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Line bundles with $c_1(L)^2 = 0$. Higher order obstruction

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Topologia. — *Line bundles with $c_1(L)^2 = 0$. Higher order obstructions.* Nota di STEFANO DE MICHELIS, presentata (*) dal Corrisp. C. PROCESI.

ABSTRACT. — We study secondary obstructions to representing a line bundle as the pull-back of a line bundle on S^2 and we interpret them geometrically.

KEY WORDS: Obstruction; Homotopy groups; Postnikov system.

RIASSUNTO. — *Fibrati in rette con $c_1(L)^2 = 0$. Ostruzioni di ordine superiore.* Si studiano le ostruzioni secondarie al rappresentare fibrati di linea come preimmagini di una funzione su S^2. E se ne dà una interpretazione geometrica.

In a previous paper [1] we studied the obstructions for a line bundle L with $c_1(L)^2 = 0$ to be the pull-back of some power of the Hopf bundle on S^2. We proved that they all vanish over the rationals and we exhibited some examples in which they are non trivial over the integers. We studied in more detail the first non obvious obstruction in H^5 and we remarked the curious fact that, while it always vanishes for 5-manifolds, there are 5-dimensional Poincaré complexes on which it is non-zero. To see whether such a phenomenon appears also in higher dimensions one has to study the higher dimensional obstructions and their indeterminacies. This is what will be done in this paper.

THE OBSTRUCTION ν

Let M be the manifold (or CW complex) we want to study, L the line bundle on it and $\varphi: M \to CP^\infty$ its classifying map. As in [1] the obstructions to factorizing the map through a $\psi: M \to S^2 \hookrightarrow CP^\infty$ come from the Postnikov invariants of the tower

$$\begin{array}{c} K(Z/2;4) \to X_4 \to K(Z/2;6) \\ \downarrow \\ K(Z;3) \to X_3 \to X_3 \to K(Z/2;5) \\ \downarrow \\ M \to CP^\infty = K(Z;2) = X_2 \to K(Z;4). \end{array}$$

We already proved that the first obstruction, apart from $c_1(L)^2 = 0$ is in $H^5(M;Z/2)$ and that its indeterminacy is

$$Im\, Sq^2 \circ \pi \qquad H^3(M;Z) \xrightarrow{\pi} H^3(M;Z/2) \xrightarrow{Sq^2} H^5(M;Z/2).$$

(*) Nella seduta del 15 dicembre 1990.

The problem of finding the next obstruction can be dealt with in essentially the same way: that is, studying the lifting problem for $X_4 \to X_3$.

There are, however, some additional technical difficulties due to the fact that the map $M \to X_2$, when it exists, is not uniquely defined. The indeterminacy lies in $H^3(M; Z)$. To avoid unnecessary complications we will assume $H^3(M; Z) = 0$. In this case there is a unique map to X_2 and, if the lifting obstruction in H^5 vanishes, we can define the next one in $H^6(M; Z/2)$. Its indeterminacy comes from the composition

$$K(Z/2; 4) \to X_4 \to K(Z/2; 6).$$

In order to compute this map we need some information on the cohomology of X_4 and X_3. Let us start with X_3; it is a $K(Z; 3)$ fibration over $K(Z; 2)$.

The cohomology with coefficients in $Z/2$ of the fibre is given by Serre's theorem and is generated up to degree seven by:

$$\begin{array}{ll} i_3 & \text{in } H^3, \\ Sq^2 i_3 & \text{in } H^5, \\ i_3 \cup i_3 = Sq^3 i_3 & \text{in } H^6. \end{array}$$

Both H^4 and H^7 are trivial; for a proof see [1].

The E_2 term of the spectral sequence, always with $Z/2$ coefficients is:

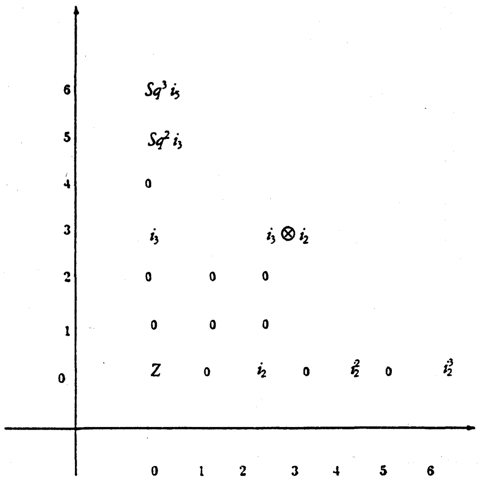

Fig. 1.

The interesting differentials are:

$$d_4(i_3 \otimes u^l) = u^{l+2}; \; d_6(Sq^2 i_3) = Sq^2(d_4 i_3) = Sq^2 u^2 = 0; \; d_7(i_3^2) = d_7(Sq^3 u^2) = 0.$$

See [1] for further details.

It follows that up to dimension 7 the cohomology of X_3 is generated by: $u \in H^2$; $Sq^2 i_3 \in H^5$; $i_3 \vee i_3 \in H^6$; $Sq^2 i_3 \cup u \in H^7$, where we denote with $Sq^2 i_3$ a cohomology class on X_3 which restricts to $Sq^2 i_3$ on $K(Z; 3)$.

Now we apply the same trick to compute $H^*(X_4; Z/2)$. The E^2 term of the spectral sequence for the fibration

$$\begin{array}{c} K(Z/2;4) \to X_4 \\ \downarrow \\ X_3 \end{array}$$

is:

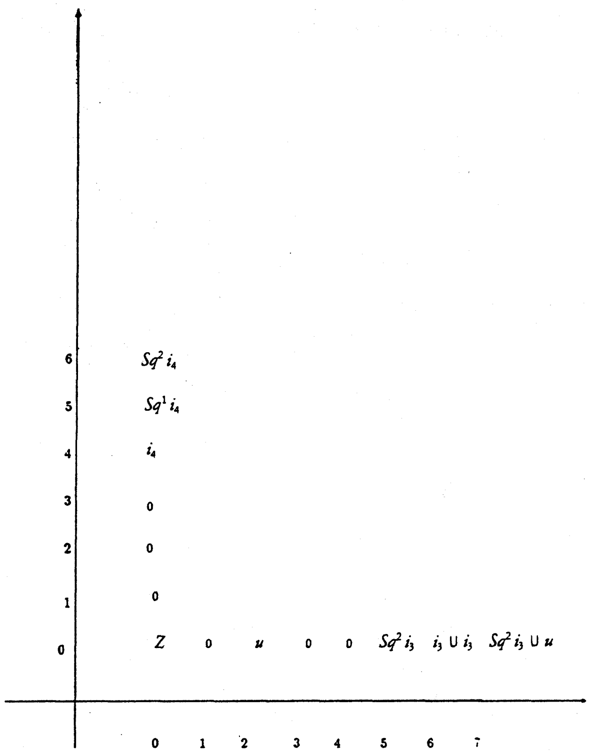

Fig. 2.

As usual, the cohomology of the fibre $K(Z/2; 4)$ follows from Serre's theorem. The differentials are as follows: $d_5(i_4) = Sq^2 i_3$, because $Sq^2 i_3$ gives the classifying map $X_3 \to K(Z/2; 5)$.

$$d_6(S_q^1 i_4) = Sq^1 (d_5 i_4) = Sq^1 [Sq^2 i_3] = Sq^3 i_3 = i_3 \cup i_3,$$

where we have used the transgression theorem of [2] and the Adem relation $Sq^1 Sq^2 = Sq^3$,

$$d_6(Sq^2 i_4) = Sq^2 [Sq^2 i_3] = 0,$$

$$d_5(i_4 \cup u) = d_5(i_4) \cup u = Sq^2 i_3 \cup u.$$

It follows that the cohomology mod 2 of X_4 is generated by u in H^5 and $Sq^2(i_4)$ in H^6.

This gives us the result we needed, *i.e.*

PROPOSITION 1. *The composition* $K(Z/2;4) \to X_4 \to K(Z/2;6)$ *corresponds to the class* $Sq^2 i_4 \in H^6(K(Z/2;4);Z/2)$.

As in [1], we deduce immediately the result:

THEOREM. *Let M be a CW complex with a line bundle L on it such that:*

a) $c_1(L)^2 = 0$ *and L on the five-dimensional skeleton is induced by a map onto* S^2;

b) $H^3(M;Z) = 0$.

Then the first obstruction to L being induced by a map onto S^2 *lies in* $H^6(M;Z/2)/Sq^2 H^4(M;Z/2)$.

GEOMETRIC INTERPRETATION OF THE OBSTRUCTION

Now we interpret geometrically the obstruction defined above, the first step is to find a six-skelton of X_4. We know that the inclusion $S^2 \to X_4$ is five connected, hence a six-skelton will be given by $S^2 \cup [\cup e_i^6]$, a look at the cohomology of X_4 tells us that the simplest choice would be $(X_4)^6 = S^2 \bigcup_f e^6$, where f is the nontrivial element in $\pi_5(S^2)$.

Now let us assume that M is a manifold of dimension 6 satisfying the hypotheses of Theorem 1. We have a map $M \to X_f$ which can be pushed to a map $M \xrightarrow{\psi} (X_4)^6$. Moreover, the diagram

$$\begin{array}{ccc} & & (X_4)^6 \\ & & \downarrow \\ M & \xrightarrow{\varphi} & CP^\infty \end{array}$$

commutes up to homotopy.

The obstruction ν is given by the pull-back $\psi^*(h_6)$, where h_6 is the restriction of $Sq^2 i_4$ to $(X_4)^6$. The fact that ψ is not unique reflects the indeterminacy of ν in H^6. Now we give explicitly the map π: $(X_4)^6 \to CP^\infty$. First, by the general position, we can assume $\pi(X_4)^6 \subset CP^3$. Moreover, it is easy to check that $\pi^*(u^3) = 0$ so that π can be represented by a map from $(X_4)^6$ to CP^2. This map must induce an isomorphism on H^2, so we can assume that $\pi|S^2$ is the inclusion $S^2 = CP^1 \hookrightarrow CP^2$.

The extensions of π to a map of all $(X_4)^6$ are in $1-1$ correspondence to elements $\mathcal{O}_6$ of $\pi_6(CP^2;S^2)$ such that in the long exact sequence

$$\pi_6(S^2) \to \pi_6(CP^2) \to \pi_6(CP^2;S^2) \xrightarrow{\partial} \pi_5(S^2) \to \pi_5(CP^2).$$

$\mathcal{O}_6$ gives the generator for $\pi_5(S^2)$. The existence of such a map follows from the following Lemma:

LEMMA 1. *The map* $\pi_5(S^2) \to \pi_5(CP^2)$ *is zero.*

PROOF. The inclusion $S^2 \hookrightarrow CP^2$ is covered by the inclusion $S^3 \hookrightarrow S^5$ in the universal S^1 bundle over CP^2. So we have the commutative diagram:

$$\begin{array}{ccc} \pi_5(S^3) & \to & \pi_5(S^5) \\ \| & & \| \\ \pi_5(S^2) & \to & \pi_5(CP^2). \end{array}$$

The map on the top is obviously zero, and so is the lower map.

The extension is not unique, since $Z/2 \simeq \pi_6(CP^6)$ injects into $\pi_6(CP^2; S^2)$. So, finally, we have the result: $\pi(X_4)^6 \to CP^\infty$ can be represented by the maps $(X_4)^6 \to CP^4$. We give an explicit description of them, which will be needed in the following.

Let f be a generator for $\pi_5(S^2)$ and lift it to a map $\tilde{f}$: $S^5 \to S^3$. We have the commutative diagram:

$$\begin{array}{ccc} S^5 & \xrightarrow{\tilde{f}} & S^3 \\ \downarrow f & & \downarrow p \\ S^2 & = & S^2, \end{array}$$

where p is the Hopf bundle projection. This induces a map π between the mapping cones of f and p which are, respectively, $(X_4)^6$ and CP^2. The «other» map π' : $(X_4)^6 \to CP^2$ is given by the composition: $(X_4)^6 \to (X_4)^6 \vee S^6 \xrightarrow{(\pi \vee g)} CP^2$, where g is the generator of $\pi_6(CP^2)$.

Now assume π is smooth on $e^6 \subset (X_4)^6$, we want to see what is the preimage of a generic point p on CP^2. This is the same as the generic preimage of a point in S^4 with respect to the map $\sum \tilde{f}$ in the diagram:

$$\begin{array}{ccccccc} S^5 & \longrightarrow & S^2 & \longrightarrow & (X_4)^6 & \longrightarrow & S^6 \\ \downarrow \tilde{f} & & \| & & \downarrow \tilde{f} & & \downarrow \Sigma\tilde{f} \\ S^3 & \xrightarrow{p} & S^2 & \longrightarrow & CP^2 & \longrightarrow & S^4 \end{array}$$

From the description of the homotopy groups of spheres in terms of framed cobordism we see that the latter preimage is a 2-dimensional framed manifold and that the framing gives a spin structure not cobordant to zero. An example is given by the torus T^2 with the Lie group framing. If we take the other map π: $(X_4)^6 \to CP^2$, the preimage in the framed cobordism group will differ from the former by the representative for the non trivial map $S^6 \to CP^2$, this is the same as the preimage of $S^6 \to CP^2 \to$ $\to CP^2/S^2 = S^4$. But this map is trivial in homotopy. To see it note that it factorizes through $S^5 \xrightarrow{p} CP^2 \to S^4$ with p the Hopf bundle, and that p is trivial, as proved in the appendix.

We have now all the ingredients to define the obstruction geometrically: let

φ: $M^6 \to CP^\infty$ be the classifying map for L, use the fact that $c_1(L)^3 = c_1(L)^2 = 0$ to push it to a map φ: $M^6 \to CP^2$. Consider now the framed preimage of a generic point, it will give an element of $\Omega_k^2(pt) \simeq Z/2$, if M^6 satisfies the hypotheses of Theorem 1, this will be a representative for the obstruction in $H^6(M; Z/2)/Sq^2 H^4 \simeq Z/2/Sq^2 H^4(M; Z/2)$. As in [1] we could also express it in terms of zero set of sections into L.

Appendix

We prove the following Lemma:

LEMMA. *Let p: $S^5 \to CP^2$ be the Hopf fibration, and let n be the composition:*

$$n\colon S^5 \xrightarrow{p} CP^2 \to S^4 = CP^2/CP^1 .$$

Then n is trivial in homotopy.

PROOF. Considering the diagram:

$$\begin{array}{ccccc} S^5 & \xrightarrow{p} & CP^2 & \xrightarrow{C_p} & CP^6 \\ \| & & \downarrow & & \downarrow h \\ S^5 & \xrightarrow{n} & S^4 & \xrightarrow{C_n} & S^4 \cup_n e^6. \end{array}$$

Let i_6 be a generator for $H_6(CP^6; Z/2)$, so that $h_* i_6$ generates $H^6\left(S^4 \cup_n e^6; Z/2\right)$ and let v be a generator for $M^4\left(S^4 \cup_n e^6\right)$. To show that n is trivial it is enough to show that $Sq^2 v$ is zero. But we have: $\langle Sq^2 v; h_* i_6 \rangle = \langle h^* Sq^2 v; i_6 \rangle = \langle Sq^2 h^* v; i_6 \rangle$. Now $h^* v = u^2 = Sq^2 u$ with u generating $H^2(CP^6; Z/2)$. So $Sq^2 v = Sq^2 Sq^2 u = Sq^3 Sq^1 u = 0$ because $H^3(CP^6) = 0$. This ends the proof.

References

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