Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei Matematica e Applicazioni

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CR-structures on a real Lie algebra

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **2** (1991), n.3, p. 203–205.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1991_9_2_3_203_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1991.

Rend. Mat. Acc. Lincei
s. 9, v. 2:203-205 (1991)

Geometria. — *CR-structures on a real Lie algebra.* Nota di Giuliana Gigante e Giuseppe Tomassini, presentata (*) dal Socio E. Vesentini.

Abstract. — Given the notion of CR-structures without torsion on a real $2n+1$ dimensional Lie algebra $\mathcal{L}_0$ we study the problem of their classification when $\mathcal{L}_0$ is a reductive algebra.

Key words: Lie algebra; Deformations of special (*e.g.* CR) structures; Analytic moduli problems.

Riassunto. — *CR-strutture su un'algebra di Lie reale.* Data la nozione di CR-strutture senza torsione su un'algebra di Lie reale $\mathcal{L}_0$, di dimensione $2n+1$ se ne studia il problema della classificazione quando $\mathcal{L}_0$ è un'algebra riduttiva.

1. Preliminaries

1. Let $\mathcal{L}_0$ be a real Lie algebra of dimension $2n+1$ and $\mathcal{L} = \mathcal{L}_0 \otimes_{\mathbf{R}} \mathbf{C}$ its complexification. We will denote by $z = x + iy$, $x, y \in \mathcal{L}_0$ the elements of $\mathcal{L}$ and by τ the involution $x + iy \to x - iy$.

A CR-*structure* on $\mathcal{L}_0$ is a complex subalgebra $\boldsymbol{q}$ of $\mathcal{L}$ such that $\boldsymbol{q} \cap \bar{\boldsymbol{q}} = \{0\}$, $(\bar{\boldsymbol{q}} = \tau \boldsymbol{q})$. Two CR-structures $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ on $\mathcal{L}_0$ will be said to be *equivalent* if there is $\sigma \in \mathrm{Aut}\, \mathcal{L}$ such that $\sigma\tau = \tau\sigma$ and $\sigma \boldsymbol{q}_1 = \boldsymbol{q}_2$.

Let $\boldsymbol{q}$ be a CR-structure on $\mathcal{L}_0$ and let $\wp = \{x \in \mathcal{L}_0 : x + iy \in \boldsymbol{q}\}$; $\wp$ is a vector subspace of $\mathcal{L}_0$ and for a given $x \in \wp$ there is a unique y such that $x + iy \in \boldsymbol{q}$. It follows that the $\mathbf{R}$-linear map $J \colon \wp \to \wp$ given by $Jx = -y$ verifies $J^2 = -\mathrm{id}$ and consequently it defines a complex structure on $\wp$.

Remark. In general $\wp \neq 0$ is not a subalgebra of $\mathcal{L}_0$ except when $\boldsymbol{q}$ is an ideal. In that case $\wp \neq 0$ is an ideal too and the Lie group G_0 corresponding to $\mathcal{L}_0$ admits a complex Lie subgroup G_1 of positive dimension. In particular G_0 is a Levi-flat real manifold.

More generally the CR-structure will be said to be *Levi-flat* if $[\boldsymbol{q}, \bar{\boldsymbol{q}}] \subseteq \boldsymbol{q} \oplus \bar{\boldsymbol{q}}$.

If $\boldsymbol{q}$ is a CR-structure on $\mathcal{L}_0$, then the Nijenhuis tensor of $\boldsymbol{q}$ vanishes and consequently the following identity $N(x, y) = [x, y] + [Jx, y] + [x, Jy] - [Jx, Jy] = 0$ holds for $x, y \in \wp$.

Then it is a simple matter to verify that a CR-structure on $\mathcal{L}_0$ is determined by the following assignment: a vector subspace $\wp$ of $\mathcal{L}_0$ with an endomorphism $J \colon \wp \to \wp$ such that $J^2 = -\mathrm{id}$, $[x, y] - [Jx, y] \in \wp$, $N(x, y) = 0$ for $x, y \in \wp$.

2. Let $\mathcal{L}_0$ be reductive. As usual let $\boldsymbol{h}$ be a Cartan subalgebra (CSA) of $\mathcal{L}_0$, Δ the root system determined by $\boldsymbol{h}$, S a simple system of roots, $\Delta^+ = \{\alpha \in \Delta : \alpha > 0\}$, $\Delta^- = \{\alpha \in \Delta : \alpha < 0\}$ ([1]).

(*) Nella seduta del 9 febbraio 1991.

Assume $\boldsymbol{h}$ is τ-stable and $\tau\Delta^+ = \Delta^-$; then $\mathcal{L}_0$ will be defined as a real Lie algebra of *category* I if τ operates on S like $-\mathrm{id}$.

It can be proved ([2], [4]) that simple Lie algebras of category I are real forms of complex simple Lie algebras except for $\boldsymbol{sl}(n+1, \mathbf{R})$, $n \geqslant 2$, $\boldsymbol{su}^*(2n)$, $n \geqslant 2$ and $\boldsymbol{so}(p, 2n-p)$, p odd, $n \geqslant 4$.

2. CR-structures without torsion

1. The family of CR-structures on a given real Lie algebra $\mathcal{L}_0$ is too large thus, in order to study the classification problem for these structures we restrict ourselves to consider CR-structures with additional properties.

In the case when $\dim_{\mathbf{C}} \boldsymbol{q} = n$ $(2n+1 = \dim_{\mathbf{R}} \mathcal{L}_0)$ it is reasonable to consider CR-structures satisfying the following condition: there exists $x \in \mathcal{L}_0 \setminus (\boldsymbol{q} \oplus \overline{\boldsymbol{q}})$ such that $[\boldsymbol{q}, x] \subset \overline{\boldsymbol{q}} \oplus \mathbf{C}x$. Such a CR-structure will be defined as without torsion structure. The main step in the classification problem for CR-structures without torsion is then contained in the following

Theorem 1. *There exist a τ-stable* CSA, $\boldsymbol{h}$ *and a simple system of roots S such that*: (i) $\tau\Delta^+ = \Delta^-$; (ii) $\boldsymbol{q}$ *induces the* CR-*structure* $\boldsymbol{q} \cap \boldsymbol{h}$ *on* $\boldsymbol{h}_0 = \boldsymbol{h} \cap \mathcal{L}_0$; (iii) $[\boldsymbol{q}, \boldsymbol{h}'] \subset \boldsymbol{q}$ *where* $\boldsymbol{h}' = (\boldsymbol{h} \cap \boldsymbol{q}) \oplus (\boldsymbol{h} \cap (\overline{\boldsymbol{q}}))$.

Now let $\mathcal{L}_0, \boldsymbol{q}, \boldsymbol{h}, \boldsymbol{S}, \Delta^+, \Delta^-$ be as in Theorem 1 and denote by $\mathcal{L}(\alpha) = \mathbf{C}x_\alpha$ the root space of $\alpha \in \Delta$: here $\{x_\alpha\}_{\alpha \in \Delta}$ is a Weyl base for $\mathcal{L}_0$ [1].

Then we have

Theorem 2. *The following two cases are possible*:

$$(1) \quad \boldsymbol{q} = (\boldsymbol{q} \cap \boldsymbol{h}) \oplus \left(\bigoplus_{\alpha \in \Delta^+} \mathcal{L}(\alpha) \right); \qquad (2) \quad \boldsymbol{q} = (\boldsymbol{q} \cap \boldsymbol{h}) \oplus \left(\bigoplus_{\substack{\alpha \in \Delta^+ \\ \alpha \neq \mu}} \mathcal{L}(\alpha) \right) \oplus \mathbf{C}(x_\mu + x);$$

where $\mu \in \boldsymbol{S}$, $x \in \mathcal{L}_0 \cap \boldsymbol{h} \setminus \boldsymbol{q} \oplus \overline{\boldsymbol{q}}$ *and* $\mu|_{\boldsymbol{h}'} = 0$.

Starting from this result it is easy to describe completely the moduli space for CR-structures without torsion.

Remark. In case (1) we have $[\boldsymbol{q}, \boldsymbol{h}] \subset \boldsymbol{q} + \mathbf{C}x$. This corresponds, in the theory of CR-hypersurfaces to the case when the Webster connexion has no torsion [5].

2. Let us end this paper describing the situation for $\boldsymbol{sl}(2, \mathbf{R})$. In order to that we fix the standard base

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and we consider a CR-structure $\boldsymbol{q} = \mathbf{C}v$.

Let $z = -i[v, \bar{v}]$ and assume that $\{v, \bar{v}, z\}$ is a base for $\boldsymbol{sl}(2, \mathbf{C})$. The matrix of this new base is of the following type

$$A(a, r) = \begin{bmatrix} a & ir & 0 \\ ir & -\bar{a} & 0 \\ 0 & 0 & i \end{bmatrix}, \qquad a \in \mathbf{C}, \ r \in \mathbf{R}.$$

If we replace v by $W = \lambda v$, $\lambda \in \mathbf{C}^*$ we get a non singular matrix

$$\begin{bmatrix} a\bar{\lambda}^2 & ir|\lambda|^2 & 0 \\ ir|\lambda|^2 & -\bar{a}\lambda^2 & 0 \\ 0 & 0 & i \end{bmatrix}, \qquad a, \ \lambda \in \mathbf{C}, \ r \in \mathbf{R}.$$

Thus, to any non Levi-flat CR-structure on $\boldsymbol{sl}(2, \mathbf{R})$ we can associate a family of invertible matrices $\{A(a, r)\}$ and two such families coincide if and only if the corresponding CR-structures are equivalent. Moreover any family contains only one matrix $A(s, r)$ where $s, r \in \mathbf{R}$, $s \geq 0$, $r^2 - s^2 = 1$. It follows that the non Levi-flat CR-structures $\boldsymbol{sl}(2, \mathbf{R})$ are parametrized by $\{(r, s) \in \mathbf{R}^2 \colon r^2 - s^2 = 1, \ r \leq 0, \ s \geq 0\} \cup \cup \{(r, s) \in \mathbf{R}^2 \colon r^2 - s^2 = -1, \ s > 0\}$. All such structures have no torsion.

Moreover there is only one Levi-flat CR-structure given by $\boldsymbol{q} = \mathbf{C}(x - ih)$.

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