)} \qquad \min_{\dot{\boldsymbol{e}}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{u}}, \dot{\lambda}} \left\{ \Xi = \frac{1}{2} \int_{\Omega} \dot{\boldsymbol{e}}^T \frac{\partial^2 U}{\partial \boldsymbol{e}\, \partial \boldsymbol{e}^T} \dot{\boldsymbol{e}}\, d\Omega + \frac{1}{2} \int_{\Omega_p} \dot{\boldsymbol{\eta}}^T \frac{\partial^2 V}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}^T} \dot{\boldsymbol{\eta}}\, d\Omega - \int_{\Omega} \dot{\boldsymbol{F}}^T \dot{\boldsymbol{u}}\, d\Omega - \int_{\Gamma_t} \dot{\boldsymbol{t}}^T \dot{\boldsymbol{u}}\, d\Gamma \right\}$$

subject to:

(9) $$\dot{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{0}, \qquad \boldsymbol{x} \in \Gamma_u,$$

(10) $$\boldsymbol{C}\,\dot{\boldsymbol{u}}(\boldsymbol{x}) - \dot{\boldsymbol{e}}(\boldsymbol{x}) = \boldsymbol{0}, \qquad \boldsymbol{x} \in \Omega_e,$$

(11) $$\boldsymbol{C}\,\dot{\boldsymbol{u}}(\boldsymbol{x}) - \dot{\boldsymbol{e}}(\boldsymbol{x}) - \frac{\partial \phi}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \boldsymbol{\chi})\,\dot{\lambda}(\boldsymbol{x}) = \boldsymbol{0}, \qquad \boldsymbol{x} \in \Omega_p,$$

(12) $$\dot{\boldsymbol{\eta}}(\boldsymbol{x}) + \frac{\partial \phi}{\partial \boldsymbol{\chi}}(\boldsymbol{\sigma}, \boldsymbol{\chi})\,\dot{\lambda}(\boldsymbol{x}) = \boldsymbol{0}, \;\; \dot{\lambda}(\boldsymbol{x}) \geqslant 0, \qquad \boldsymbol{x} \in \Omega_p.$$

The quadratic functional (8) is convex in view of the assumed convexity of U and V.

This extremum property can be regarded as a generalization to the present internal variable model of a theorem presented by Capurso and Maier in [12]. The equivalence of the rate problem (1)-(7) with the minimization problem (8)-(12) can be shown by writing its optimality conditions according to the Lagrange multiplier method.

(13) $$\int_{\Omega} \left[\frac{\partial^2 U}{\partial \boldsymbol{e}\, \partial \boldsymbol{e}^T}\,\dot{\boldsymbol{e}} - \boldsymbol{s} \right]^T \delta\dot{\boldsymbol{e}}\, d\Omega = 0, \qquad \int_{\Omega_p} \left[\frac{\partial^2 V}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}^T}\,\dot{\boldsymbol{\eta}} + \boldsymbol{c} \right]^T \delta\dot{\boldsymbol{\eta}}\, d\Omega = 0,$$

(14) $$-\int_{\Omega} \dot{\boldsymbol{F}}^T \delta\dot{\boldsymbol{u}}\, d\Omega - \int_{\Gamma_t} \dot{\boldsymbol{t}}^T \delta\dot{\boldsymbol{u}}\, d\Gamma + \int_{\Omega} [\boldsymbol{C}^T \boldsymbol{s}]^T \delta\dot{\boldsymbol{u}}\, d\Omega = 0,$$

(15) $$\int_{\Omega_p} \left[-\frac{\partial \phi}{\partial \boldsymbol{\sigma}^T}\boldsymbol{s} + \frac{\partial \phi}{\partial \boldsymbol{\chi}^T}\boldsymbol{c} - f \right]^T \delta\dot{\lambda}\, d\Omega = 0,$$

(16) $$f(\boldsymbol{x}) \geqslant 0, \qquad f(\boldsymbol{x})\,\dot{\lambda}(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega_p.$$

These conditions need to be supplemented by eqs. (9)-(12) which define the feasible domain. In eqs. (13)-(16), $\boldsymbol{s}(\boldsymbol{x})$, $\boldsymbol{c}(\boldsymbol{x})$ and $f(\boldsymbol{x})$ are Lagrange multiplier functions; $\delta\dot{\boldsymbol{e}}$, $\delta\dot{\boldsymbol{\eta}}$ and $\delta\dot{\lambda}$ are free variations in Ω of $\dot{\boldsymbol{e}}$, $\dot{\boldsymbol{\eta}}$, and $\dot{\lambda}$ respectively; $\delta\dot{\boldsymbol{u}}$ is a variation in Ω of $\dot{\boldsymbol{u}}$ such that $\delta\dot{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in \Gamma_u$. Enforcement of eqs. (13) for arbitrary $\delta\dot{\boldsymbol{e}}$ and $\delta\dot{\boldsymbol{\eta}}$ implies that the corresponding terms in square brackets vanish. Therefore, $\boldsymbol{s}(\boldsymbol{x})$ and $-\boldsymbol{c}(\boldsymbol{x})$ can be identified with stress rates $\dot{\boldsymbol{\sigma}}(\boldsymbol{x})$ and static internal variables $\dot{\boldsymbol{\chi}}(\boldsymbol{x})$. Since $\delta\dot{\boldsymbol{u}}$ is by definition compatible, eq. (14) can be interpreted as a virtual work equation expressing equilibrium in Ω and on Γ_t. From eqs. (15) and (16*a*) it follows that $f(\boldsymbol{x}) \equiv -\dot{\phi}(\boldsymbol{x}) \geqslant 0$ in Ω_p and, hence, eq. (16*b*) coincides with eq. (6).

An equivalent unconstrained formulation of the constrained minimum problem (8)-(12) can be obtained as follows. First, transform the inequality constraint (12*b*) into an equality by introducing the slack variable $\alpha^2(\boldsymbol{x})$ so that $\dot{\lambda} - \alpha^2 = 0$. Then, write the Lagrangean functional $\mathcal{L}_{\Xi}$ associated to the original problem, taking into account the above constraint modification:

$$\mathcal{L}_{\Xi}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{e}}, \dot{\boldsymbol{\eta}}, \dot{\lambda}, s, c, f, \alpha) = \frac{1}{2}\int_{\Omega} \dot{\boldsymbol{e}}^T \frac{\partial^2 U}{\partial \boldsymbol{e}\, \partial \boldsymbol{e}^T} \dot{\boldsymbol{e}}\, d\Omega + \frac{1}{2}\int_{\Omega_p} \dot{\boldsymbol{\eta}}^T \frac{\partial^2 V}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}^T} \dot{\boldsymbol{\eta}}\, d\Omega - \tag{17}$$

$$- \int_{\Omega} \dot{\boldsymbol{F}}^T \dot{\boldsymbol{u}}\, d\Omega - \int_{\Gamma_t} \dot{\boldsymbol{t}}^T \dot{\boldsymbol{u}}\, d\Gamma + \int_{\Omega_e} [\boldsymbol{C}\dot{\boldsymbol{u}} - \dot{\boldsymbol{e}}]^T \boldsymbol{s}\, d\Omega +$$

$$+ \int_{\Omega_p} \left[\boldsymbol{C}\dot{\boldsymbol{u}} - \dot{\boldsymbol{e}} - \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, \boldsymbol{\chi})\, \dot{\boldsymbol{\lambda}}\right]^T \boldsymbol{s}\, d\Omega + \int_{\Omega_p} \left[\dot{\boldsymbol{\eta}} + \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\chi}}(\boldsymbol{\sigma}, \boldsymbol{\chi})\, \dot{\boldsymbol{\lambda}}\right]^T \boldsymbol{c}\, d\Omega + \int_{\Omega_p} [\dot{\lambda} - \alpha^2] f\, d\Omega .$$

Enforce now the condition of vanishing first variation of $\mathcal{L}_{\Xi}$ with respect to arbitrary variations of $\dot{\boldsymbol{e}}$, $\dot{\boldsymbol{\eta}}$, $\dot{\lambda}$, s, c, f, α in Ω and to variations of $\dot{\boldsymbol{u}}$ such that $\dot{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in \Gamma_u$. A stationary point of $\mathcal{L}_{\Xi}$ such that the second variation with respect to α is nonnegative minimizes Ξ and viceversa. In a more compact form, it can be stated that the following unconstrained saddle-point problem:

$$\min_{\dot{\boldsymbol{u}}, \dot{\boldsymbol{e}}, \dot{\boldsymbol{\eta}}, \dot{\lambda}, \alpha} \; \max_{s, c, f} \{\mathcal{L}_{\Xi}\} \tag{18}$$

is equivalent to the original constrained minimization problem (8)-(12).

3. Discretized elastoplastic rate problem

Let us subdivide the domain Ω in finite elements and introduce in $\mathcal{L}_{\Xi}$ independent interpolations of all fields (including α^2) over each element e.

$$\dot{\boldsymbol{u}}^h(\boldsymbol{x}) = \boldsymbol{N}^e_u(\boldsymbol{x})\, \dot{\bar{\boldsymbol{u}}}^e, \qquad [\alpha^2(\boldsymbol{x})]^h = \boldsymbol{N}^e_\alpha(\boldsymbol{x})(\bar{\boldsymbol{\alpha}}^2)^e, \qquad \left(\bar{\boldsymbol{\alpha}}^2 = \begin{Bmatrix} \cdots \\ \bar{\alpha}^2_i \\ \cdots \end{Bmatrix}\right); \tag{19}$$

$$\dot{\boldsymbol{e}}^h(\boldsymbol{x}) = \boldsymbol{N}^e_e(\boldsymbol{x})\, \dot{\bar{\boldsymbol{e}}}^e, \qquad \dot{\boldsymbol{\eta}}^h(\boldsymbol{x}) = \boldsymbol{N}^e_\eta(\boldsymbol{x})\, \dot{\bar{\boldsymbol{\eta}}}^e, \qquad \dot{\lambda}^h(\boldsymbol{x}) = \boldsymbol{N}^e_\lambda(\boldsymbol{x})\, \dot{\bar{\boldsymbol{\lambda}}}^e; \tag{20}$$

$$\boldsymbol{s}^h(\boldsymbol{x}) = \boldsymbol{N}^e_\sigma(\boldsymbol{x})\, \dot{\bar{\boldsymbol{\sigma}}}^e, \qquad -\boldsymbol{c}^h(\boldsymbol{x}) = \boldsymbol{N}^e_\chi(\boldsymbol{x})\, \dot{\bar{\boldsymbol{\chi}}}^e, \qquad f^h(\boldsymbol{x}) = \boldsymbol{N}^e_\phi(\boldsymbol{x})\, \dot{\bar{\boldsymbol{\phi}}}^e . \tag{21}$$

In eqs. (19)-(21) a superscript h marks modelled fields over the considered element; barred symbols denote independent parameters, matrices $\boldsymbol{N}^e_{[\cdot]}(\boldsymbol{x})$ collect suitable interpolation functions whose features are to be specified later; the notation used in eqs. (21) has been chosen because of the special meaning that the Lagrange multipliers assume at the solution. In what follows $\dot{\bar{\boldsymbol{u}}}^e$ will be interpreted as a vector of displacement nodal values, *i.e.* its components can be identified as the displacements of specific material points; on the contrary, the components of the other barred vectors not necessarily will be given a physical meaning.

Henceforth, all symbols without the superscript e will denote «global» vectors relevant to the whole assembled aggregate of finite elements. Consider now the discretized form $\overline{\mathcal{L}}_\Xi$ of $\mathcal{L}_\Xi$:

$$(22)\quad \overline{\mathcal{L}}_\Xi(\dot{\bar{u}}, \dot{\bar{e}}, \dot{\bar{\eta}}, \dot{\bar{\lambda}}, \dot{\bar{\sigma}}, \dot{\bar{\chi}}, \dot{\bar{\phi}}, \bar{\alpha}) = \frac{1}{2}\int_\Omega \dot{\bar{e}}^T N_e^T \frac{\partial^2 U}{\partial e \, \partial e^T} N_e \dot{\bar{e}} \, d\Omega +$$

$$+\frac{1}{2}\int_{\Omega_p} \dot{\bar{\eta}}^T N_\eta^T \frac{\partial^2 V}{\partial \eta \, \partial \eta^T} N_\eta \dot{\bar{\eta}} \, d\Omega - \int_\Omega \dot{F}^T N_u \dot{\bar{u}} \, d\Omega - \int_{\Gamma_t} \dot{t}^T N_u \dot{\bar{u}} \, d\Gamma +$$

$$+ \int_{\Omega_e} [C N_u \dot{\bar{u}} - N_e \dot{\bar{e}}]^T N_\sigma \dot{\bar{\sigma}} \, d\Omega + \int_{\Omega_p} \left[C N_u \dot{\bar{u}} - N_e \dot{\bar{e}} - \frac{\partial \phi}{\partial \sigma} N_\lambda \dot{\bar{\lambda}} \right]^T N_\sigma \dot{\bar{\sigma}} \, d\Omega -$$

$$- \int_{\Omega_p} \left[N_\eta \dot{\bar{\eta}} + \frac{\partial \phi}{\partial \chi} N_\lambda \dot{\bar{\lambda}} \right]^T N_\chi \dot{\bar{\chi}} \, d\Omega + \int_{\Omega_p} [N_\lambda \dot{\bar{\lambda}} - N_\alpha \bar{\alpha}^2]^T N_\phi \dot{\bar{\phi}} \, d\Omega .$$

The solution of the boundary value rate problem discretized by eqs. (19)-(21), is obtained by enforcing the first derivatives of $\overline{\mathcal{L}}_\Xi$ to vanish and its second variation with respect to $\bar{\alpha}$ to be nonnegative:

$$(23)\quad \frac{\partial \overline{\mathcal{L}}_\Xi}{\partial \dot{\bar{u}}} = \int_\Omega N_u^T C^T N_\sigma \dot{\bar{\sigma}} \, d\Omega - \int_\Omega N_u^T \dot{F} \, d\Omega - \int_{\Gamma_t} N_u^T \dot{t} \, d\Gamma = \mathbf{0},$$

$$(24a)\quad \frac{\partial \overline{\mathcal{L}}_\Xi}{\partial \dot{\bar{e}}} = \int_\Omega N_e^T \left(\frac{\partial^2 U}{\partial e \, \partial e^T} N_e \dot{\bar{e}} - N_\sigma \dot{\bar{\sigma}} \right) d\Omega = \mathbf{0},$$

$$(24b)\quad \frac{\partial \overline{\mathcal{L}}_\Xi}{\partial \dot{\bar{\eta}}} = \int_{\Omega_p} N_\eta^T \left(\frac{\partial^2 V}{\partial \eta \, \partial \eta^T} N_\eta \dot{\bar{\eta}} - N_\chi \dot{\bar{\chi}} \right) d\Omega = \mathbf{0},$$

$$(25)\quad \frac{\partial \overline{\mathcal{L}}_\Xi}{\partial \dot{\bar{\lambda}}} = \int_{\Omega_p} N_\lambda^T \left(- \frac{\partial \phi}{\partial \sigma^T} N_\sigma \dot{\bar{\sigma}} - \frac{\partial \phi}{\partial \chi^T} N_\chi \dot{\bar{\chi}} + N_\phi \dot{\bar{\phi}} \right) d\Omega = \mathbf{0},$$

$$(26)\quad \frac{\partial \overline{\mathcal{L}}_\Xi}{\partial \dot{\bar{\sigma}}} = \int_{\Omega_e} N_\sigma^T [C N_u \dot{\bar{u}} - N_e \dot{\bar{e}}] \, d\Omega + \int_{\Omega_p} N_\sigma^T \left[C N_u \dot{\bar{u}} - N_e \dot{\bar{e}} - \frac{\partial \phi}{\partial \sigma} N_\lambda \dot{\bar{\lambda}} \right] d\Omega = \mathbf{0},$$

$$(27)\quad \frac{\partial \overline{\mathcal{L}}_\Xi}{\partial \dot{\bar{\chi}}} = \int_{\Omega_p} N_\chi^T \left[N_\eta \dot{\bar{\eta}} + \frac{\partial \phi}{\partial \chi} N_\lambda \dot{\bar{\lambda}} \right] d\Omega = \mathbf{0},$$

$$(28)\quad \frac{\partial \overline{\mathcal{L}}_\Xi}{\partial \dot{\bar{\phi}}} = \int_{\Omega_p} N_\phi^T (N_\lambda \dot{\bar{\lambda}} - N_\alpha \bar{\alpha}^2) \, d\Omega = \mathbf{0},$$

$$(29)\quad \frac{\partial \overline{\mathcal{L}}_\Xi}{\partial \bar{\alpha}} = - \int_{\Omega_p} \mathrm{diag}\,[2\bar{\alpha}_i] \, N_\alpha^T N_\phi \dot{\bar{\phi}} \, d\Omega = \mathbf{0},$$

$$(30)\quad \delta\bar{\alpha}^T \frac{\partial^2 \overline{\mathcal{L}}_\Xi}{\partial \bar{\alpha} \, \partial \bar{\alpha}^T} \delta\bar{\alpha} = -\delta\bar{\alpha}^T \int_{\Omega_p} \mathrm{diag}\,[A_i] \, d\Omega \, \delta\bar{\alpha} \geq 0 \qquad \forall \delta\bar{\alpha}, \qquad A \equiv N_\alpha^T N_\phi \dot{\bar{\phi}},$$

where A_i in (30*a*) denote the *i*-th component of the column vector A defined in (30*c*). In eq. (29) it was taken into account that $\partial \bar{\boldsymbol{\alpha}}^{2^T}/\partial \bar{\boldsymbol{\alpha}} = \mathrm{diag}\,[2\bar{\alpha}_i]$. Equations (23)-(30) can also be regarded as characterizing the solution of an unconstrained saddle-point problem:

$$\min_{\dot{\bar{u}},\dot{\bar{e}},\dot{\bar{\eta}},\bar{\lambda},\bar{\alpha}} \; \max_{\dot{\bar{\sigma}},\dot{\bar{\chi}},\dot{\bar{\phi}}} \{\bar{\mathcal{L}}_{\Xi}\}. \tag{31}$$

So far, the interpolation matrices $N_{[\cdot]}(\boldsymbol{x})$ were kept purposedly unspecified. A set of governing relations formally similar to the one governing the continuum rate problem can be obtained in terms of the barred quantities by a suitable choice of the interpolation matrices. Let us assume that: *i*) two barred vectors relative to conjugate static and kinematic quantities (*i.e.* $\dot{\bar{\boldsymbol{\sigma}}}$ and $\dot{\bar{\boldsymbol{e}}}$, $\dot{\bar{\boldsymbol{\chi}}}$ and $\dot{\bar{\boldsymbol{\eta}}}$, $\dot{\bar{\boldsymbol{\phi}}}$ and $\dot{\bar{\boldsymbol{\lambda}}}$) have the same number of components; *ii*) the product of the shape matrices pertinent to conjugate fields gives a nonsingular matrix $\left(e.g.\ \det\left(\int_\Omega N_e^T N_\sigma\, d\Omega\right) \neq 0\right)$.

Considering, *e.g.*, eq. (24*a*) one has:

$$\dot{\bar{\boldsymbol{\sigma}}} = \left[\int_\Omega N_e^T N_\sigma\, d\Omega\right]^{-1} \int_\Omega N_e^T \frac{\partial^2 U}{\partial \boldsymbol{e}\, \partial \boldsymbol{e}^T} N_e\, d\Omega\, \dot{\bar{\boldsymbol{e}}} \equiv \bar{\boldsymbol{E}}(\bar{\boldsymbol{e}})\, \dot{\bar{\boldsymbol{e}}}. \tag{32}$$

Equation (32) can be thought of as the discrete counterpart of Hooke's law, eq. (3*a*), $\bar{\boldsymbol{E}}$ playing the role of the matrix of current tangent elastic moduli. However, it is worth noting that, for an arbitrary choice of N_e and N_σ complying with *i*) and *ii*), $\bar{\boldsymbol{E}}$ is not symmetric. On the other hand, if the shape matrices are assumed to satisfy the additional «orthogonality condition»:

$$\int_\Omega N_e^T N_\sigma\, d\Omega = \boldsymbol{I}, \qquad \boldsymbol{I} = \mathrm{diag}\,[1], \tag{33}$$

$\bar{\boldsymbol{E}}$ turns out to be symmetric. Similar considerations, applied to the other equations of set (23)-(30), suggest that the shape functions in the couples $N_\eta \div N_\chi$ and $N_\lambda \div N_\phi$ be chosen such as:

$$\int_\Omega N_\eta^T N_\chi\, d\Omega = \boldsymbol{I}, \qquad \int_\Omega N_\lambda^T N_\phi\, d\Omega = \boldsymbol{I}. \tag{34}$$

By introducing the additional conditions (33)-(34) in (23)-(30) and taking $N_\alpha \equiv N_\lambda$ one obtains:

$$\bar{\boldsymbol{C}}^T \dot{\bar{\boldsymbol{\sigma}}} = \dot{\bar{\boldsymbol{F}}}, \quad \text{having set: } \bar{\boldsymbol{C}} \equiv \int_\Omega N_\sigma^T \boldsymbol{C} N_u\, d\Omega, \qquad \dot{\bar{\boldsymbol{F}}} \equiv \int_\Omega N_u^T \dot{\boldsymbol{F}}\, d\Omega + \int_{\Gamma_t} N_u^T \dot{\boldsymbol{t}}\, d\Gamma, \tag{35}$$

$$\dot{\bar{\boldsymbol{\sigma}}} = \left[\int_\Omega N_e^T \frac{\partial^2 U}{\partial \boldsymbol{e}\, \partial \boldsymbol{e}^T} N_e\, d\Omega\right] \dot{\bar{\boldsymbol{e}}}, \tag{36}$$

(37) $$\dot{\chi}=\left[\int_{\Omega_p} N_\eta^T \frac{\partial^2 V}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}^T} N_\eta \, d\Omega\right] \dot{\boldsymbol{\eta}},$$

(38) $$\dot{\boldsymbol{\phi}}=\left[\int_{\Omega_p} N_\lambda^T \frac{\partial \phi}{\partial \boldsymbol{\sigma}^T} N_\sigma \, d\Omega\right] \dot{\boldsymbol{\sigma}}+\left[\int_{\Omega_p} N_\lambda^T \frac{\partial \phi}{\partial \boldsymbol{\chi}^T} N_\chi \, d\Omega\right] \dot{\boldsymbol{\chi}},$$

(39) $$\overline{\boldsymbol{C}}\dot{\boldsymbol{u}}=\dot{\boldsymbol{e}} \text{ in } \Omega_e, \qquad \overline{\boldsymbol{C}}\dot{\boldsymbol{u}}=\dot{\boldsymbol{e}}+\left[\int_{\Omega_p} N_\sigma^T \frac{\partial \phi}{\partial \boldsymbol{\sigma}} N_\lambda \, d\Omega\right] \dot{\boldsymbol{\lambda}} \text{ in } \Omega_p,$$

(40) $$\dot{\boldsymbol{\eta}}=-\left[\int_{\Omega_p} N_\chi^T \frac{\partial \phi}{\partial \boldsymbol{\chi}} N_\lambda \, d\Omega\right] \dot{\boldsymbol{\lambda}} \text{ in } \Omega_p,$$

(41) $$\dot{\boldsymbol{\lambda}}=\overline{\boldsymbol{\alpha}}^2 \geqslant \mathbf{0}, \qquad 2\overline{\alpha}_i \dot{\phi}_i=0 \Rightarrow \dot{\lambda}_i \dot{\phi}_i=0, \qquad \dot{\boldsymbol{\phi}} \leqslant \mathbf{0} \text{ in } \Omega_p.$$

It is worth noting that, at the considered instant, $U[\boldsymbol{e}(\boldsymbol{x})]$, $V[\boldsymbol{\eta}(\boldsymbol{x})]$ and $\phi[\boldsymbol{\sigma}(\boldsymbol{x}), \boldsymbol{\chi}(\boldsymbol{x})]$ are known functions of the position and therefore need not be modelled. However, a substantial simplification in eqs. (36)-(40) can be achieved by modelling $\boldsymbol{e}(\boldsymbol{x})$, $\boldsymbol{\eta}(\boldsymbol{x})$, $\boldsymbol{\sigma}(\boldsymbol{x})$ and $\boldsymbol{\chi}(\boldsymbol{x})$ by the same interpolation functions used for $\dot{\boldsymbol{e}}(\boldsymbol{x})$, $\dot{\boldsymbol{\eta}}(\boldsymbol{x})$, $\boldsymbol{s}(\boldsymbol{x})$ and $-\boldsymbol{c}(\boldsymbol{x})$, respectively. In fact, the following relations can be easily shown to hold:

(42) $$U[\boldsymbol{e}^h(\boldsymbol{x})]=U[N_e(\boldsymbol{x})\,\overline{\boldsymbol{e}}] \Rightarrow \int_{\Omega} N_e^T \frac{\partial^2 U}{\partial \boldsymbol{e}^h \partial \boldsymbol{e}^{hT}} N_e \, d\Omega=\frac{\partial^2 \overline{U}}{\partial \overline{\boldsymbol{e}}\, \partial \overline{\boldsymbol{e}}^T}, \quad \overline{U} \equiv \int_{\Omega} U[\boldsymbol{e}^h(\boldsymbol{x})] \, d\Omega,$$

(43) $$V[\boldsymbol{\eta}^h(\boldsymbol{x})]=V[N_\eta(\boldsymbol{x})\,\overline{\boldsymbol{\eta}}] \Rightarrow \int_{\Omega_p} N_\eta^T \frac{\partial^2 V}{\partial \boldsymbol{\eta}^h \partial \boldsymbol{\eta}^{hT}} N_\eta \, d\Omega=\frac{\partial^2 \overline{V}}{\partial \overline{\boldsymbol{\eta}}\, \partial \overline{\boldsymbol{\eta}}^T}, \quad \overline{V} \equiv \int_{\Omega_p} V[\boldsymbol{\eta}^h(\boldsymbol{x})] \, d\Omega,$$

(44) $$\phi[\boldsymbol{\sigma}^h(\boldsymbol{x}), \boldsymbol{\chi}^h(\boldsymbol{x})]=\phi[N_\sigma(\boldsymbol{x})\,\overline{\boldsymbol{\sigma}}, N_\chi(\boldsymbol{x})\,\overline{\boldsymbol{\chi}}] \Rightarrow \int_{\Omega_p} N_\lambda^T \frac{\partial \phi}{\partial \boldsymbol{\sigma}^{hT}} N_\sigma \, d\Omega=\frac{\partial \overline{\boldsymbol{\phi}}}{\partial \overline{\boldsymbol{\sigma}}^T},$$
$$\int_{\Omega_p} N_\lambda^T \frac{\partial \phi}{\partial \boldsymbol{\chi}^{hT}} N_\chi \, d\Omega=\frac{\partial \overline{\boldsymbol{\phi}}}{\partial \overline{\boldsymbol{\chi}}^T}, \qquad \overline{\boldsymbol{\phi}} \equiv \int_{\Omega_p} N_\lambda^T \phi[\boldsymbol{\sigma}^h(\boldsymbol{x}), \boldsymbol{\chi}^h(\boldsymbol{x})] \, d\Omega.$$

With the additional interpolations, eqs. (36)-(40) can be written in the form:

(45) $$\dot{\boldsymbol{\sigma}}=\frac{\partial^2 \overline{U}}{\partial \overline{\boldsymbol{e}}\, \partial \overline{\boldsymbol{e}}^T} \dot{\boldsymbol{e}}, \qquad \dot{\boldsymbol{\chi}}=\frac{\partial^2 \overline{V}}{\partial \overline{\boldsymbol{\eta}}\, \partial \overline{\boldsymbol{\eta}}^T} \dot{\boldsymbol{\eta}},$$

(46) $$\dot{\boldsymbol{\phi}}=\frac{\partial \overline{\boldsymbol{\phi}}}{\partial \overline{\boldsymbol{\sigma}}^T} \dot{\boldsymbol{\sigma}}+\frac{\partial \overline{\boldsymbol{\phi}}}{\partial \overline{\boldsymbol{\chi}}^T} \dot{\boldsymbol{\chi}},$$

(47) $$\overline{\boldsymbol{C}}\dot{\boldsymbol{u}}=\dot{\boldsymbol{e}} \text{ in } \Omega_e, \qquad \overline{\boldsymbol{C}}\dot{\boldsymbol{u}}=\dot{\boldsymbol{e}}+\frac{\partial \overline{\boldsymbol{\phi}}^T}{\partial \overline{\boldsymbol{\sigma}}} \dot{\boldsymbol{\lambda}} \text{ in } \Omega_p, \qquad \dot{\boldsymbol{\eta}}=-\frac{\partial \overline{\boldsymbol{\phi}}^T}{\partial \overline{\boldsymbol{\chi}}} \dot{\boldsymbol{\lambda}} \text{ in } \Omega_p.$$

Equations (35), (41) and (45)-(47) are the discrete counterparts of the relations which govern the continuum rate problem. Equations (35) and (47*a,b*) express equilibrium and compatibility in terms of the unknown interpolation parameters. All other relations (41), (45)-(46) and (47*c*), can be regarded as a «global» constitutive law in the sense that they are written in terms of a discrete number of parameters concerning the whole structure rather than single material points chosen for the enforcement of the material model.

4. Discretized elastoplastic rate problem in generalized variables

4.1. «*Generalized*» *variables.* – The barred variables appearing as parameters in the interpolations (19)-(21) are «generalized» in Prager's sense [4], when the adopted interpolations are such as to satisfy conditions (33) and (34). In fact, according to Prager's definition, a static and a conjugate kinematic variables (typically moments and rotations) relative to one element are said to be «generalized» if their scalar product equals the virtual work done over the same element by the corresponding continuous fields. With reference to eqs. (20) and (21), this means that:

$$\dot{\bar{\boldsymbol{\sigma}}}^T \dot{\bar{\boldsymbol{e}}} = \int_{\Omega} [\dot{\boldsymbol{\sigma}}^b(\boldsymbol{x})]^T \dot{\boldsymbol{e}}^b(\boldsymbol{x})\, d\Omega \Leftrightarrow \int_{\Omega} \boldsymbol{N}_\sigma(\boldsymbol{x})^T \boldsymbol{N}_e(\boldsymbol{x})\, d\Omega = \boldsymbol{I}, \tag{48}$$

$$\dot{\bar{\boldsymbol{\chi}}}^T \dot{\bar{\boldsymbol{\eta}}} = \int_{\Omega_p} [\dot{\boldsymbol{\chi}}^b(\boldsymbol{x})]^T \dot{\boldsymbol{\eta}}^b(\boldsymbol{x})\, d\Omega \Leftrightarrow \int_{\Omega_p} \boldsymbol{N}_\chi(\boldsymbol{x})^T \boldsymbol{N}_\eta(\boldsymbol{x})\, d\Omega = \boldsymbol{I}, \tag{49}$$

$$\dot{\bar{\boldsymbol{\phi}}}^T \dot{\bar{\boldsymbol{\lambda}}} = \int_{\Omega_p} [\dot{\boldsymbol{\phi}}^b(\boldsymbol{x})]\, \dot{\boldsymbol{\lambda}}^b(\boldsymbol{x})\, d\Omega \Leftrightarrow \int_{\Omega_p} \boldsymbol{N}_\phi(\boldsymbol{x})^T \boldsymbol{N}_\lambda(\boldsymbol{x})\, d\Omega = \boldsymbol{I}. \tag{50}$$

A possible way of satisfying the ortogonality condition (48*b*) is to choose $\boldsymbol{N}_\sigma$ as follows, for given $\boldsymbol{N}_e$:

$$\boldsymbol{N}_\sigma(\boldsymbol{x}) = \boldsymbol{N}_e(\boldsymbol{x}) \left[\int_{\Omega} \boldsymbol{N}_e(\boldsymbol{x})^T \boldsymbol{N}_e(\boldsymbol{x})\, d\Omega \right]^{-1}, \tag{51}$$

and viceversa. Similarly for (49*b*) and (50*b*).

By substituting into the integral in eq. (48*a*) the interpolation (21*a*) of $\dot{\boldsymbol{\sigma}}^b(\boldsymbol{x})$ in terms of $\dot{\bar{\boldsymbol{\sigma}}}$, one has:

$$\dot{\bar{\boldsymbol{\sigma}}}^T \dot{\bar{\boldsymbol{e}}} = \dot{\bar{\boldsymbol{\sigma}}} \int_{\Omega} \boldsymbol{N}_\sigma^T(\boldsymbol{x})\, \dot{\boldsymbol{e}}^b(\boldsymbol{x})\, d\Omega \qquad \forall \dot{\bar{\boldsymbol{\sigma}}} \;\Rightarrow\; \dot{\bar{\boldsymbol{e}}} = \int_{\Omega} \boldsymbol{N}_\sigma^T(\boldsymbol{x})\, \dot{\boldsymbol{e}}^b(\boldsymbol{x})\, d\Omega. \tag{52}$$

Similar substitutions for the other fields (both static and kinematic) provide analogous expressions for all generalized variables. Therefore, all generalized components can be interpreted as particular weighted averages of the corresponding interpolated fields over the domain Ω, the weights being the interpolation functions of the conjugate fields. As a consequence, it can be shown that the generalized governing relations (35), (41) and (45)-(47) do not imply a pointwise fulfillment of the corresponding conditions in terms of continuous fields. Consider, *e.g.*, the compatibility eq. (47*a*) which comes from the stationarity of $\bar{\mathcal{L}}_{\Xi}$ with respect to $\dot{\bar{\boldsymbol{\sigma}}}$ (eq. (26)):

$$\int_{\Omega_e} \boldsymbol{N}_\sigma^T(\boldsymbol{x})\,[\boldsymbol{C}\,\dot{\boldsymbol{u}}^b(\boldsymbol{x}) - \dot{\boldsymbol{e}}^b(\boldsymbol{x})]\,d\Omega = \boldsymbol{0}. \tag{53}$$

As it can be easily seen, the compatibility constraint between interpolated displacements and elastic strains in Ω_e turns out to be enforced only in a weighted average sense, $\boldsymbol{N}_\sigma^T$ being the matrix of the weight functions.

REMARK 4.1. Let us assume that the interpolated fields satisfy pointwise eqs. (1)-(7). This implies fulfillment of the corresponding equalities in eqs. (35), (41) and (45)-(47) in terms of generalized variables. It is easy to show that this does not apply to inequalities. For instance, define $\dot{\bar{\boldsymbol{\lambda}}}$ by an expression of the type of (52*b*):

$$\dot{\bar{\boldsymbol{\lambda}}} = \int_{\Omega_p} \boldsymbol{N}_\phi^T(\boldsymbol{x})\,\dot{\lambda}^b(\boldsymbol{x})\,d\Omega. \tag{54}$$

Since the interpolation functions $\boldsymbol{N}_\phi(\boldsymbol{x})$ are not necessarily sign-constrained, the nonnegativeness of $\dot{\lambda}^b(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \Omega_p$, is not sufficient to guarantee the nonnegativeness of $\dot{\bar{\boldsymbol{\lambda}}}$.

REMARK 4.2. As noted at the end of Sect. 3, if no special assumptions are made on the interpolation functions, the relations describing the constitutive law at a «global» level would exhibit a coupled dependence on generalized variables pertaining to different portions of the domain. The constitutive coupling shows up even if the orthogonality conditions (48*b*)-(50*b*) are fulfilled. The portion of space to which a generalized variable pertains is here intended as that part of the domain where the shape function which multiplies that variable does not vanish identically.

This coupling is present at difference from what happens for the constitutive law of the continuum where only values relative to a single material point are involved. Furthermore, this coupling entails a most unfavourable computational burden when an integration in time is carried out along the process of loading [11]. For this reason, desirable choices of the interpolation functions are such that the «global» constitutive law can be written as an ensemble of decoupled relation sets, each involving only generalized variables pertaining to a single finite element. Assume, in addition, that the kinematic fields are interpolated in terms of their local values at the Gauss points (the static fields are interpolated according to (48*b*)-(50*b*)) and the integrations are carried out numerically by using the same Gauss points. In this case a complete decoupling occurs and the constitutive law can be written for each Gauss point separately, also when expressed in terms of generalized variables [11].

4.2. *On the convexity of the generalized yield functions.* – If a given elastoplastic material obeys the principle of maximum dissipation [14], its yield function can be easily shown to be a convex function of stresses and static internal variables. It is here proved that, if such a material is considered, its generalized yield functions are convex in the generalized (stress and static internal) variables.

The principle of maximum dissipation can be formulated at a local level as follows ($\dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{\eta}}$ being assigned fields):

$$\max_{\boldsymbol{\sigma},\boldsymbol{\chi}} \{D(\boldsymbol{x}) = \boldsymbol{\sigma}^T(\boldsymbol{x})\dot{\boldsymbol{p}}(\boldsymbol{x}) - \boldsymbol{\chi}^T(\boldsymbol{x})\dot{\boldsymbol{\eta}}(\boldsymbol{x}) \,|\, \phi(\boldsymbol{\sigma},\boldsymbol{\chi},\boldsymbol{x}) \leqslant 0 \ \ \forall \boldsymbol{x} \in \Omega_p\}, \tag{55}$$

D denoting the dissipation function.

Integrating the dissipation rate over the volume and using the classical Lagrange multipliers method as in Sect. 2, one is led to consider the following functional:

$$\mathcal{L}_D = -\int_{\Omega_p} [\boldsymbol{\sigma}^T(\boldsymbol{x})\dot{\boldsymbol{p}}(\boldsymbol{x}) - \boldsymbol{\chi}^T(\boldsymbol{x})\dot{\boldsymbol{\eta}}(\boldsymbol{x})]\, d\Omega + \int_{\Omega_p} [\phi(\boldsymbol{\sigma},\boldsymbol{\chi},\boldsymbol{x}) + \beta^2(\boldsymbol{x})]\,\dot{\lambda}(\boldsymbol{x})\, d\Omega, \tag{56}$$

where β^2 is a slack variable and $\dot{\lambda}(\boldsymbol{x})$ is a Lagrange multiplier. It is easy to show that the stationary point of $\mathcal{L}_D$ maximizes $D(\boldsymbol{x})$ under the condition that the second variation of $\mathcal{L}_D$ with respect to β is nonnegative. Let us model all fields in (56) by using for $\dot{\boldsymbol{\eta}}$ and $\dot{\lambda}$ the interpolations (20*b, c*) and for $\boldsymbol{\sigma}$ and $\boldsymbol{\chi}$ the interpolation matrices $\boldsymbol{N}_\sigma$ and $\boldsymbol{N}_\chi$ of (21*a, b*). Moreover, let us set:

$$\dot{\boldsymbol{p}}^b(\boldsymbol{x}) = \boldsymbol{N}_e(\boldsymbol{x})\dot{\bar{\boldsymbol{p}}}, \qquad [\beta^2(\boldsymbol{x})]^b = \boldsymbol{N}_\phi(\boldsymbol{x})\bar{\boldsymbol{\beta}}^2. \tag{57}$$

Taking into account the orthogonality conditions (48*b*)-(50*b*) and making use of the definition (44*d*) of $\bar{\boldsymbol{\phi}}$, the discretized version of $\mathcal{L}_D$ reads:

$$\bar{\mathcal{L}}_D(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}},\dot{\bar{\boldsymbol{\lambda}}},\bar{\boldsymbol{\beta}}) = -\bar{\boldsymbol{\sigma}}^T\dot{\bar{\boldsymbol{p}}} + \bar{\boldsymbol{\chi}}^T\dot{\bar{\boldsymbol{\eta}}} + (\bar{\boldsymbol{\phi}}(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}}) + \bar{\boldsymbol{\beta}}^2)^T\dot{\bar{\boldsymbol{\lambda}}}. \tag{58}$$

For given $\dot{\bar{\boldsymbol{p}}}$ and $\dot{\bar{\boldsymbol{\eta}}}$, a saddle point of $\bar{\mathcal{L}}_D$ (*max* with respect to $\dot{\bar{\boldsymbol{\lambda}}}$ and *min* with respect to $\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}},\bar{\boldsymbol{\beta}}$) is obtained from the following conditions:

$$-\dot{\bar{\boldsymbol{p}}} + \frac{\partial\bar{\boldsymbol{\phi}}^T}{\partial\bar{\boldsymbol{\sigma}}}\dot{\bar{\boldsymbol{\lambda}}} = \boldsymbol{0}; \qquad \dot{\bar{\boldsymbol{\eta}}} + \frac{\partial\bar{\boldsymbol{\phi}}^T}{\partial\bar{\boldsymbol{\chi}}}\dot{\bar{\boldsymbol{\lambda}}} = \boldsymbol{0}; \tag{59}$$

$$\bar{\boldsymbol{\phi}} + \bar{\boldsymbol{\beta}}^2 = \boldsymbol{0}; \qquad \operatorname{diag}[2\bar{\beta}_i]\,\dot{\bar{\boldsymbol{\lambda}}} = 0 \Rightarrow \bar{\phi}_i\dot{\bar{\lambda}}_i = 0; \qquad 2\dot{\bar{\lambda}}_i \geqslant 0. \tag{60}$$

Let us mark by an asterisk * the solution of problem (59)-(60). The following inequality holds:

$$\bar{\mathcal{L}}_D(\bar{\boldsymbol{\sigma}}^*,\bar{\boldsymbol{\chi}}^*,\dot{\bar{\boldsymbol{\lambda}}}^*,\bar{\boldsymbol{\beta}}^*) \leqslant \bar{\mathcal{L}}_D(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}},\dot{\bar{\boldsymbol{\lambda}}}^*,\bar{\boldsymbol{\beta}}^*). \tag{61}$$

Making use of the definition (58) of $\bar{\mathcal{L}}_D$, of eqs. (59) and (60*d*), one has:

$$-\left.\frac{\partial\bar{\boldsymbol{\phi}}}{\partial\bar{\boldsymbol{\sigma}}^T}\right|_{\bar{\sigma}^*,\bar{\chi}^*}(\bar{\boldsymbol{\sigma}}^* - \bar{\boldsymbol{\sigma}}) - \left.\frac{\partial\bar{\boldsymbol{\phi}}}{\partial\bar{\boldsymbol{\chi}}^T}\right|_{\bar{\sigma}^*,\bar{\chi}^*}(\bar{\boldsymbol{\chi}}^* - \bar{\boldsymbol{\chi}}) \leqslant -\bar{\boldsymbol{\phi}}(\bar{\boldsymbol{\sigma}}^*,\bar{\boldsymbol{\chi}}^*) + \bar{\boldsymbol{\phi}}(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}}). \tag{62}$$

Equation (62) expresses the convexity of $\bar{\boldsymbol{\phi}}(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}})$.

4.3. *Minimum properties in generalized variables.* – The problem of finding a stationary point of $\bar{\mathcal{L}}_\Xi$ can be reformulated as a constrained minimization problem by reversing the procedure followed in Sect. 2. Namely, some independent variables in the expression (22) of $\bar{\mathcal{L}}_\Xi$ can be removed by introducing suitable constraints on the remaining variables. The addition of supplementary constraints is admissible as long as they are satisfied at a stationary point of $\bar{\mathcal{L}}_\Xi$. Depending on the nature of the variables which appear in the resulting function, either a kinematic (*i*) or a static (*ii*) extremum property is obtained.

(*i*) Eliminate the Lagrange multipliers $\boldsymbol{s}$, $\boldsymbol{c}$ and f from the expression (22) of $\bar{\mathcal{L}}_{\Xi}$ by adding the constraints (26), (27) and (28). Eliminate $\bar{\boldsymbol{\alpha}}^2$ by transforming the constraint (28) in the inequality $\dot{\bar{\boldsymbol{\lambda}}} \geqslant \mathbf{0}$. The following constrained minimization problem is thus obtained:

$$\min_{\dot{\bar{\boldsymbol{u}}}, \dot{\bar{\boldsymbol{e}}}, \dot{\bar{\boldsymbol{\eta}}}, \dot{\bar{\boldsymbol{\lambda}}}} \left\{ \bar{\Theta} = \frac{1}{2} \dot{\bar{\boldsymbol{e}}}^T \frac{\partial^2 \bar{U}}{\partial \bar{\boldsymbol{e}} \partial \bar{\boldsymbol{e}}^T} \dot{\bar{\boldsymbol{e}}} + \frac{1}{2} \dot{\bar{\boldsymbol{\eta}}}^T \frac{\partial^2 \bar{V}}{\partial \bar{\boldsymbol{\eta}} \partial \bar{\boldsymbol{\eta}}^T} \dot{\bar{\boldsymbol{\eta}}} - \dot{\bar{\boldsymbol{F}}}^T \dot{\bar{\boldsymbol{u}}} \right\} \tag{63}$$

subject to:

$$\bar{\boldsymbol{C}} \dot{\bar{\boldsymbol{u}}} = \dot{\bar{\boldsymbol{e}}} \text{ in } \Omega_e, \qquad \bar{\boldsymbol{C}} \dot{\bar{\boldsymbol{u}}} = \dot{\bar{\boldsymbol{e}}} + \frac{\partial \bar{\boldsymbol{\phi}}^T}{\partial \bar{\boldsymbol{\sigma}}} \dot{\bar{\boldsymbol{\lambda}}} \text{ in } \Omega_p, \tag{64}$$

$$\dot{\bar{\boldsymbol{\eta}}} = - \frac{\partial \bar{\boldsymbol{\phi}}^T}{\partial \bar{\boldsymbol{\chi}}} \dot{\bar{\boldsymbol{\lambda}}} \text{ in } \Omega_p, \qquad \dot{\bar{\boldsymbol{\lambda}}} \geqslant \mathbf{0} \text{ in } \Omega_p. \tag{65}$$

(*ii*) A second minimization problem is obtained by considering the following feasible domain:

$$\bar{\boldsymbol{C}}^T \dot{\bar{\boldsymbol{\sigma}}} - \dot{\bar{\boldsymbol{F}}} = \mathbf{0} \text{ in } \Omega, \qquad \dot{\bar{\boldsymbol{\phi}}} = \frac{\partial \bar{\boldsymbol{\phi}}}{\partial \bar{\boldsymbol{\sigma}}^T} \dot{\bar{\boldsymbol{\sigma}}} + \frac{\partial \bar{\boldsymbol{\phi}}}{\partial \bar{\boldsymbol{\chi}}^T} \dot{\bar{\boldsymbol{\chi}}} \leqslant \mathbf{0} \text{ in } \Omega_p. \tag{66}$$

A stationary point of $\bar{\mathcal{L}}_{\Xi}$ satisfies eq. (66) because they coincide with eqs. (23), (25) and (30*b*), respectively. Introducing eqs. (66) in the expression (22) of $\bar{\mathcal{L}}_{\Xi}$, the saddle point problem (31) is transformed as follows:

$$\min_{\dot{\bar{\boldsymbol{e}}}, \dot{\bar{\boldsymbol{\eta}}}, \bar{\boldsymbol{\alpha}}} \max_{\dot{\bar{\boldsymbol{\sigma}}}, \dot{\bar{\boldsymbol{\chi}}}} \left\{ \frac{1}{2} \dot{\bar{\boldsymbol{e}}}^T \frac{\partial^2 \bar{U}}{\partial \bar{\boldsymbol{e}} \partial \bar{\boldsymbol{e}}^T} \dot{\bar{\boldsymbol{e}}} + \frac{1}{2} \dot{\bar{\boldsymbol{\eta}}}^T \frac{\partial^2 \bar{V}}{\partial \bar{\boldsymbol{\eta}} \partial \bar{\boldsymbol{\eta}}^T} \dot{\bar{\boldsymbol{\eta}}} - \dot{\bar{\boldsymbol{e}}}^T \dot{\bar{\boldsymbol{\sigma}}} - \dot{\bar{\boldsymbol{\eta}}}^T \dot{\bar{\boldsymbol{\chi}}} - \bar{\boldsymbol{\alpha}}^{2^T} \left(\frac{\partial \bar{\boldsymbol{\phi}}}{\partial \bar{\boldsymbol{\sigma}}^T} \dot{\bar{\boldsymbol{\sigma}}} + \frac{\partial \bar{\boldsymbol{\phi}}}{\partial \bar{\boldsymbol{\chi}}^T} \dot{\bar{\boldsymbol{\chi}}} \right) \right\} \tag{67}$$

subject to constraints (66).

The kinematic variables $\dot{\bar{\boldsymbol{e}}}$ and $\dot{\bar{\boldsymbol{\eta}}}$ are eliminated by defining $\bar{U}_c(\bar{\boldsymbol{\sigma}})$ and $\bar{V}_c(\bar{\boldsymbol{\chi}})$ as Legendre transforms of $\bar{U}(\bar{\boldsymbol{e}})$ and $\bar{V}(\bar{\boldsymbol{\eta}})$, respectively:

$$\bar{U}_c(\bar{\boldsymbol{\sigma}}) = -\bar{U}(\bar{\boldsymbol{e}}) + \bar{\boldsymbol{\sigma}}^T \bar{\boldsymbol{e}}, \qquad \bar{V}_c(\bar{\boldsymbol{\chi}}) = -\bar{V}(\bar{\boldsymbol{\eta}}) + \bar{\boldsymbol{\chi}}^T \bar{\boldsymbol{\eta}}. \tag{68}$$

Since the last term in (67) is always nonnegative and $\bar{\boldsymbol{\alpha}}^2$ does not play any role in other addends, at the solution it must be $\bar{\boldsymbol{\alpha}}^{2^T} \left(\frac{\partial \bar{\boldsymbol{\phi}}}{\partial \bar{\boldsymbol{\sigma}}^T} \dot{\bar{\boldsymbol{\sigma}}} + \frac{\partial \bar{\boldsymbol{\phi}}}{\partial \bar{\boldsymbol{\chi}}^T} \dot{\bar{\boldsymbol{\chi}}} \right) = 0$. Therefore, problem (67)-(66) can be recast in the form:

$$\max_{\dot{\bar{\boldsymbol{\sigma}}}, \dot{\bar{\boldsymbol{\chi}}}} \left\{ \bar{\Theta}_c = - \frac{1}{2} \dot{\bar{\boldsymbol{\sigma}}}^T \frac{\partial^2 \bar{U}_c}{\partial \bar{\boldsymbol{\sigma}} \partial \bar{\boldsymbol{\sigma}}^T} \dot{\bar{\boldsymbol{\sigma}}} - \frac{1}{2} \dot{\bar{\boldsymbol{\chi}}}^T \frac{\partial^2 \bar{V}_c}{\partial \bar{\boldsymbol{\chi}} \partial \bar{\boldsymbol{\chi}}^T} \dot{\bar{\boldsymbol{\chi}}} \right\} \tag{69}$$

subject to constraints (66).

REMARK 4.3. The kinematic extremum property, eqs. (63)-(65), and the static one, eqs. (69) and (66), are formulated in terms of rates. A formulation in terms of finite increments can be achieved subdividing the history of loading in finite time intervals and introducing an approximate integration scheme. In particular, it can be shown that the adoption of a backward-difference integration would lead to the same kinematic and static theorems presented by Comi *et al.* in [11]. The existence of extremum properties for the finite-step problem is of practical interest inasmuch as it allows to provide convergence conditions for a commonly used predictor-corrector iteration algorithm [7, 15].

REMARK 4.4. It is worth noting that a direct discretization in terms of generalized variables of the objective function Ξ (eq. (8)) and of the constraints (9)-(12) of the minimum problem of Sect. 2, would not lead to the same kinematic problem (*i*) (eqs. (63)-(65)) of this Section. In fact, while the objective function would be the same ($\bar{\Xi} \equiv \bar{\Theta}$), the discretized version of the constraints (9)-(12) would not coincide with (64)-(65).

5. CONCLUSIONS

This paper was aimed at a variationally consistent formulation of the elastoplastic boundary value rate problem in generalized variables. By «variationally consistent» we mean here that all governing relations are obtained from the stationarity conditions of a variational principle. The path of reasoning adopted to this purpose can be summarized as follows: *a*) starting from an extension of Capurso-Maier's kinematic theorem, the elastoplastic continuum rate problem is formulated as an unconstrained saddle-point problem; *b*) a finite element independent discretization of all fields is introduced; *c*) the discretized version of the governing relations is obtained by the saddle-point conditions on the discretized functional; *d*) at this stage, the generalized variable interpolation appears to be the most natural choice, as it provides discretized constitutive equations formally identical to the ones for the continuum problem; *e*) the convexity of the generalized yield functions is proved by discretization of the principle of maximum dissipation which holds when the material is stable.

As a further result, a couple of minimum properties in generalized variables has been presented. The adoption of a finite-step backward-difference time integration would transform these minimum principles into those presented in [11].

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