Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Stefano De Michelis

On manifolds diffeomorphic on the complement of a point

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **2** (1991), n.3, p. 229–233.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1991_9_2_3_229_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1991.

Rend. Mat. Acc. Lincei
s. 9, v. 2:229-233 (1991)

Topologia. — *On manifolds diffeomorphic on the complement of a point.* Nota di
Stefano De Michelis, presentata (*) dal Corrisp. C. Procesi.

Abstract. — We prove that four manifolds diffeomorphic on the complement of a point have the same Donaldson invariants.

Key words: Manifolds; Instantons; Donaldson polynomials.

Riassunto. — *Sulle varietà diffeomorfe sul complemento di un punto.* Si dimostra che varietà di dimensione 4 diffeomorfe sul complemento di un punto hanno gli stessi polinomi di Donaldson.

In this *Note* we consider the following question:

Question. *Let M and M' be two smooth manifolds and φ: $M - \{p\} \to M' - \{p'\}$ a diffeomorphism on the complement of a point. When are M and M' diffeomorphic?*

It is trivial to see that they are homeomorphic and so the question is interesting only in dimension four or greater. The answer in dimension five, or more, follows from classical differential topology and is not necessarily always yes as the following example shows:

Example. Let M be the non-standard S^7 described in [4] and M' the euclidean S^7. The complements of a point in both are homeomorphic, and hence diffeomorphic, to R^7. So M and M' satisfy the hypothesis of Q but are not diffeomorphic. Actually, one can prove that this is essentially the only possible obstruction to an affirmative answer to Q.

Proposition. Let M and M' satisfy Q. Then M' is diffeomorphic to $M \# \Sigma$, with Σ an homotopy sphere.

Proof. First consider the case of $\dim \geq 5$. Choose a coordinate neighborhood around p and p', U and U' such that the image under φ of a standard embedded sphere S^{n-1} in U is contained in U'.

The component of $M' - \varphi(S^{n-1})$ containing k' is homeomorphic to D^n and hence diffeomorphic to it. See [4]. Hence M' is obtained from M by cutting out a disk and gluing it back with a diffeomorphism. So the «difference» between M and M' lies in $\{\mathrm{Diff}\, S^{n-1}\} / \{\mathrm{Diff}\, B^n \,|S^{n-1}\}$. This is exactly the group of «homotopy spheres» studied in [3]. In particular, note that this group is finite in $\dim \geq 5$.

(*) Nella seduta del 15 dicembre 1990.

In dimension 4 the proof fails because it is not known if an S^3 in R^4 bounds a smooth ball. This is the generalized Schoenfliess' conjecture.

We can, however, prove that the proposition still holds. In this case the Σ is obtained by attaching a B^4 to the possibly «fake» B_Γ^4 containing P'.

Note, however, that there are substantial differences between dimension 4 and the others.

In dim $\geq 5, \varphi$ can be made isotopic to a diffeomorphism commuting with radial dilations around P and P'. To see this, consider the h-cobordism between $\varphi(S^{n-1})$ and some standard S^{n-1} of small enough radius. Observe that it has a smooth product structure, even if $n=5$ by [4], and use this to isotope φ to a compatible diffeomorphism. If this were possible in dim 4, Cerf's theorem would imply that the answer to Q would be yes.

Using the technique of «furling» we can, at least, obtain a partial result, which will be of substantial help in the following section. Blow up both M and M' so that the ends become diffeomorphic to $S^3 \times R$ Then:

Proposition. The diffeomorphism φ can be arranged so that it commutes with translation by one on the end.

Proof. It follows from the following fact: any diffeomorphism of $Y \times R$ into itself commutes with translations by one. For the proof see [6].

The Proposition is useful because it allows us to assume that φ is a diffeomorphism of «periodic ends», and so to apply the «technology» of [5].

Analytical details

We first fix some notation.

Let t and t' be the R coordinate of the end of M and M' respectively. Note that if φ is end periodic $|t - t' \circ \varphi| \leq 1$. We also fix $SU(2)$ bundles E and E' such that $\hat{\varphi}^*(E') = E$ where $\hat{\varphi}$ is the continuous extension of φ to all of M.

Given a metric g on M, $\tilde{g}$ will be the conformal blow up of it which gives the almost cylindrical metric on $M - \{p\}$. We will denote by $\mathcal{M}_g$ the moduli space of anti self dual connections on (M, R) with respect to the metric g. We will denote by K the subset of $\overline{\mathcal{M}}_g$ (the compactification of $\mathcal{M}_g$) consisting of «ideal» connection with at least a unit of charge in p.

A similar notation will hold for M'.

B_0 will be the quotient of the space of smooth connections on $M - \{p\}$ such that $1/2 \int |F_A|^2 \, d\mu = p_1(E)$ divided by the group of gauge equivalences which are «asymptotically constant» (in the sense of [5]) on the end.

The spaces $\mathcal{M}_g$ and $\mathcal{M}_{g'}$ map into a one-to-one fashion into B_0, and it is not hard to see that B_0 is homotopy equivalent to $B_M = \{\text{connections of } M\}/\{\text{gauge equivalences}\}$ or $B_{M'}$ and that the inclusions of $\mathcal{M}_g$ or $\mathcal{M}_{g'}$ are compatible with this equivalence. In particular one can compute Donaldson's polynomials for M and M' using B_0.

PROPOSITION. Let M and M' be four manifolds and let φ be a diffeomorphism on the complement of a point. There exist a metric g on M and metrics g_n on M' such that:

a) g_n converges on compact subsets of $M' - \{p\}$ to $(\varphi^{-1})^* g$;

b) $\mathcal{M}_{g_n}$ in B_0 converges uniformly on compact subsets of $\overline{\mathcal{M}}_{g_n} - K'$ to $\mathcal{M}_g - K$.

In point *b*) by «converges» we mean that, on compact subsets, $\mathcal{M}_{g_n}$ for n large is a small C^1 perturbation of $\mathcal{M}_g$. In particular not only the points are close but also their tangent spaces.

PROOF. On M' we define g_n as follows. Take $\varphi^*(g)$ on $K(n)$. Connect to the flat metric on the end in $M - K(n+l)$ with a partition of the unity and then perturb it so that the corresponding moduli space is smooth.

We define maps from $\overline{\mathcal{M}}_{g_n} - K'$ into $\overline{\mathcal{M}}_g - K$ and vice versa. Let A' be a connection in $\mathcal{M}_{g_n}$ and pull it back to $\varphi^*(A') = A$: a connection on $M - \{p\}$, via φ. A satisfies

a) $F^- A = 0$ on $K(n)$;

b) $|F_A^-|_{g_n} \leq |F_A|_{g_n}$ on M with $\int\limits_0^\infty e^{2t} |F_A|^2 \, d\mu_g < +\infty$.

a) is true because $\varphi^* g_n = g$ on $K(n)$. To see that *b*) holds remember that $|d\varphi|_{\tilde{g}}$ is uniformly bounded on the end, due to the end periodicity and that $|t - t'| < 1$. It follows that $\int e^{2t} |F_A|^2 \, d\mu_{g_n}$ and $\int e^{2t'} |F_{A'}|^2 \, d\mu_g$ differ by a constant. The convergence of the last integral can be proved in several ways. One would be to follow the proof of appendix $L^2 \to L^2_\delta$ of [1] in the case of the cylinder and write the δ involved as the eigenvalue of curl^{-1} on S^3. The other is to exploit the conformal invariance of L^2 norms on two forms and compute everything on the ball, using the fact that F_A is bounded. The last method gives us a more precise result.

PROPOSITION. Given a compact subset C of $\overline{\mathcal{M}} - K$ and ε, there exists an n such that $\int\limits_{M - K(n)} e^{2\tau} |F_A|^2 \, d\mu_{g_n} < \varepsilon$.

PROOF. For such a compact set there exists an n' such that $|F_A|$ is uniformly bounded on $M - K(n')$ with respect to g. Take $n = n' + l$. Note that the integral in the Proposition goes to zero for $l \to +\infty$.

We now apply the iteration procedure of [5].

Let $D_A : \Omega(\mathrm{ad}A) \to \Omega^2_+(\mathrm{ad}A)$ and $D_A^* : \Omega^2_+(\mathrm{ad}A) \to \Omega^1(\mathrm{ad}A)$. It is adjoint in the L^2_δ inner product. We are looking for a $u \in \Omega^2_+(\mathrm{ad}A)$ such that:

$$T_A(u) = D_A D_A^* u + (D_A^* u \wedge D_A^* u)^+ = -F_A . \tag{1}$$

We will solve the equation with the continuity method, solving for the right hand side equal to $-tF_A$.

Observe that, by the genericity condition $|D_A D_A^* u|_{L^2_{\delta=2}} > \lambda(A)|u|_{L^2_{\delta=2}}$ with $\lambda(A)$ uniformly bounded on C. Let $0 < \lambda = \inf_C \lambda(A)$.

We will use the following weighted Sobolev inequality for the proof. See [5], Ch.s 5 and 7.

LEMMA. For A as above and $u \in L^2_{2,\delta=2}$ we have

$$|D^* u|_{L^4_{\delta=4}} \leq C|u|_{L^2_{2,\delta=2}}.$$

This implies that there exist an ε such that if $|u|_{L^2_{2,\delta=2}} < \varepsilon$, the map $T_A(u)$: $L^2_{2,\delta=2} \to L^2_{2,\delta=2}$ has an invertible derivative at u. The derivative is

$$h \to D_A D_A^* h + 2(D^* u \wedge D^* h)^+ .$$

We know the first term is invertible. We estimate the second as:

$$|(D^* u \wedge D^* h)^+|_{L^2_{0,\delta=2}} \leq |D^* u|_{L^4_{0,\delta=4}} |D^* h|_{L^4_{0,\delta=4}} \leq |u|_{L^2_{2,\delta=2}} \leq |u|_{L^2_{2,\delta=2}} |h|_{L^2_{2,\delta=2}} .$$

The claim follows for u small. We can now prove:

PROPOSITION. There exist ε' such that, if $\int e^{2\tau\delta}|F_A^+|^2 < \varepsilon'$, then (1) has a solution for $|u|_{L^2_{2,\delta=2}} < \delta'$.

PROOF. Embed the problem in a continuous set of problems u_A with the constraint:

$$|u_t|_{L^2_{2,\delta=2}} \leq \delta' .$$

First we prove that the set of t with a solution is closed. Let u_{t_n} be a sequence of solutions. We have $u_{t_n} \to u_t$ weakly in $L^2_{2,2\delta=2}$ and $D^* u_{t_n} \to D^* u_t$ strongly in $L^{4-\varepsilon}_{0,\delta=4}$. The multiplication theorem for Sobolev spaces implies that u_t is a solution of 1_t.

It is also open, in fact, by the implicit function theorem. Given a solution u_t we have a solution $u_{t+\alpha}$ with $|u_{t+\alpha}| \leq 2\delta'$. But now:

$$|u_{t+\alpha}|_{L^2_{2,\delta}} < \lambda^{-1}|D_A D_A^* u_t|_{L^2_{0,\delta}} \leq \lambda^{-1}(|D^* u_t \wedge D^* u_t|_{L^2_{0,\delta}} + t|F_A^-|_{L^2_{0,\delta}}) \leq \lambda^{-1}(4c^2\delta^{12} + \varepsilon).$$

If δ is small enough the result follows.

This gives the map f from $\mathcal{M}_{g_n}$ to $\mathcal{M}_g$. A similar argument gives a map from $\mathcal{M}_g$ to $\mathcal{M}_{g_n}$. Note that the bounds on $|u|$ imply that f and g displace points by a small amount. This gives «C^0» convergence of $\mathcal{M}_{g_n}$ to $\mathcal{M}_g$. To get «C^1» convergence note that the operator $(D_A^*; D_A)\,\Omega^1 \to \Omega^0 \oplus \Omega^2_+$ can be used to define a continuous bundle on B_0 which restricts to the tangent bundle on $\mathcal{M}_{g_n}$ and $\mathcal{M}_g$. Details follow from the proof of point 3 in [1].

The Corollary is the main result of this *Note*.

COROLLARY. M and M' have the same Donaldson polynomials, whenever these are defined.

REMARK. This result should not be surprising in the light of [2]. The recent results of Donaldson and Sullivan which extend gauge theory to the quasi-conformal category. In fact it is not hard to prove that the homeomorphism between M and M' can be made quasi-conformal.

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Department of Mathematics - Harvard University
1, Oxford Street - 02138 Cambridge, Masss. (U.S.A.)