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Stefano De Michelis

Line bundles with $c_1(L)^2 = 0$. A six dimensional example

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Topologia. — *Line bundles with* $c_1(L)^2 = 0$. *A six dimensional example.* Nota di
Stefano De Michelis, presentata (*) dal Corrisp. C. Procesi.

Abstract. — We exhibit a six dimensional manifold with a line bundle on it which is not the pull-back of a bundle on S^2.

Key words: Surgery; Manifold; Line bundles.

Riassunto. — *Fibrati in rette con* $c_1(L)^2 = 0$. *Un esempio di dimensione* 6. Si costruisce una varietà di dimensione 6 con fibrato di linea che non è la preimmagine di una mappa su S^2.

In a previous paper [2] we considered the following question: Let L be a line bundle on an n-dimensional oriented manifold M and let $c_1(L)^2 = 0$. When is L the pull-back of a line bundle over S^2?

We proved that the answer is always «yes» if $n \leqslant 5$ (the case of $n \leqslant 4$ is trivial) while there are examples with non vanishing obstructions for $n \geqslant 7$. The case of $n = 6$ was left open. In this paper we shall prove that there exists a 6-dimensional manifold and a line bundle over it with nonvanishing obstruction. The construction is somewhat indirect: we first construct a Poincaré type and then we prove that it can be realized as a smooth manifold using obstruction theory and the surgery techniques of Browder-Novikov. It would be interesting to exhibit it explicitly.

The Poincaré duality space

Consider the wedge $S^2 \vee S^4$. Its homotopy groups are given by the Hilton-Milnor theorem and in particular we have $\pi_5(S^2 \vee S^4) = Z \oplus Z/2 \oplus Z/2$. The generators are respectively $[i_2 ; i_4]; \alpha; \eta$ with i_2, i_4 the identities, α the nontrivial element of $\pi_5(S^2)$ and η the double suspension of the Hopf map, generating $\pi_5(S^4)$.

The six-dimensional complex will be obtained by attaching a 6-cell e^6 to the wedge according to some map f: $S^5 \to S^2 \times S^4$. The homotopy class of the complex will depend on the class of $f \in \pi_5(S^2 \vee S^4)$. Observe that if we want the complex to satisfy Poincaré duality we have to choose the component of f along the Whitehead product to be one. There are four possibilities, depending on the $Z/2 \oplus Z/2$ factor. Let f be $[i_2 ; i_4] + \varepsilon\alpha + \delta\eta$, $\varepsilon, \delta = 0, 1$, we will analyze the possible candidates:

If $\delta = 1$, $Sq^2 : H^4(M; Z/2) \to H^6(M; Z/2)$ will be surjective on the Poincaré complex. We have proved in [1] that the six-dimensional obstruction is determinate only up to the image of $Sq^2 : H^4(M; Z/2) \to H^6(M; Z/2)$, so to find an obstruction we need $s = 0$. If $\varepsilon = 0$, too, we have the homotopy type of $S^2 \times S^4$, which is not interesting. We are left with $\varepsilon = 1$, $\delta = 0$.

We call the corresponding space X.

(*) Nella seduta del 15 dicembre 1990.

THEOREM 1. *There is a line bundle L over X such that $c_1(L)^2 = 0$ but L is not the pull-back of any line bundle on S^2.*

PROOF. Let L be the line bundle with $c_1(L)$ dual to the homology class of S^2 in X. The couple $(M; L)$ satisfy the hypotheses of Theorem 1 of [1]; so we have to prove that the obstruction defined there does not vanish.

Consider the retract r: $X \to S^2 \bigcup_\alpha e^6$. There is a line bundle L' on $S^2 \bigcup_\alpha e^6$ such that $r^*(L')$ is L. Moreover the obstruction ν of L will be the pullback of the obstruction associated to L'; since the latter does not vanish (almost by construction, see [1]) and r^* is an isomorphism on H^6, the result follows.

CONSTRUCTION OF A MANIFOLD WITH THE HOMOTOPY TYPE OF X

In this section we prove:

THEOREM 2. *The space X of Theorem 1 can be assumed to be an orientable, simply-connected piecewise linear manifold..*

In order to make X a manifold we will associate to it a surgery problem and we will prove that the latter is solvable; a general reference for the method is [2].

We recall that given a homotopy type we can associate to it its «Spivak normal bundle», which has structural group G: the inductive limit for N going to infinity of the homotopy equivalences of S^N. If X has to be a manifold a first requirement is that this bundle has a reduction to a PL-bundle. The latter would play the role of the stable normal bundle. Such a reduction exists if and only if there exists a section of the associated G/PL-bundle. We refer to [1] for details. The obstructions to constructing such a section lie in $H^{n+1}(X; \pi_n(G/PL))$. Recall that the homotopy groups of G/PL are known from [3].

$$\pi_i(G/PL) = \begin{cases} 0 & \text{if } i \text{ odd,} \\ Z/2 & \text{if } i = 4k+2, \\ Z & \text{if } i = 4k. \end{cases}$$

It follows that in our case all the obstruction groups vanish. The next step is to consider the surgery exact sequence in dimension 6, see [4].

$$(*) \qquad \ldots \to L_7(1) \xrightarrow{\partial} S_{PL}(X) \xrightarrow{\eta} [X; G/PL] \xrightarrow{\vartheta} L_6(1) \to .$$

X will have at least a manifold structure if and only if the surgery obstruction map in $(*)$ is zero. As observed in [1] we can always use the plumbing construction to make the obstruction vanish in the PL category. Notice that we may change the reduction from G to PL.

This ends the proof of Theorem 2.

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Department of Mathematics - Harvard University
1, Oxford Street - 02138 Cambridge, Mass. (U.S.A.)