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Geometria differenziale. — *Comparison of metrics on three-dimensional Lie groups.* Nota di FEDERICO G. LASTARIA, presentata (*) dal Socio E. MARCHIONNA.

ABSTRACT. — We study local equivalence of left-invariant metrics with the same curvature on Lie groups G and $\overline{G}$ of dimension three, when G is unimodular and $\overline{G}$ is non-unimodular.

KEY WORDS: Invariant metric; Invariant isometry; Singer invariant.

RIASSUNTO. — *Confronto di metriche su gruppi di Lie di dimensione tre.* Si studia l'equivalenza locale di metriche invarianti a sinistra con la stessa curvatura su gruppi di Lie di dimensione tre G e $\overline{G}$, con G unimodulare e $\overline{G}$ non-unimodulare.

1. INTRODUCTION

Two homogeneous Riemannian manifolds (M, g), $(\overline{M}, \overline{g})$ are said *to have the same curvature* when for some (hence, by homogeneity, for any) couple of points $p \in M$, $\bar{p} \in \overline{M}$ there exists a linear isometry $F\colon T_p M \to T_{\bar{p}} \overline{M}$ which preserves the Riemann curvature tensors, *i.e.* such that $F^* \overline{R}_{\bar{p}} = R_p$. Here R_p and $\overline{R}_{\bar{p}}$ are the values of the Riemann curvature tensors R, $\overline{R}$ of g and $\overline{g}$ at the points p, $\bar{p}$.

In this paper we consider three-dimensional Lie groups (G, g), $(\overline{G}, \overline{g})$ endowed with left-invariant (therefore homogeneous) metrics with the same curvature, with G unimodular and $\overline{G}$ non-unimodular. We give a necessary and sufficient condition these metrics must satisfy in order to be locally equivalent. As a consequence, there follows the existence of homogeneous Riemannian manifolds which have the same curvature but are not locally isometric (see also [2] for results with both groups unimodular and [3] for further information).

2. EQUIVALENCE OF METRICS

We need the following:

LEMMA 1.

(1) *For all $a, b, c > 0$, with $b > c$, there exist metrics g, $\overline{g}$ defined on three-dimensional Lie algebras $\mathfrak{g}$, $\overline{\mathfrak{g}}$ respectively, with $\mathfrak{g}$ unimodular and $\overline{\mathfrak{g}}$ non-unimodular, both with the same principal Ricci curvatures $(-a^2, -b^2, c^2)$, and therefore with the same curvature.*

(2) *The signature $(-, -, +)$ is the only one admissible for the Ricci tensor of left-invariant metrics g, $\overline{g}$ with the same curvature, respectively defined on a unimodular and a non-unimodular group, both of dimension three.*

A proof of this Lemma, based on the analysis carried over in [4], may be found in [3, p. 62].

(*) Nella seduta del 9 marzo 1991.

We fix our notations. Henceforth, we will denote by r_{11}, r_{22}, r_{33} the common principal Ricci curvatures of the metrics g and $\bar{g}$. We will assume r_{11}, $r_{22} < 0$, and $r_{33} > 0$. Furthermore, r_{11} *will always denote the principal Ricci curvature relative to the orthogonal complement of the unimodular kernel of the Lie algebra* $\bar{\mathfrak{g}}$ *of* $\bar{G}$.

THEOREM 2. *The left-invariant metrics* g, $\bar{g}$ *with the same principal Ricci curvatures* r_{11}, r_{22}, r_{33} *are locally equivalent if and only if* $r_{11} = r_{22}$.

PROOF.

(A) *The condition is necessary*

Suppose that there exists a local isometry mapping the identity $e \in G$ to the identity $\bar{e} \in \bar{G}$. Let F: $T_e G \to T_{\bar{e}} \bar{G}$ be its differential at the identity e of G, and suppose, by absurd, $r_{11} \neq r_{22}$.

Let $\bar{e}_1, \bar{e}_2, \bar{e}_3$ be an orthonormal basis of the Lie algebra $\bar{\mathfrak{g}}$ of $\bar{G}$, such that $[\bar{e}_1, \bar{e}_2] = \alpha \bar{e}_2 + \beta \bar{e}_3$; $[\bar{e}_1, \bar{e}_3] = \gamma \bar{e}_2 + \delta \bar{e}_3$; $[\bar{e}_2, \bar{e}_3] = 0$; $\alpha + \delta > 0$; $\alpha\gamma + \beta\delta = 0$. (As to the existence of such a basis, we refer to [4]). The basis $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$, diagonalizes the Ricci tensor $\bar{r}$ of $\bar{g}$. Since $F^* \bar{r} = r$ (r = Ricci tensor of g), the orthonormal basis $e_1 = F^{-1}(\bar{e}_2)$, $e_2 = F^{-1}(\bar{e}_1)$, $e_3 = F^{-1}(\bar{e}_3)$ of $T_e G \simeq \mathfrak{g}$ diagonalizes r. It is easy to prove (see [3, p. 26]) that there exist λ_1, λ_2, $\lambda_3 \in R$ such that $[e_2, e_3] = \lambda_1 e_1$; $[e_3, e_1] = \lambda_2 e_2$; $[e_1, e_2] = \lambda_3 e_3$.

Let $\bar{\theta}^1$, $\bar{\theta}^2$, $\bar{\theta}^3$, and θ^1, θ^2, θ^3 be the dual coframes. Note that $F^* \bar{\theta}^i = \theta^i$, $i = 1, 2, 3$.

A straightforward computation gives the following expressions for the covariant differentials Dr and $\bar{D}\bar{r}$ (D and $\bar{D}$ are the Levi Civita connections of g and $\bar{g}$):

$$Dr = \mu_1 (r_{22} - r_{33}) \theta^1 \otimes (\theta^2 \otimes \theta^3 + \theta^3 \otimes \theta^2) + \\ + \mu_2 (r_{33} - r_{11}) \theta^2 \otimes (\theta^1 \otimes \theta^3 + \theta^3 \otimes \theta^1) + \mu_3 (r_{11} - r_{22}) \theta^3 \otimes (\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1),$$

where $\mu_i = (\lambda_1 + \lambda_2 + \lambda_3)/2 - \lambda_i$ and

$$\begin{aligned} \bar{D}\bar{r} = \alpha(r_{22} - r_{11}) (\bar{\theta}^2 \otimes \bar{\theta}^2 \otimes \bar{\theta}^1 + \bar{\theta}^2 \otimes \bar{\theta}^1 \otimes \bar{\theta}^2) + & \\ + (\beta + \gamma)(r_{22} - r_{11}) (\bar{\theta}^3 \otimes \bar{\theta}^1 \otimes \bar{\theta}^2 + \bar{\theta}^3 \otimes \bar{\theta}^2 \otimes \bar{\theta}^1)/2 + & \\ + (\beta + \gamma)(r_{33} - r_{11}) (\bar{\theta}^2 \otimes \bar{\theta}^3 \otimes \bar{\theta}^1 + \bar{\theta}^2 \otimes \bar{\theta}^1 \otimes \bar{\theta}^3)/2 + & \\ + \delta(r_{33} - r_{11}) (\bar{\theta}^3 \otimes \bar{\theta}^1 \otimes \bar{\theta}^3 + \bar{\theta}^3 \otimes \bar{\theta}^3 \otimes \bar{\theta}^1) + & \\ + (\beta - \gamma)(r_{22} - r_{33}) (\bar{\theta}^1 \otimes \bar{\theta}^2 \otimes \bar{\theta}^3 + \bar{\theta}^1 \otimes \bar{\theta}^3 \otimes \bar{\theta}^2)/2 . & \end{aligned}$$

Then the condition $F^* \bar{D}\bar{r} = Dr$ implies:

$$\alpha(r_{22} - r_{11}) \theta^2 \otimes \theta^1 \otimes \theta^2 + \delta(r_{33} - r_{11}) \theta^3 \otimes \theta^3 \otimes \theta^1 + \\ + (\text{terms not containing } \theta^1 \otimes \theta^2 \otimes \theta^1 \text{ and } \theta^3 \otimes \theta^2 \otimes \theta^3) = 0 .$$

Since, by absurd, $r_{11} \neq r_{22}$ and $r_{33} - r_{11} > 0$, we must have $\alpha = \delta = 0$. But this contradicts the condition $\alpha + \delta \neq 0$.

(B) *The condition is sufficient*

We need the notion of *Singer invariant* of a homogeneous metric, introduced by

I.M. Singer in [6]. Let (M, g) be a homogeneous Riemannian manifold and let $V = T_p M$ be the tangent space at an arbitrary point $p \in M$. Furthermore, let $\mathfrak{so}(V)$ be the Lie algebra of skew-symmetric endomorphisms of (V, g_p). For each integer $k \geqslant 0$ define a Lie subalgebra $\mathfrak{g}(k)$ of $\mathfrak{so}(V)$ by $\mathfrak{g}(k) = \{A \in \mathfrak{so}(V) | A \cdot R_p = A \cdot (DR)_p = = \ldots = A \cdot (D^k R)_p = 0\}$. Here $(D^s R)_p$, $s = 0, 1, \ldots, k$, is the value at p of the s-th covariant differential of the Riemann curvature tensor R, and the endomorphism A acts on the tensor algebra of V as a derivation. By definition, $D^0 R = R$. The *Singer invariant* of the homogeneous metric g is the integer k_g defined by $k_g = \min\{k \in N | \mathfrak{g}(k) = = \mathfrak{g}(k+1)\}$.

LEMMA 3. *Let (M, g), $(\overline{M}, \overline{g})$ be homogeneous Riemannian manifolds. Assume that*:

(1) *(M, g) and $(\overline{M}, \overline{g})$ have the same Singer invariant $k_g = k_{\overline{g}} (= k)$;*

(2) *there exist $p \in M$, $\overline{p} \in \overline{M}$, and there exists a linear isometry F: $T_p M \to T_{\overline{p}} \overline{M}$ such that $F^* (\overline{D}^s \overline{R})_{\overline{p}} = (D^s R)_p$, for all $0 \leqslant s \leqslant k+1$. Then (M, g) is locally isometric to $(\overline{M}, \overline{g})$.*

(For the proof of this Lemma, see [5, Th. 2.5]).

By hypothesis, (G, g) and $(\overline{G}, \overline{g})$ have the same curvature, *i.e.* there exists a linear isometry F: $T_e G \to T_{\overline{e}} \overline{G}$ such that $F^* (\overline{R})_{\overline{e}} = R_e$, where R and $\overline{R}$ are the Riemann tensors of g and $\overline{g}$ respectively. To prove the local equivalence of g and $\overline{g}$, it is sufficient, by the Lemma above, to prove that $F^* (\overline{DR})_{\overline{e}} = (DR)_e$, because $k_g = k_{\overline{g}} = 0$ (see [2, 3]). Choose orthonormal bases $\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$ of $T_{\overline{e}} \overline{G} \simeq \overline{\mathfrak{g}}$, and e_1, e_2, e_3 of $T_e G \simeq \mathfrak{g}$ as in Part (A). The hypotheses $r_{11} = r_{22} < 0$, $r_{33} > 0$ are equivalent, for $\overline{g}$ and g respectively, to the following conditions: $\alpha \neq 0$, $\gamma = \delta = 0$, $\beta \neq 0$ and $\mu_1 = \mu_2 =: \mu$, $\mu\mu_3 < 0$.

We may always assume $\beta > 0$, $\mu > 0$ and $\mu_3 < 0$.

The Ricci principal curvatures of g and $\overline{g}$ are then given respectively by $(2\mu\mu_3, 2\mu\mu_3, 2\mu^2)$ and $(-\alpha^2 - \beta^2/2, -\alpha^2 - \beta^2/2, \beta^2/2)$.

Since g and $\overline{g}$ have the same curvature, we have $\mu = \beta/2$ and $\mu_3 = -(\alpha^2 + + \beta^2/2)\beta^{-1}$.

An explicit calculation of the covariant differentials Dr and $\overline{D}\overline{r}$ gives

$$Dr = \mu(r_{11} - r_{33})(\theta^1 \otimes \theta^2 \otimes \theta^3 + \theta^1 \otimes \theta^3 \otimes \theta^2 - \theta^2 \otimes \theta^1 \otimes \theta^3 - \theta^2 \otimes \theta^3 \otimes \theta^1);$$

$$\overline{D}\overline{r} = \beta(r_{11} - r_{33})(\overline{\theta}^1 \otimes \overline{\theta}^2 \otimes \overline{\theta}^3 + \overline{\theta}^1 \otimes \overline{\theta}^3 \otimes \overline{\theta}^2 - \overline{\theta}^2 \otimes \overline{\theta}^1 \otimes \overline{\theta}^3 - \overline{\theta}^2 \otimes \overline{\theta}^3 \otimes \overline{\theta}^1)/2.$$

Since $\mu = \beta/2$, we have $F^* (\overline{D}\overline{r})_{\overline{e}} = (Dr)_e$.

By Lemma 3, it is now sufficient to prove that $F^* (\overline{D}\overline{r})_{\overline{e}} = (Dr)_e$ implies $F^* (\overline{D}\overline{R})_{\overline{e}} = = (DR)_e$.

Now it is well known that $\overline{R} = (-\overline{s}\,\overline{g}/4 + \overline{r}) \barwedge \overline{g}$, where $\overline{s}$ is the scalar curvature of $\overline{g}$ and «$\barwedge$» is the Kulkarni-Nomizu product (see *e.g.* [1, p. 49]). Then, for each $\overline{X} \in T_{\overline{e}} \overline{G}$, $(\overline{D}_{\overline{X}} \overline{R})_{\overline{e}} = (\overline{D}_{\overline{X}} \overline{r})_{\overline{e}} \barwedge \overline{g}_{\overline{e}}$.

Set $X = F^{-1} \overline{X}$. Then $F^* (\overline{D}_{\overline{X}} \overline{R})_{\overline{e}} = F^* (\overline{D}_{\overline{X}} \overline{r})_{\overline{e}} \barwedge F^* \overline{g}_{\overline{e}} = (D_X r)_e \barwedge g_e = (D_X R)_e$.

Therefore $F^*(\overline{D}_{\overline{X}}\overline{R})_{\bar{e}} = (D_X R)_e$, which implies $F^*(\overline{D}\,\overline{R})_{\bar{e}} = (DR)_e$. This ends the proof. ■

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