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Internal constraints and linear constitutive relations for transversely isotropic materials

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Meccanica dei continui. — *Internal constraints and linear constitutive relations for transversely isotropic materials.* Nota di Paolo Podio-Guidugli e Maurizio Vianello, presentata (*) dal Corrisp. C. Cercignani.

Abstract. — All internal constraints compatible with transverse isotropy are determined and representation formulae are given for the constitutive relations of arbitrarily constrained, transversely isotropic materials.

Key words: Linear elasticity; Internal constraints; Material symmetries.

Riassunto. — *Vincoli interni ed equazioni costitutive lineari per materiali trasversalmente isotropi.* Vengono determinati tutti i vincoli interni compatibili con l'isotropia trasversa. Si danno altresì formule di rappresentazione per le equazioni costitutive di materiali trasversalmente isotropi arbitrariamente vincolati.

1. Introduction

The constitutive relation for a linearly elastic material is prescribed through an *elasticity tensor* $\mathbb{C}$ which gives for each value of the (infinitesimal) strain $\boldsymbol{E}$ the corresponding stress $\boldsymbol{T} = \mathbb{C}[\boldsymbol{E}]$ both $\boldsymbol{E}$ and $\boldsymbol{T}$ belonging to the space Sym of symmetric tensors. Hyperelastic materials obtain when $\mathbb{C}$ is a self-adjoint transformation of Sym, but we do not confine ourselves to that important special case in this paper. A relevant algebraic structure connected with $\mathbb{C}$ is the *material symmetry group*, defined as the collection of all rotations $\boldsymbol{Q}$ such that $\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}^T$ yields the stress $\boldsymbol{Q}\mathbb{C}[\boldsymbol{E}]\,\boldsymbol{Q}^T$ for each $\boldsymbol{E}$ (cf. Gurtin [1; §21]). When *each* rotation belongs to such a group, $\mathbb{C}$ is isotropic. Cases of lesser symmetry have a well-established classification [1; §21 and §27], [2]; in particular, when the symmetry group includes *all* rotations about a fixed axis the material is *transversely isotropic.*

The Representation Problem consists in the explicit determination of all elasticity tensors sharing a given symmetry group. As is well known, there is a classical and simple representation formula for isotropy [1; §22], while the more challenging problem for transverse isotropy has been only recently solved by Walpole [3] and, with a different and simpler approach, by Podio-Guidugli and Virga [4].

An *internal constraint* (such as, for instance, incompressibility or inextensibility in a given direction) can be viewed as a subspace $\mathcal{D}$ to which the values of the linear strain $\boldsymbol{E}$ must belong. The constraint induces an indetermination of the stress, in the sense that to each value of the stress corresponding to an admissible strain $\boldsymbol{E} \in \mathcal{D}$ we may add an arbitrary element in the reaction space $\mathcal{D}^{\perp}$, the orthogonal complement of $\mathcal{D}$ (cf. [5, §30; 6]). As a consequence, the value $\mathbb{C}[\boldsymbol{E}]$ is interpreted as the active part of the stress, and it is quite natural to expect it to belong to $\mathcal{D}$.

In a constrained context, it is necessary to include in the definition of symmetry

(*) Nella seduta del 15 dicembre 1990.

group the request that the constraint be invariant with respect to all rotations $\boldsymbol{Q}$ in the group itself, in the sense that

$$
\boldsymbol{Q}\mathcal{D}\boldsymbol{Q}^T = \mathcal{D}. \tag{1}
$$

This observation motivates the concept of compatibility between a constraint and a symmetry group. Our first result in this paper is that we explicitly determine the class of all constraints compatible with transverse isotropy, for which the above equality holds for all rotations about an axis.

In the presence of an internal constraint the representation problem takes a new aspect, too. In fact, one now wishes to determine the class of all elasticity tensors, with a given symmetry, that map $\mathcal{D}$ into itself. As shown in [6], such a problem can always be reduced to the equivalent unconstrained case, in that the solution of the first can be deduced from the solution of the second by a formula having a constructive character. Thus, starting from the unconstrained representation given in [4], as a second result we are here able to provide the totality of representation formulae for transversely isotropic materials, *however* constrained.

2. Constraints and transverse isotropy

Let $\mathcal{V}$ be a three-dimensional inner product space, and let Lin be the space of all linear transformations $\boldsymbol{A}, \boldsymbol{B}, \dots$ on $\mathcal{V}$ with $\boldsymbol{I}$ the identity. Lin is made an inner product space by $\boldsymbol{A}\cdot\boldsymbol{B} := \mathrm{tr}\,(\boldsymbol{A}\boldsymbol{B}^T)$, where tr denotes the trace mapping and $\boldsymbol{B}^T$ is the transpose of $\boldsymbol{B}$. We denote by Sym, as mentioned, the subspace of all symmetric ($\boldsymbol{A} = \boldsymbol{A}^T$) elements of Lin, and by Rot the subgroup of all rotations.

Let $(\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g})$ be an orthonormal triad orienting $\mathcal{V}$. We denote with Rot$(\boldsymbol{e})$ the group of all rotations about $\boldsymbol{e}$, and write $\boldsymbol{Q}(\boldsymbol{e},\theta)$ for the rotation about $\boldsymbol{e}$ of an angle θ, for which we have:

$$
\begin{cases} \boldsymbol{Q}(\boldsymbol{e},\theta)\boldsymbol{f} = \cos\theta\,\boldsymbol{f} + \sin\theta\,\boldsymbol{g}, \\ \boldsymbol{Q}(\boldsymbol{e},\theta)\boldsymbol{g} = -\sin\theta\,\boldsymbol{f} + \cos\theta\,\boldsymbol{g}. \end{cases} \tag{2}
$$

In the space Ela (Sym) of *elasticity tensors, i.e.*, of all linear maps of Sym, the identity is denoted by $\mathbb{I}$. Ela (Sym) contains $\mathbb{O}$rth, the group of maps of Sym onto itself that preserve inner products. To each $\boldsymbol{Q}$ in Rot we associate the element $\mathbb{Q}$ in $\mathbb{O}$rth such that

$$
\mathbb{Q}\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T \qquad \forall \boldsymbol{A} \in \mathrm{Sym} \tag{3}
$$

and denote by $\mathbb{R}$ot the collection of all such $\mathbb{Q}$ when $\boldsymbol{Q}$ ranges over Rot. Finally, we denote by $\mathbb{R}$ot$(\boldsymbol{e})$ the subgroup of $\mathbb{R}$ot generated by Rot$(\boldsymbol{e})$; in particular, we write $\mathbb{Q}(\boldsymbol{e},\theta)$ for the element generated by $\boldsymbol{Q}(\boldsymbol{e},\theta)$.

The material symmetry group for a given elasticity tensor $\mathbb{C} \in$ Ela (Sym) is defined as the set of all rotations $\boldsymbol{Q}$ such that

$$
\mathbb{C}\,[\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}^T] = \boldsymbol{Q}\mathbb{C}\,[\boldsymbol{E}]\,\boldsymbol{Q}^T \qquad \forall \boldsymbol{E} \in \mathrm{Sym}. \tag{4}
$$

In view of (3), we may give a more compact formulation to this condition and formally introduce the symmetry group $\mathcal{G}(\mathbb{C})$ as a subgroup of Rot (cf. [5]):

$$\mathcal{G}(\mathbb{C}) := \{\mathrm{Q} \in \mathrm{Rot} \mid \mathrm{Q}\mathbb{C} = \mathbb{C}\mathrm{Q}\} \tag{5}$$

The introduction of a constraint subspace $\mathcal{D}$ of Sym is frequently used to model particular mechanical situations such as incompressibility, inextensibility in a given direction, or impossibility of shearing between a pair of orthogonal directions:

$$\begin{gathered} \mathcal{D} = \{\boldsymbol{E} \in \mathrm{Sym} \mid \mathrm{tr}\,\boldsymbol{E} = 0\}, \qquad \mathcal{D} = \{\boldsymbol{E} \in \mathrm{Sym} \mid \boldsymbol{E} \cdot \boldsymbol{e} \otimes \boldsymbol{e} = 0\}, \\ \mathcal{D} = \{\boldsymbol{E} \in \mathrm{Sym} \mid \boldsymbol{E} \cdot (\boldsymbol{e} \otimes \boldsymbol{f} + \boldsymbol{f} \otimes \boldsymbol{e}) = 0\}. \end{gathered} \tag{6}$$

In the presence of an internal constraint we distinguish the total stress into an *active* part, given by $\boldsymbol{T}_A = \mathbb{C}[\boldsymbol{E}]$, $\boldsymbol{E} \in \mathcal{D}$, and a *reactive* part, given by an arbitrary element of $\mathcal{D}^{\perp}$, the orthogonal complement of $\mathcal{D}$ in Sym. Thus, we are led to consider elasticity tensors in Ela $(\mathcal{D})$, the space of linear transformations of $\mathcal{D}$ onto itself. For an element of Ela $(\mathcal{D})$ the definition of symmetry group has to be modified, because, for (5) to be meaningful, the constraint must be invariant under the action of all rotations in the group. This requirement suggests the following definition of the *constraint symmetry group* $\mathcal{S}(\mathcal{D})$:

$$\mathcal{S}(\mathcal{D}) := \{\mathrm{Q} \in \mathrm{Rot} \mid \mathrm{Q}\mathcal{D} = \mathcal{D}\} \tag{7}$$

(cf. (1)). For $\mathbb{C} \in \mathrm{Ela}(\mathcal{D})$, $\mathcal{G}(\mathbb{C})$ is defined to be

$$\mathcal{G}(\mathbb{C}) := \{\mathrm{Q} \in \mathcal{S}(\mathcal{D}) \mid \mathrm{Q}\mathbb{C} = \mathbb{C}\mathrm{Q}|_{\mathcal{D}}\}. \tag{8}$$

In this paper we are interested in transversely isotropic elasticity tensor, for which we have

$$\mathcal{G}(\mathbb{C}) \supset \mathrm{Rot}(\boldsymbol{e}). \tag{9}$$

Our first goal here is to determine explicitly the class of constraints $\mathcal{D}$ *compatible* with transverse isotropy, in the sense that

$$\mathcal{S}(\mathcal{D}) \supset \mathrm{Rot}(\boldsymbol{e}). \tag{10}$$

The analogous but simpler problem of compatibility with isotropy was solved, as a side result, in [8]. It turns out that there are only two nontrivial compatible constraints, namely, preservation of volume, given by $(6)_1$, and preservation of shape $\mathcal{D} = \{\boldsymbol{E} \in \mathrm{Sym} \mid \boldsymbol{E} = \varepsilon \boldsymbol{I},\ \varepsilon \in \mathbb{R}\}$. Transversely isotropic materials admit a greater variety of constraints.

Consider the following orthonormal basis for Sym:

$$\begin{cases} \boldsymbol{P}_1 := \boldsymbol{e} \otimes \boldsymbol{e}, & \boldsymbol{P}_2 := \sqrt{2}/2\,(\boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e}), \\ \boldsymbol{A}_1 := \sqrt{2}/2\,(\boldsymbol{e} \otimes \boldsymbol{f} + \boldsymbol{f} \otimes \boldsymbol{e}), & \boldsymbol{A}_2 := \sqrt{2}/2\,(\boldsymbol{e} \otimes \boldsymbol{g} + \boldsymbol{g} \otimes \boldsymbol{e}), \\ \boldsymbol{B}_1 := \sqrt{2}/2\,(\boldsymbol{f} \otimes \boldsymbol{f} - \boldsymbol{g} \otimes \boldsymbol{g}), & \boldsymbol{B}_2 := \sqrt{2}/2\,(\boldsymbol{f} \otimes \boldsymbol{g} + \boldsymbol{g} \otimes \boldsymbol{f}). \end{cases} \tag{11}$$

Let $\mathcal{T}_\alpha := \mathrm{span}(\boldsymbol{P}_\alpha)$ $(\alpha = 1,2)$; $\mathcal{S}_1 := \mathcal{T}_1 \oplus \mathcal{T}_2 = \mathrm{span}(\boldsymbol{P}_1, \boldsymbol{P}_2)$, $\mathcal{S}_2 := \mathrm{span}(\boldsymbol{A}_1, \boldsymbol{A}_2)$, $\mathcal{S}_3 := \mathrm{span}(\boldsymbol{B}_1, \boldsymbol{B}_2)$. Then,

$$\mathrm{Sym} = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3. \tag{12}$$

We shall be using a number of orthogonal projections of Sym onto one or another of the subspaces we have just introduced: $\mathbb{S}_i$ onto $\mathcal{S}_i$ $(i = 1,2,3)$; $\mathbb{S}_{i,j}$ onto $\mathcal{S}_i \oplus \mathcal{S}_j$ $(i,j = 1,2,3$ and $i<j)$; $\mathbb{T}_\alpha$ onto $\mathcal{T}_\alpha$ $(\alpha = 1,2)$.

PROPOSITION 1. *Let* $\mathrm{Q} = \mathrm{Q}(\boldsymbol{e}, \theta) \in \mathrm{Rot}(\boldsymbol{e})$ *be given. Then, each* $\mathcal{T}_\alpha$ *and each* $\mathcal{S}_i$ *is invariant under the action of* Q, *i.e.,*

(13) $$\mathrm{Q}\mathcal{T}_\alpha = \mathcal{T}_\alpha, \quad \mathrm{Q}\mathcal{S}_i = \mathcal{S}_i.$$

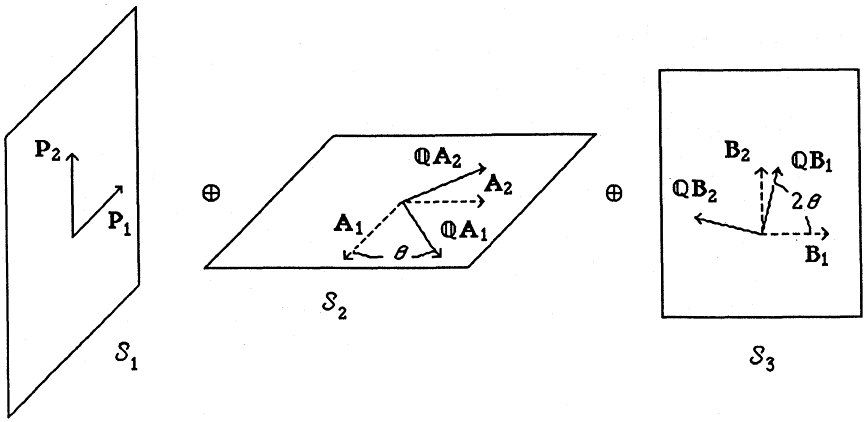
Precisely, Q *acts as the identity on (each* $\mathcal{T}_\alpha$ *and therefore on)* $\mathcal{S}_1$*; as a rotation of an angle* θ *on* $\mathcal{S}_2$*; and as a rotation of an angle* 2θ *on* $\mathcal{S}_3$*.*

Moreover, all projections commute with Q:

(14) $$\mathrm{T}_\alpha \mathrm{Q} = \mathrm{Q}\mathrm{T}_\alpha, \quad \mathrm{S}_i \mathrm{Q} = \mathrm{Q}\mathrm{S}_i, \quad \mathrm{S}_{i,j}\mathrm{Q} = \mathrm{Q}\mathrm{S}_{i,j}.$$

PROOF. We begin to check that Q acts as the identity on $\mathcal{T}_1$: $\mathrm{Q}\boldsymbol{P}_1 = \boldsymbol{Q}(\boldsymbol{e} \otimes \boldsymbol{e})\boldsymbol{Q}^T = \boldsymbol{e} \otimes \boldsymbol{e} = \boldsymbol{P}_1$. But, as $\boldsymbol{P}_2$ is the complementary projection of $\boldsymbol{P}_1$ on $\mathcal{S}_1$, Q must act as the identity on $\mathcal{T}_2$ and $\mathcal{S}_1$ as well. Next, we note that, in view of (2), we have: $\mathrm{Q}\boldsymbol{A}_1 = \sqrt{2}/2\,(\boldsymbol{Qe} \otimes \boldsymbol{Qf} + \boldsymbol{Qf} \otimes \boldsymbol{Qe}) = \cos\theta\,\boldsymbol{A}_1 + \sin\theta\,\boldsymbol{A}_2$; $\mathrm{Q}\boldsymbol{A}_2 = \sqrt{2}/2\,(\boldsymbol{Qe} \otimes \boldsymbol{Qg} + \boldsymbol{Qg} \otimes \boldsymbol{Qe}) = -\sin\theta\,\boldsymbol{A}_1 + \cos\theta\,\boldsymbol{A}_2$. Thus, Q acts as a rotation of an angle θ on $\mathcal{S}_2$. Moreover, $\mathrm{Q}\boldsymbol{B}_1 = \sqrt{2}/2\,(\boldsymbol{Qf} \otimes \boldsymbol{Qf} - \boldsymbol{Qg} \otimes \boldsymbol{Qg}) = (\cos^2\theta - \sin^2\theta)\sqrt{2}/2\,(\boldsymbol{f} \otimes \boldsymbol{f} - \boldsymbol{g} \otimes \boldsymbol{g}) + 2\sin\theta\cos\theta\sqrt{2}/2\,(\boldsymbol{f} \otimes \boldsymbol{g} + \boldsymbol{g} \otimes \boldsymbol{f}) = \cos 2\theta\,\boldsymbol{B}_1 + \sin 2\theta\,\boldsymbol{B}_2$. Analogously, $\mathrm{Q}\boldsymbol{B}_2 = \sqrt{2}/2\,(\boldsymbol{Qf} \otimes \boldsymbol{Qg} + \boldsymbol{Qg} \otimes \boldsymbol{Qf}) = -\sin 2\theta\,\boldsymbol{B}_1 + \cos 2\theta\,\boldsymbol{B}_2$, and we conclude that Q acts as a rotation of an angle 2θ on $\mathcal{S}_3$. Finally, these results imply that Q is reduced by the pairs $(\mathcal{S}_i, \mathcal{S}_i^\perp)$, $(\mathcal{T}_\alpha, \mathcal{T}_\alpha^\perp)$ in the sense of Halmos [9, § 40, § 43 Thm. 2], which proves (14). □

We may visualize the action of Q on Sym through a picture:



Hereafter we say that a nontrivial subspace $\mathcal{D}$ of Sym is *invariant* when (10) holds, *i.e.*, when $\mathcal{D}$ is invariant under the action of Rot$(\boldsymbol{e})$. For $\mathcal{D}_i := \mathrm{S}_i\mathcal{D}$ and $\mathcal{D}_{i,j} := \mathrm{S}_{i,j}\mathcal{D}$ notice that, in view of (14), we have:

(15) $$\{\mathrm{Q}\mathcal{D} = \mathcal{D}\} \Rightarrow \begin{cases} \mathrm{Q}\mathcal{D}_i = \mathrm{Q}\mathrm{S}_i\mathcal{D} = \mathrm{S}_i\mathrm{Q}\mathcal{D} = \mathrm{S}_i\mathcal{D} = \mathcal{D}_i, \\ \mathrm{Q}\mathcal{D}_{i,j} = \mathrm{Q}\mathrm{S}_{i,j}\mathcal{D} = \mathrm{S}_{i,j}\mathrm{Q}\mathcal{D} = \mathrm{S}_{i,j}\mathcal{D} = \mathcal{D}_{i,j}, \end{cases}$$

Thus, invariance of $\mathcal{D}$ implies invariance of $\mathcal{D}_i, \mathcal{D}_{i,j}$ (and of $\mathcal{D}^\perp$).

We now state our main result, namely, a complete classification of all invariant subspaces of Sym.

THEOREM. *Let $\mathcal{D}$ be a nontrivial invariant subspace of* Sym. *Then, one of the following holds*:

(*a*) *$\mathcal{D}$ is a one-dimensional subspace of $\mathcal{S}_1$*;

(*b*) *$\mathcal{D} = \mathcal{S}_i (i = 1,2,3)$*;

(*c*) *$\mathcal{D}$ is the direct sum of a one-dimensional subspace of $\mathcal{S}_1$ with $\mathcal{S}_2$ or $\mathcal{S}_3$*;

(*d*) *$\mathcal{D} = \mathcal{S}_i \oplus \mathcal{S}_j (i,j = 1,2,3$ and $i < j)$*;

(*e*) *$\mathcal{D}$ is the direct sum of a one-dimensional subspace of $\mathcal{S}_1$ with $\mathcal{S}_2 \oplus \mathcal{S}_3$.*

We shall find the following lemmata particularly useful for the proof.

LEMMA 1. *There is no invariant one-dimensional subspace of $\mathcal{S}_2$ (or $\mathcal{S}_3$).*

The geometric insight given by the figure makes a formal proof superfluous.

LEMMA 2. *Let $\mathcal{D}$ be a nontrivial invariant subspace of $\mathcal{S}_2 \oplus \mathcal{S}_3$. Then, either $\mathcal{D} = \mathcal{S}_2$ or $\mathcal{D} = \mathcal{S}_3$.*

PROOF. Lemma 1, invariance of $\mathcal{D}^{\perp}$ and the obvious inequality

$$\dim \mathcal{D}_i \leq \dim \mathcal{D} \tag{16}$$

imply that $\dim \mathcal{D} = 2$. We are left with three choices:

(*a*) $\dim \mathcal{D}_2 = 2$, $\dim \mathcal{D}_3 = 0$;

(*b*) $\dim \mathcal{D}_2 = 0$, $\dim \mathcal{D}_3 = 2$;

(*c*) $\dim \mathcal{D}_2 = \dim \mathcal{D}_3 = 2$.

Obviously, (*a*) and (*b*) imply, respectively, $\mathcal{D} = \mathcal{S}_2$ and $\mathcal{D} = \mathcal{S}_3$. If (*c*) holds, we may use bases $(\boldsymbol{A}_1, \boldsymbol{A}_2)$ and $(\boldsymbol{B}_1, \boldsymbol{B}_2)$ to identify elements of $\mathcal{S}_2, \mathcal{S}_3$ and vectors $\boldsymbol{x}, \boldsymbol{y}$ of $\mathbb{R}^2$. Thus, with slight abuse of notation, we may write

$$\mathcal{D} = \{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \mathcal{A}\boldsymbol{x} + \mathcal{B}\boldsymbol{y} = \boldsymbol{0}\}, \tag{17}$$

for some pair of 2×2 real matrices $\mathcal{A}$ and $\mathcal{B}$. In view of the geometric interpretation of the action of $Q(\boldsymbol{e}, \pi)$ we have that

$$(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{D} \Leftrightarrow (-\boldsymbol{x},\boldsymbol{y}) \in \mathcal{D}. \tag{18}$$

Thus, (17) and (18) combined give

$$(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{D} \Rightarrow \begin{cases} \mathcal{A}\boldsymbol{x} = \boldsymbol{0}, \\ \mathcal{B}\boldsymbol{y} = \boldsymbol{0}. \end{cases} \tag{19}$$

Since (*c*) implies the existence for each $\boldsymbol{x}$ (resp. $\boldsymbol{y}$) of $\boldsymbol{y}$ (resp. $\boldsymbol{x}$) such that $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{D}$, from (19) we deduce that $\mathcal{A} = \mathcal{B} = 0$, which contradicts $\dim \mathcal{D} = 2$. $\square$

LEMMA 3. *Let $\mathcal{D}$ be a nontrivial invariant subspace of $\mathcal{S}_1 \oplus \mathcal{S}_2$. Then, one of the following holds*:

(*a*) *$\mathcal{D}$ is a one-dimensional subspace of $\mathcal{S}_1$*;

(*b*) $\mathcal{D} = \mathcal{S}_i\,(i = 1, 2)$;

(*c*) *$\mathcal{D}$ is the direct sum of a one-dimensional subspace of $\mathcal{S}_1$ with $\mathcal{S}_2$.*

PROOF. Suppose $\dim \mathcal{D} = 1$. Lemma 1, invariance of $\mathcal{D}_2$ and (16) force $\dim \mathcal{D}_2 = 0$ and (*a*) holds.

Next, suppose $\dim \mathcal{D} = 2$. As in Lemma 2, we use bases $(\boldsymbol{P}_1, \boldsymbol{P}_2)$ and $(\boldsymbol{A}_1, \boldsymbol{A}_2)$ to identify $\mathcal{D}$ with a plane in $\mathbb{R}^2 \times \mathbb{R}^2$. Again, invariance of $\mathcal{D}$ under rotation $\mathrm{Q}(\boldsymbol{e}, \pi)$ gives

$$(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{D} \Rightarrow \begin{cases} \mathcal{A}\boldsymbol{x} = \boldsymbol{0}, \\ \mathcal{B}\boldsymbol{y} = \boldsymbol{0}. \end{cases} \tag{20}$$

Lemma 1 implies that either $\dim \mathcal{D}_2 = 0$, in which case we have $\mathcal{D} = \mathcal{S}_1$, or $\dim \mathcal{D}_2 = 2$, which means that for each $\boldsymbol{y}$ there is an $\boldsymbol{x}$ such that $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{D}$. In view of (20), this implies that $\mathcal{B} = 0$ and $\mathcal{D} = \mathcal{S}_2$.

Finally, suppose $\dim \mathcal{D} = 3$, and let $\mathcal{D}^\perp$ be the orthogonal complement of $\mathcal{D}$ in $\mathcal{S}_1 \oplus \mathcal{S}_2$. Since $\dim \mathcal{D}^\perp = 1$, from the first part of this discussion we conclude that $\mathcal{D}^\perp \subset \mathcal{S}_1$ and (*c*) holds. □

REMARK 1. A result completely analogous to Lemma 3 holds with $\mathcal{S}_2$ in place of $\mathcal{S}_3$.

PROOF OF THEOREM. We proceed case by case, according to the dimension of $\mathcal{D}$.

(α) $\dim \mathcal{D} = 1$. The invariance of $\mathcal{D}_{2,3}$, Lemma 1 and the inequality $\dim \mathcal{D}_{i,j} \leqslant \dim \mathcal{D}$ give $\dim \mathcal{D}_{2,3} = 0$, which implies (*a*).

(β) $\dim \mathcal{D} = 2$. The same argument used just above leaves us with two choices:

(β_1) $\dim \mathcal{D}_{2,3} = 0$, which implies $\mathcal{D} = \mathcal{S}_1$;

(β_2) $\dim \mathcal{D}_{2,3} = 2$, which, in view of Lemma 2, gives either $\mathcal{D}_{2,3} = \mathcal{S}_2$ or $\mathcal{D}_{2,3} = \mathcal{S}_3$. In the first case, we have $\mathcal{D} \subset \mathcal{S}_1 \oplus \mathcal{S}_2$, and Lemma 3 implies either $\mathcal{D} = \mathcal{S}_1$ or $\mathcal{D} = \mathcal{S}_2$; in the second case, we have $\mathcal{D} \subset \mathcal{S}_1 \oplus \mathcal{S}_3$ and Remark 1 implies either $\mathcal{D} = \mathcal{S}_1$ or $\mathcal{D} = \mathcal{S}_3$.

Thus, (β) implies (*b*).

(γ) $\dim \mathcal{D} = 3$. Necessarily we have $\dim \mathcal{D}_{2,3} > 0$, which, in view of Lemma 2 gives $\dim \mathcal{D}_{2,3} = 2$, and leaves us with two subcases:

(γ_1) $\mathcal{D}_{2,3} = \mathcal{S}_2$ and $\mathcal{D} \subset \mathcal{S}_1 \oplus \mathcal{S}_2$. Lemma 3 gives $\mathcal{D} = \mathcal{K} \oplus \mathcal{S}_2$, with $\mathcal{K}$ a one dimensional subspace of $\mathcal{S}_1$;

(γ_2) $\mathcal{D}_{2,3} = \mathcal{S}_3$ and $\mathcal{D} \subset \mathcal{S}_1 \oplus \mathcal{S}_3$. In view of Remark 1 we have $\mathcal{D} = \mathcal{K} \oplus \mathcal{S}_3$.

Thus, (γ) implies (*c*).

(δ) $\dim \mathcal{D} = 4$. Let $\mathcal{D}^\perp$ be the orthogonal complement of $\mathcal{D}$ in Sym. Since $\dim \mathcal{D}^\perp = 2$, from case ($\beta$) we immediately prove (*d*).

(ε) $\dim \mathcal{D} = 5$. As above, consideration of $\mathcal{D}^\perp$ and use of (α) give (*e*). □

REMARK 2. As there are infinitely many one-dimensional invariant subspaces $\mathcal{H}$ of $\mathcal{S}_1$, there are infinitely many constraint spaces of types $\mathcal{D} = \mathcal{H}$; $\mathcal{H} \oplus \mathcal{S}_2$, $\mathcal{H} \oplus \mathcal{S}_3$; $\mathcal{H} \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$; they have dimensions 1, 3, and 5, respectively. In addition, there are three constraint spaces of dimension 2 ($\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$), and three of dimension 4 ($\mathcal{S}_1 \oplus \mathcal{S}_2, \mathcal{S}_1 \oplus \mathcal{S}_3, \mathcal{S}_2 \oplus \mathcal{S}_3$).

3. THE CONSTRAINED REPRESENTATION PROBLEM

This section is devoted to the solution of the representation problem for *constrained* transversely isotropic, linearly elastic materials. Since the Theorem lists all possible constraints for such materials, we are left with a finite number of cases to discuss. More precisely, for any given constraint $\mathcal{D}$ satisfying (10), we define the space of transversely isotropic elasticity tensors on $\mathcal{D}$ as

$$\mathrm{Tis}(\mathcal{D}) := \{\mathbb{C} \in \mathrm{Ela}(\mathcal{D}) \,|\, \mathbb{Q}\mathbb{C} = \mathbb{C}\mathbb{Q}|_{\mathcal{D}} \qquad \forall Q \in \mathrm{Rot}(e)\}. \tag{21}$$

Thus, the Constrained Representation Problem can be formulated as follows:

($\mathcal{CRP}$) Given an admissible constraint space $\mathcal{D}$, find a basis for $\mathrm{Tis}(\mathcal{D})$.

If $\mathcal{D} = \mathrm{Sym}$, we have the unconstrained representation problem, solved in [3] and [4]. In particular, in [4] the elements of Tis (Sym) are represented in a basis whose geometric interpretation is especially suitable for our present developments, and indeed inspired them to some extent. The eight elements forming such a basis are:

$$\begin{cases} \mathbb{P}_1 := \boldsymbol{P}_1 \otimes \boldsymbol{P}_1, & \mathbb{P}_2 := \boldsymbol{P}_2 \otimes \boldsymbol{P}_2, \\ \mathbb{E}_1 := \boldsymbol{A}_1 \otimes \boldsymbol{A}_1 + \boldsymbol{A}_2 \otimes \boldsymbol{A}_2, & \mathbb{E}_2 := \boldsymbol{B}_1 \otimes \boldsymbol{B}_1 + \boldsymbol{B}_2 \otimes \boldsymbol{B}_2, \\ \mathbb{P}_3 := \boldsymbol{P}_1 \otimes \boldsymbol{P}_2 + \boldsymbol{P}_2 \otimes \boldsymbol{P}_1, & \mathbb{P}_4 := \boldsymbol{P}_2 \otimes \boldsymbol{P}_1 - \boldsymbol{P}_1 \otimes \boldsymbol{P}_2, \\ \mathbb{E}_3 := \boldsymbol{A}_2 \otimes \boldsymbol{A}_1 - \boldsymbol{A}_1 \otimes \boldsymbol{A}_2, & \mathbb{E}_4 := \boldsymbol{B}_2 \otimes \boldsymbol{B}_1 - \boldsymbol{B}_1 \otimes \boldsymbol{B}_2, \end{cases} \tag{22}$$

Notice that $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{E}_1$, and $\mathbb{E}_2$ are, respectively, the orthogonal projections $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{S}_2$, and $\mathbb{S}_3$ of Sym onto $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{S}_2$ and $\mathcal{S}_3$. Moreover, $\mathbb{P}_3$ and $\mathbb{P}_4$ are the orthogonal projection $\mathbb{S}_1$ onto $\mathcal{S}_1$ followed, respectively, by a reflection with respect to the diagonal of $\mathcal{S}_1$ spanned by $(\boldsymbol{P}_1 + \boldsymbol{P}_2)$, and by a rotation of an angle $\pi/2$ in $\mathcal{S}_1$; $\mathbb{E}_3$ and $\mathbb{E}_4$ are, respectively, the orthogonal projections $\mathbb{S}_2$ and $\mathbb{S}_3$ onto $\mathcal{S}_2$ and $\mathcal{S}_3$ followed by rotations of $\pi/2$ in the same planes.

From [6] we know that, for each admissible constraint $\mathcal{D}$, the map $\mathbb{C} \mapsto \mathbb{P}_{\mathcal{D}}\mathbb{C}|_{\mathcal{D}}$, where $\mathbb{P}_{\mathcal{D}}$ is the orthogonal projection onto $\mathcal{D}$, is surjective from Tis (Sym) onto $\mathrm{Tis}(\mathcal{D})$, *i.e.*,

$$\mathbb{P}_{\mathcal{D}}[\mathrm{Tis}(\mathrm{Sym})]|_{\mathcal{D}} = \mathrm{Tis}(\mathcal{D}). \tag{23}$$

We may combine formula (23) with the result of the Theorem and obtain a solution of problem ($\mathcal{CRP}$) for any constraint compatible with transverse isotropy. Our findings are organized in the following table, where $\mathcal{H}$ denotes a generic one-dimensional subspace of $\mathcal{S}_1$ and all mappings are understood to have domain restricted to the corresponding constraint $\mathcal{D}$.

$\mathcal{D}$	Basis for Tis $(\mathcal{D})$
$\mathcal{H}$	$\{\mathbb{I}\}$,
$\mathcal{S}_1$	$\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4\}$,
$\mathcal{S}_2$	$\{\mathbb{I}, \mathbb{E}_3\}$,
$\mathcal{S}_3$	$\{\mathbb{I}, \mathbb{E}_4\}$,
$\mathcal{H} \oplus \mathcal{S}_2$	$\{\mathbb{P}_1 + \mathbb{P}_2, \mathbb{E}_1, \mathbb{E}_3\}$,
$\mathcal{H} \oplus \mathcal{S}_3$	$\{\mathbb{P}_1 + \mathbb{P}_2, \mathbb{E}_2, \mathbb{E}_4\}$,
$\mathcal{S}_1 \oplus \mathcal{S}_2$	$\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4, \mathbb{E}_1, \mathbb{E}_3\}$,
$\mathcal{S}_2 \oplus \mathcal{S}_3$	$\{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3, \mathbb{E}_4\}$,
$\mathcal{S}_1 \oplus \mathcal{S}_3$	$\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4, \mathbb{E}_2, \mathbb{E}_4\}$,
$\mathcal{H} \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$	$\{\mathbb{P}_1 + \mathbb{P}_2, \mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3, \mathbb{E}_4\}$.

REMARK 3. For hyperelastic materials, the elasticity tensors $\mathbb{C} \in$ Tis (Sym) can be represented in a basis of only five elements, namely, $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$, $\mathbb{E}_1$ and $\mathbb{E}_2$, (cf. *e.g.* [4]). The corresponding table for Tis $(\mathcal{D})$ can be read off from the above table by deleting $\mathbb{P}_4$, $\mathbb{E}_3$, $\mathbb{E}_4$, throughout.

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