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On tensor functions whose gradients have some skew-symmetries

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Fisica matematica. — *On tensor functions whose gradients have some skew-symmetries.* Nota di Adriano Montanaro, presentata (*) dal Corrisp. A. Bressan.

Abstract. — Let $\mathcal{V}_n$ be a real inner product space of any dimension; and let $Q^{\alpha_1 \dots \alpha_\upsilon} = \hat{Q}^{\alpha_1 \dots \alpha_\upsilon}(X_{\beta_1 \dots \beta_\tau})$ be a C^2-map relating any two tensor spaces on $\mathcal{V}_n$. We study the consequences imposed on the form of this function by the condition that its gradient should be skew-symmetric with respect to some pairs (α_μ, β_η) of indexes. Any such a condition is written as a system of linear partial differential equations, with constant coefficients, which is symmetric with respect to certain couples of independent variables. The solutions of these systems appear useful to characterize the possible indeterminations in the admissible systems of constitutive equations for various continuous media.

Key words: Tensor calculus; Partial differential equations; Continuous media.

Riassunto. — *Sulle funzioni tensoriali il cui gradiente ha qualche antisimmetria.* Siano $\mathcal{V}_n$ uno spazio lineare di qualunque dimensione, dotato di prodotto interno, e $Q^{\alpha_1 \dots \alpha_\upsilon} = \hat{Q}^{\alpha_1 \dots \alpha_\upsilon}(X_{\beta_1 \dots \beta_\tau})$ una funzione di classe C^2 tra due arbitrari spazi tensoriali su $\mathcal{V}_n$. Si trova l'espressione di questa funzione dovuta alla condizione di antisimmetria del suo gradiente rispetto a coppie di indici (α_μ, β_η). Una tale condizione equivale a un certo sistema di equazioni lineari a coefficienti costanti, alle derivate parziali, simmetrico rispetto a una coppia di variabili indipendenti. La conoscenza delle soluzioni di tale sistema è utile per caratterizzare le possibili indeterminazioni degli ammissibili sistemi di equazioni costitutive per vari mezzi continui.

1. Introduction

We consider a tensor function $\mathbf{Q} = \hat{\mathbf{Q}}(X) = [\hat{Q}^{\alpha_1 \dots \alpha_\upsilon}(X_{\beta_1 \dots \beta_\tau})]$ (see (2.1)-(2.3)) relating arbitrary tensor spaces on an inner product space of arbitrary dimension into another such a space; and we study the general smooth solution to the symmetric first order system of equalities (2.4), which is equivalent to a condition of the kind below.

(1.A) *The tensor* Grad $\mathbf{Q}$ *is skew-symmetric in the fixed indexes* α_μ *and* β_η *where* $1 \leq \mu \leq \upsilon$ *and* $1 \leq \eta \leq \tau$ (see (2.1)-(2.4)) (1).

To solve system (2.4), points (*i*) through (*iii*) below are performed. (*i*) It is pointed out that any smooth solution to it also solves a certain second order system of linear partial differential equations, that is system (2.5) (see Lemma 2.1). (*ii*) This second order system is solved (see Theorem 3.1.). (*iii*) Among the solutions of the latter system those of the former are selected (see Theorem 4.1.). Roughly, one can say that any symmetry of this system induces certain skew-symmetries in the tensor coefficients of the independent variables in the general solution (see (4.C) and Theorem 4.1.). This is

(*) Nella seduta del 10 novembre 1990.

(1) Some skew-symmetry properties for tensors, that are considered here (see *e.g.* (4.C)) are similar to some properties involved in [3]; however the theorems proved here are completely unrelated to those presented there.

the reason why, roughly, the solutions are very few in comparison with the values that n, υ and τ can take; *e.g.* if τ is odd, then only linear solutions exist and, if there is a symmetry in a pair $(\alpha_\eta, \alpha_\mu)$ of indexes, then only constant solution exist (see Corollaries 4.1-4.3.). To show some examples, at the end of N. 4 the solutions to (2.4) are explicitly found in the cases $n = 3$, $\upsilon = 1 = \tau$, and $\upsilon = 2 = \tau$. In N. 5 multiple systems, which arise by coupling systems of the above kind, are quickly studied.

The systems being studied arise in the problem of characterizing the indeterminations of the admissible constitutive equations for a given continuous medium. This problem has been considered in [2] for purely mechanical simple bodies and in [4,5] within thermodynamics. Note that these indetermination questions are usually considered only in connection with entropy. Knowing the solutions of such systems is useful to study these indeterminations, for all simple and some nonsimple bodies (for more details see N. 6). Indeed, I think that the natural «algorithm» to perform these studies is given by the general solutions to these systems. For instance in [4, 5] the solutions to the above systems for $\{\upsilon, \tau\} = \{1,2\}$ and $n = 3$, together with the characterization of certain physically remarkable solution classes, have allowed the author to characterize the indetermination of the response function for the heat flux and to prove some uniqueness theorems for the response functions of stress, internal energy, and entropy, in thermo-elastic bodies; furthermore in [7-9] the solutions to some of these systems, coupled together, have allowed the authors to extend some of the above results to large classes of differential bodies.

2. Some symmetric systems for tensor functions

For $n, p, q \in \{1, 2, \ldots\}$ let $\mathcal{V}_n$ be a vector space of dimension n on the real field $\boldsymbol{R}$ and let $\mathcal{T}^p_q(\mathcal{V}_n)$ be the vector space of tensors of contravariant order p and covariant order q on $\mathcal{V}_n$. Throughout this paper we shall assume that $\mathcal{V}_n$, and hence its associated tensor spaces, are equipped with an inner product; thus one can identify tensor spaces of the same total order by means of various canonical isomorphisms (see *e.g.* [1]). If one refers $\mathcal{V}_n$ to an orthonormal basis, then the covariant, contravariant and mixed representations of a same tensor coincide. For $n, \upsilon, \tau \in \{1, 2, \ldots\}$ let us set

$$\alpha = (\alpha_1, \ldots, \alpha_\upsilon) \in \{1, \ldots, n\}^\upsilon, \qquad \beta = (\beta_1, \ldots, \beta_\tau) \in \{1, \ldots, n\}^\tau. \tag{2.1}$$

Let $\mathcal{U}_\tau$ be an open connected subset of $\mathcal{T}_\tau(\mathcal{V}_n)$, and let

$$\hat{Q}\colon \mathcal{U}_\tau \to \mathcal{T}^\upsilon(\mathcal{V}_n) \qquad (Q = \hat{Q}(X)). \tag{2.2}$$

Let $[X_\beta]$ and $[Q^\alpha]$ be the covariant and contravariant representations of $\boldsymbol{X}$ and $\boldsymbol{Q}$ respectively, with respect to a given vector basis. We do not wish to stress the dependence on such a basis; hence the component form of $(2.2)_2$ briefly writes as

$$Q^\alpha = \hat{Q}^\alpha(X_\beta). \tag{2.3}$$

Choose $\mu \in \{1, \dots, \upsilon\}$ and $\eta \in \{1, \dots, \tau\}$; consider the symmetric system of linear and homogeneous partial differential equations with constant coefficients

$$\text{(2.4)} \qquad \frac{\partial Q^{\alpha_1 \dots \alpha_{\mu-1}(\alpha_\mu \alpha_{\mu+1} \dots \alpha_\upsilon}}{\partial X_{\beta_1 \dots \beta_{\eta-1} \beta_\eta) \beta_{\eta+1} \dots \beta_\tau}} = 0 \quad i.e. \ [\mathrm{Grad}\, \mathbf{Q}]^{\alpha_1 \dots (\alpha_\mu \dots \alpha_\upsilon \beta_1 \dots \beta_\eta) \dots \beta_\tau} = 0\,(^2),$$

for each α and β as in (2.1). In words (2.4) means that *tensor* Grad $\mathbf{Q}$ *is skew-symmetric in the indexes* α_μ *and* β_η. Consider also the second order system

$$\text{(2.5)} \qquad \frac{\partial^2 Q^\alpha}{\partial X_{\beta_1 \dots \beta_{\eta-1} b \beta_{\eta+1} \dots \beta_\tau} \partial X_{\beta_1 \dots \beta_{\eta-1} c \beta_{\eta+1} \dots \beta_\tau}} = 0 \quad (\text{see (2.1)}),$$

for each $\alpha \in \{1, \dots, n\}^\upsilon$ and $\beta \in \{1, \dots, n\}^\tau$ ($\beta_\eta = b$, $\beta_\eta = c$, b, $c = 1, 2, 3$, η fixed); note that system (2.5) is equivalent to the condition $[\mathrm{Grad}^2\, \mathbf{Q}]^{\alpha\gamma\delta} = 0$ *whenever* $\alpha \in \{1, \dots, n\}^\upsilon$ *and* $\gamma, \delta \in \{1, \dots, n\}^\tau$ *satisfy*

$$\text{(2.6)} \qquad \gamma_{\{\eta\}} = \delta_{\{\eta\}},$$

where $\xi_{\{\eta\}}$ denotes the $(\tau - 1)$-tuple obtained by deleting the η-th component in $\xi \in \{1, \dots, n\}^\tau$:

$$\text{(2.7)} \qquad \xi_{\{\eta\}} := (\xi_1, \dots, \xi_{\eta-1}, \xi_{\eta+1}, \dots, \xi_\tau).$$

Next we prove that the solutions of the first order system (2.4) also solve the second order system (2.5).

LEMMA 2.1. *Any* C^2*-solution of the first order system* (2.4) *also solves the second order system* (2.5).

PROOF. Let $\hat{\mathbf{Q}}$ solve (2.4); fix $\alpha \in \{1, \dots, n\}^\upsilon$, and $\beta \in \{1, \dots, n\}^\tau$; in (2.5) set $a = \alpha_\mu$; and for $a, b, c \in \{1, \dots, n\}$ let $[a\,b\,c]$ equal the left-hand side of equality (2.5). Now (2.4) and the smoothness of $\hat{\mathbf{Q}}$ yield $[a\,b\,c] = -[b\,a\,c] = -[b\,c\,a] = [c\,b\,a] = [c\,a\,b] = -[a\,c\,b] = -[a\,b\,c]$, that is $[a\,b\,c] = 0$. Q.E.D.

3. Smooth solution of the above second order system

By the Lemma of the preceding section the class of the smooth solutions of the first order system (2.4) is embedded into the one of the second order system (2.5). In N. 4 the solutions of the former will be selected among the solutions of the latter. Next Theorem 3.1 states the mutual equivalence of assertions (3.A)-(3.C) below and gives a characterization of the solutions to system (2.5).

(3.A) [(3.B)] *The function* $\hat{\mathbf{Q}}$ *in* (2.2) *is a* C^2-[C^∞-] *solution on* $\mathcal{U}_\tau$ *of the second order system* (2.5).

(3.C) *For* $k = 0, \dots, m$, *with* $m = n^{\tau-1}$, *there are tensors* $\mathbf{R}^{[k]} \in \mathcal{T}^{\upsilon + k\tau}(\mathcal{V}_n)$ *such*

(^{2}) For any tensor X, its *symmetric* and *skew-symmetric parts* with respect to a and b are denoted by $X^{\dots(a\dots b)\dots} = (X^{\dots a \dots b \dots} + X^{\dots b \dots a \dots})/2$ and $X^{\dots[a\dots b]\dots} = (X^{\dots a \dots b \dots} - X^{\dots b \dots a \dots})/2$, respectively.

that equalities (3.1)-(3.2) *below hold.*

$$(3.1)\qquad Q^{\alpha}=\sum_{k=0}^{m} R^{[k]\alpha\beta^{[1]}\beta^{[2]}\ldots\beta^{[k]}} X_{\beta^{[1]}} X_{\beta^{[2]}}\ldots X_{\beta^{[k]}} \qquad (m=n^{\tau-1})\,(^3);$$

$$(3.2a)\qquad R^{[k]\alpha\beta^{[1]}\ldots\beta^{[p]}\ldots\beta^{[q]}\ldots\beta^{[k]}}=R^{[k]\alpha\beta^{[1]}\ldots\beta^{[q]}\ldots\beta^{[p]}\ldots\beta^{[k]}} \qquad (p,q=2,\ldots,k;\,k\geqslant 2);$$

$$(3.2b)\qquad R^{[k]\alpha\beta^{[1]}\ldots\beta^{[p-1]}\gamma\beta^{[p+1]}\ldots\beta^{[q-1]}\delta\beta^{[q+1]}\ldots\beta^{[k]}}=0 \quad \textit{whenever } \gamma_{\{\eta\}}=\delta_{\{\eta\}} \quad (\textit{see } (2.7)).$$

THEOREM 3.1. *The three assertions* (3.A), (3.B) *and* (3.C) *are equivalent.*

PROOF. Assume (3.A); by Lemma 2.1 function $\hat{Q}$ also solves the equations

$$(3.3)\qquad (\partial^2 Q^{\alpha})/(\partial X_{\beta})^2=0 \quad \textit{for each} \quad \beta\in\{1,\ldots,n\}^{\tau};$$

hence for each α the component function $\hat{Q}^{\alpha}$ is the restriction to $\mathscr{U}_{\tau}$ of a mapping, from $\boldsymbol{R}^{n^{\tau}}$ $(\cong\mathscr{T}_{\tau}(\mathscr{V}_n))$ into $\boldsymbol{R}$, which is multilinear in all variables X_{β}; hence $\hat{Q}^{\alpha}$ is of class $\boldsymbol{C}^{\infty}$ (4), and thus (3.B) holds. In order to prove that (3.B) implies (3.C), note that, by the aforementioned multilinearity, the components C_k of any monomial solution to (2.5) are of the kind

$$(3.4)\qquad C_k=r X_{\beta^{[1]}}X_{\beta^{[2]}}\ldots X_{\beta^{[k]}} \qquad (C_0=r\in R),$$

for some integer k. By substitution in (2.5) it follows that the τ-tuples $\beta^{[p]}$ in (3.4) satisfy

(3.D) $\beta^{[p]}_{\{\eta\}}\neq\beta^{[q]}_{\{\eta\}}$ *whenever* $1\leqslant p\neq q\leqslant k$ (see (2.7)).

Since there are $n^{\tau-1}$ distinct $(\tau-1)$-tuples

(3.E) *in equality* (3.4) *number k runs from* 1 *to* $m=n^{\tau-1}$.

Any solution to (2.4) is a linear combination of monomial matrices whose components have the form (3.4) for some k, and which satisfy (3.D)-(3.E); hence

(3.F) *for* $k=0,\ldots,m$, *with* $m=n^{\tau-1}$, *there is a tensor* $\boldsymbol{R}^{[k]}\in\mathscr{T}^{\upsilon+k\tau}(\mathscr{V}_n)$ *such that any component* Q^{α} *has the form* (3.1).

If $\boldsymbol{R}^{[k]}$ does not satisfy the symmetry properties (3.2*a*), then replace it with its «symmetrization» $(1/k!)\sum_{\sigma} R^{[k]\alpha\beta^{[\sigma 1]}\ldots\beta^{[\sigma k]}}$, where the summation is taken over all permutations σ of $\{1,\ldots,k\}$; it follows that equalities (3.2*a*) hold. Now let $\gamma,\delta\in\{1,\ldots,n\}^{\tau}$; taking the derivatives of both sides of equality (3.1) with respect to X_{γ} yields

$$(3.5)\qquad \frac{\partial Q^{\alpha}}{\partial X_{\gamma}}=\sum_{k=1}^{m}\sum_{p=1}^{k} R^{[k]\alpha\beta^{[1]}\ldots\beta^{[p-1]}\gamma\beta^{[p+1]}\ldots\beta^{[k]}} X_{\beta^{[1]}}\ldots X_{\beta^{[p-1]}}X_{\beta^{[p+1]}}\ldots X_{\beta^{[k]}}=$$

(3) The representation of tensor $\boldsymbol{R}^{[k]}$ in the fixed vector basis is $[R^{[k]\alpha\beta^{[1]}\beta^{[2]}\ldots\beta^{[k]}}]$, where $\alpha==\beta^{[0]}\in\{1,\ldots,n\}^{\upsilon}$ and $\beta^{[i]}\in\{1,\ldots,n\}^{\tau}$ for $i=1,\ldots,k$.

(4) Let $\mathscr{E}$ and $\mathscr{F}$ be finite-dimensional Banach spaces. Any m-multilinear mapping from $\mathscr{E}$ to $\mathscr{F}$ is of class $\boldsymbol{C}^{\infty}$.

$$= k \sum_{k=1}^{m} R^{[k]\alpha\beta^{[1]}\ldots\beta^{[k-1]}\gamma} X_{\beta^{[1]}} \ldots X_{\beta^{[k-1]}} \; (^5),$$

where the last equality follows by the symmetry properties (3.2*a*) and by suitably renaming the dummy τ-ples. By the same arguments one finds

$$\frac{\partial^2 Q^\alpha}{\partial X_\delta \partial X_\gamma} = k(k-1) \sum_{k=2}^{m} R^{[k]\alpha\beta^{[1]}\ldots\beta^{[k-2]}\gamma\delta} X_{\beta^{[1]}} \ldots X_{\beta^{[k-2]}} . \tag{3.6}$$

Iteration of the above steps yields the formula for the h-order gradient

$$\frac{\partial^h Q^\alpha}{\partial X_{\gamma^{[1]}} \ldots \partial X_{\gamma^{[h]}}} = k(k-1)\ldots(k-h+1) \sum_{k=h}^{m} R^{[k]\alpha\beta^{[1]}\ldots\beta^{[k-h]}\gamma^{[1]}\ldots\gamma^{[h]}} X_{\beta^{[1]}} \ldots X_{\beta^{[k-h]}} . \tag{3.7}$$

But by (3.6) equalities (2.5) are equivalent to the condition

$$\sum_{k=1}^{m} R^{[k]\alpha\beta^{[1]}\ldots\beta^{[k-2]}\gamma\delta} X_{\beta^{[1]}} \ldots X_{\beta^{[k-2]}} = 0 \qquad \textit{whenever} \qquad \gamma_{\{\eta\}} = \delta_{\{\eta\}} \quad (\text{see } (2.7)). \tag{3.8}$$

Now for each $k \in \{3, \ldots, m\}$ let us successively take the derivatives of both sides of equality (3.8) with respect to $X_{\beta^{[1]}}, \ldots, X_{\beta^{[k-2]}}$, where all $\beta^{[1]}, \ldots, \beta^{[k-2]}$ satisfy (3.D) and are arbitrarily prefixed; (3.7) yields

$$R^{[k]\alpha\beta^{[1]}\ldots\beta^{[k-2]}\gamma\delta} = 0 \qquad \textit{whenever} \qquad \gamma_{\{\eta\}} = \delta_{\{\eta\}} , \tag{3.9}$$

which by (3.2*a*) implies (3.2*b*). To prove that (3.C) implies (3.A) it suffices to note that (3.1) and (3.2*a*) yield the expression for the second order gradient occurring in equalities (3.6), which by (3.2*b*) is equivalent to (2.5). Q.E.D.

4. Smooth solutions of the above first order systems. Examples

The next Theorem 4.1 states the mutual equivalence of assertions (4.A) through (4.C) below.

(4.A) [(4.B)] *The function* $\hat{Q}$ *in (2.2) is a* C^2-[C^∞-] *solution on* $\mathcal{U}_\tau$ *to the first order system* (2.4).

(4.C) *For* $k = 0, \ldots, m$, *with* $m = n - 1$, *there are tensors* $R^{[k]} \in \mathcal{T}^{\upsilon + k\tau}(\mathcal{V}_n)$ *that are totally skew-symmetric in the indexes* $\{\alpha_\mu, \beta_\eta^{[1]}, \ldots, \beta_\eta^{[k]}\}$ *and satisfy equalities* (3.1)-(3.2). *Furthermore these tensors can be chosen totally skew-symmetric in the indexes* $\{\beta_\rho^{[1]}, \ldots, \beta_\rho^{[k]}\}$ *for* $\rho \in \{1, \ldots, \tau\}$.

Furthermore the proof of the same Theorem involves the following

Lemma 4.1. *Let* $[T^{bcBC}]$ *and* $[Y_{bB}]$ *be any tensors. Then*

$$T^{(bc)[BC]} Y_{bB} Y_{cC} = 0, \qquad (\textit{hence } T^{[bc](BC)} Y_{bB} Y_{cC} = 0 \textit{ too}), \tag{4.1}$$

$$T^{bcBC} Y_{bB} Y_{cC} = T^{(bc)(BC)} Y_{bB} Y_{cC} + T^{[bc][BC]} Y_{bB} Y_{cC} \qquad (\textit{see ftn. } (^2)). \tag{4.2}$$

(5) Note that a symmetry or skew-symmetry in the indexes of X would imply that X should belong to a proper vector subspace $\mathcal{W}$ of $\mathcal{T}_\tau(\mathcal{V}_n)$; and $\mathcal{U}_\tau \subset \mathcal{W}$ would follow, in contrast to the hypothesis that $\mathcal{U}_\tau$ is an open subset of $\mathcal{T}_\tau(\mathcal{V}_n)$. Hence there are neither symmetries nor skew-symmetries in X, and thus equality $(3.5)_1$ holds.

Hence, if T is skew-symmetric in $\{B, C\}$, then

$$T^{bcBC} Y_{bB} Y_{cC} = T^{[bc]BC} Y_{bB} Y_{cC}, \qquad (i.e.\, T^{(bc)BC} Y_{bB} Y_{cC} = 0). \tag{4.3}$$

Proof. Equality $(4.1)_1$ is a consequence of the equalities

$$T^{(bc)[BC]} Y_{bB} Y_{cC} = \sum_{B<C} (T^{(bc)[BC]} Y_{bB} Y_{cC} + T^{(bc)[CB]} Y_{bC} Y_{cB}) =$$

$$= \sum_{B<C} (T^{(bc)[BC]} Y_{bB} Y_{cC} + T^{(cb)[CB]} Y_{cC} Y_{bB}) = \sum_{B<C} (T^{(bc)[BC]} + T^{(bc)[CB]}) Y_{bB} Y_{cC} = 0,$$

where the 1-st and 4-th equalities follow by skew-symmetry in $\{B, C\}$, the 2-nd is obtained by renaming dummy indexes and the 3-rd follows by symmetry in $\{b, c\}$. Equality (4.2) is a consequence of (4.1) and of equality $T^{bcBC} = T^{(bc)(BC)} + T^{(bc)[BC]} + T^{[bc](BC)} + T^{[bc][BC]}$. Equalities (4.3) are trivial consequences of (4.1)-(4.2). Q.E.D.

Theorem 4.1. *The three assertions* (4.A), (4.B) *and* (4.C) *are equivalent.*

Proof. Assume (4.A); then by Lemma 2.1 function $\hat{Q}$ also solves system (2.5); hence, by Theorem 3.1, $\hat{Q} \in C^\infty$ and satisfies (3.C); hence (4.B) holds and each component $\hat{Q}^\alpha$ can be written as in (3.1), where equalities (3.2*a*)-(3.2*b*) hold. Now by taking the derivative of both sides of equality (3.1) with respect to X_γ ($\gamma \in \{1, \ldots, n\}^\tau$) it follows (3.5), which by (2.5) yields

$$\frac{\partial Q^{\alpha_1 \ldots \alpha_{\mu-1}(\alpha_\mu \alpha_{\mu+1} \ldots \alpha_\nu}}{\partial X_{\gamma_1 \ldots \gamma_{\eta-1}\gamma_\eta)\gamma_{\eta+1} \ldots \gamma_\tau}} = \tag{4.4}$$

$$= k \sum_{k=1}^{m} R^{[k]\alpha_1 \ldots \alpha_{\mu-1}(\alpha_\mu \alpha_{\mu+1} \ldots \alpha_\nu \beta^{[1]} \ldots \beta^{[k-1]} \gamma_1 \ldots \gamma_{\eta-1}\gamma_\eta)\gamma_{\eta+1} \ldots \gamma_\tau} X_{\beta^{[1]}} \ldots X_{\beta^{[k-1]}} = 0,$$

which is equivalent to the m equalities

$$R^{[k]\alpha_1 \ldots \alpha_{\mu-1}(\alpha_\mu \alpha_{\mu+1} \ldots \alpha_\nu \beta^{[1]} \ldots \beta^{[k-1]} \gamma_1 \ldots \gamma_{\eta-1}\gamma_\eta)\gamma_{\eta+1} \ldots \gamma_\tau} X_{\beta^{[1]}} \ldots X_{\beta^{[k-1]}} = 0 \qquad (k = 1, \ldots, m). \tag{4.5}$$

Now let $\gamma^{[1]}, \ldots, \gamma^{[k-1]}$ satisfy (3.D); let us successively take the derivatives of both sides of equality (4.5) with respect to $X_{\gamma^{[p]}}, \ldots, X_{\gamma^{[k-1]}}$; by (3.7)

$$R^{[k]\alpha_1 \ldots \alpha_{\mu-1}(\alpha_\mu \alpha_{\mu+1} \ldots \alpha_\nu \gamma^{[1]} \ldots \gamma^{[k-1]} \gamma_1 \ldots \gamma_{\eta-1}\gamma_\eta)\gamma_{\eta+1} \ldots \gamma_\tau} = 0 \qquad \textit{for } k = 1, \ldots, m. \tag{4.6}$$

Hence by (4.6) and (3.2*a*), each tensor $R^{[k]}$ is skew-symmetric in any pair of indexes $\{\alpha_\mu, \beta_\eta^{[p]}\}$ for $p = 1, \ldots, k$. Thus these tensors are also totally skew-symmetric in the indexes $\{\alpha_\mu, \beta_\eta^{[1]}, \ldots, \beta_\eta^{[k]}\}$ (6). Now let $\rho \in \{1, \ldots, \tau\}$; choose $p, q \in \{1, \ldots, k\}$ with $p \neq q$; and in equality (3.1) let us set $b = \beta_\rho^{[p]}$, $B = \beta_\eta^{[p]}$, $c = \beta_\rho^{[q]}$, $C = \beta_\eta^{[q]}$, $T^{bcBC} = R^{[k]\ldots b \ldots c \ldots B \ldots C \ldots}$, and $Y_{bB} = X_{\ldots b \ldots B \ldots}$. Thus tensor T is skew-symmetric in $\{B, C\}$. Hence by Lemma 4.1 – see (4.3) – we can replace in (3.1) each tensor $R^{[k]}$ ($k \geq 2$) with its skew-symmetric part in $\{b, c\}$. Repeating this step for each choice of p, q, and ρ yields (4.C). Conversely, if $\hat{Q}$ satisfies (4.C), then by (4.6) and (4.4) function $\hat{Q}$ solves (2.4). Q.E.D.

(6) Recall that a tensor skew-symmetric in its pairs of indexes $\{a, b\}$ and $\{a, c\}$ also is skew-symmetric in $\{b, c\}$ and is totally skew-symmetric in $\{a, b, c\}$.

COROLLARY 4.1 [4.2] *In* (2.1), (2.2) *and* (2.4) *assume that* (i) $\upsilon > 1$, (ii) $\alpha_\mu \in S \subseteq \{\alpha_1, \ldots, \alpha_\upsilon\}$, ($iii$) $s = |S| > 1$ (7) *and* (iv) *function* $\hat{Q}$ *is totally skew-symmetric [symmetric] in the indexes belonging to S. Then* $\hat{Q}$ *satisfies either* (4.A) *or* (4.B) *if and only if* $\hat{Q}$ *satisfies* (4.C) *with* «$m = n - 1$» *replaced with* «$m = n - s$» [«$m = 0$», i.e. *system* (2.4) *only has constant solutions*].

PROOF. Set $K = \{\alpha_\mu, \beta_\eta^{[1]}, \ldots, \beta_\eta^{[k]}\}$, assume ($i$)-($iv$) above and either (4.A) or (4.B). As $K \cap S = \{\alpha_\mu\} \neq \emptyset$, it follows that each $\boldsymbol{R}^{[k]}$ is totally skew-symmetric in $K \cup S$ [$\boldsymbol{R}^{[k]} = 0$ for $k > 0$ and Corollary 4.2 is proved] (8). Recall that there are no tensors which are totally skew-symmetric in more than n indexes. Hence, as $|K \cup S| = k + s$, $\boldsymbol{R}^{[k]} = 0$ for each $k > n - s$. The equivalence of the three assertions (4.A) through (4.C) (see Theorem 4.1) yields the thesis. Q.E.D.

COROLLARY 4.3. *If* τ *is odd, then system* (2.4) *only has linear solutions* (9).

PROOF. Assume (4.A). By Theorem 4.1 each tensor $\boldsymbol{R}^{[k]}$ can be chosen skew-symmetric in the indexes $\beta_\rho^{[1]}, \ldots, \beta_\rho^{[k]}$ for $\rho \in \{1, \ldots, \tau\}$. Now let us interchange *e.g.* any index $\beta_\rho^{[1]}$ with $\beta_\rho^{[2]}$ for each $\rho \in \{1, \ldots, \tau\}$. The aforementioned skew-symmetry property yields $R^{[k]\alpha\beta^{[1]}\beta^{[2]}\beta^{[3]}\ldots\beta^{[k]}} = (-1)^\tau R^{[k]\alpha\beta^{[2]}\beta^{[1]}\beta^{[3]}\ldots\beta^{[k]}}$, which, by the symmetry property in (3.2a) and the hypothesis that τ is odd, yields $\boldsymbol{R}^{[k]} = 0$ for each $k \geqslant 2$. Q.E.D.

As an example, let us consider the two cases $n = 3$, $\upsilon = 1 = \tau$ and $n = 3$, $\upsilon \geqslant 1$, $\tau = 2$. System (2.4) reduces to $\partial Q^{(A} / \partial X_{B)} = 0$ and $\partial Q^{a(A} / \partial X_{bB)} = 0$, respectively, where $a \in \{1,2,3\}^{\upsilon - 1}$ and $b, A, B = 1,2,3$. Their solutions are

$$Q^A = R^A + \varepsilon^{ABC} T^C X_B \qquad \text{and} \qquad Q^{aA} = \tau^{[0]aA} + \tau^{[1]abC} \varepsilon^{ABC} X_{bB} + \tau^{[2]ab} \varepsilon^{bbc} \varepsilon^{ABC} X_{bB} X_{cC},$$

respectively, where $\tau^{[i]}$ ($i = 0, 1, 2$) are arbitrary tensors of the order determined by their indexes.

5. GRADIENTS WITH MULTIPLE SKEW-SYMMETRIES

Next let us consider multiple systems, which are composed by systems of the kind studied above. We show that the class of the solutions reduces in a way strictly related with the number of the component systems. For $u \in \{1, 2, \ldots, \upsilon\}$ and $v \in \{1, 2, \ldots, \tau\}$ let us consider the (multiple) systems below.

$$\text{(5.1)} \qquad [\mathrm{Grad}\, \mathbf{Q}]^{\alpha_1 \ldots (\alpha_{\mu_i} \ldots \alpha_\upsilon}_{\beta_1 \ldots \beta_{\eta_j}) \ldots \beta_\tau} = 0 \qquad \textit{for each } (i,j) \in \{1, \ldots, u\} \times \{1, \ldots, v\},$$

$$\textit{where} \quad 1 \leqslant \mu_1 < \mu_2 < \ldots < \mu_u \leqslant \upsilon \quad \textit{and} \quad 1 \leqslant \eta_1 < \eta_2 < \ldots < \eta_v \leqslant \tau.$$

(7) The cardinality of any set S is denoted by $|S|$.

(8) Let A and B be non-disjoint sets of indexes of a tensor $\boldsymbol{T}$; then $\boldsymbol{T} = 0$ whenever $\boldsymbol{T}$ is (totally) symmetric in A and skew-symmetric in B; furthermore $\boldsymbol{T}$ is skew-symmetric in $A \cup B$ whenever it is skew-symmetric in both A and B.

(9) Note that, if τ is odd, then, by the skew-symmetry properties of the coefficients tensors in (3.1), the general solution of (2.4) can be written as $Q^\alpha = R^\alpha + \varepsilon^{\alpha_\mu \beta_\eta C_3 \ldots C_n} T^{\alpha\{\mu\}\beta\{\eta\} C_3 \ldots C_n} X_\beta$ (see Theorem 4.1 and (2.7)), where ε is Ricci's tensor and $\boldsymbol{R}$, $\boldsymbol{T}$ are any tensors of suitable orders.

Note that this system is composed by $u \times v$ systems of the kind (2.4). In words (5.1) means that *tensor* Grad $\mathbf{Q}$ *is skew-symmetric in its pairs of indexes* $(\alpha_{\mu_i}, \beta_{\eta_j})$. Next Theorem 5.1 below, which for $u = 1 = v$ reduces to Theorem 4.1, will state the mutual equivalence of assertions (5.A)-(5.C) below.

(5.A) [(5.B)] *The function* $\hat{\mathbf{Q}}$ *in* (2.2) *is a* $\mathbf{C}^2$-$[\mathbf{C}^\infty$-] *solution on* $\mathcal{U}_\tau$ *of the multiple system* (5.1).

(5.C) *For* $k = 0, \dots, m$, *with* $m = \mathrm{Int}[(n-u)/v]$ (10), *there are tensors* $\boldsymbol{R}^{[k]} \in \mathcal{J}^{\upsilon + k\tau}(\mathcal{V}_n)$ *that are totally skew-symmetric in the set of indexes* $I^{[k]} = \{\alpha_{\mu_1}, \dots, \alpha_{\mu_u}, \beta^{[1]}_{\eta_z}, \dots, \beta^{[k]}_{\eta_z} \,|\, z = 1, \dots, v\}$ *and satisfy equalities* (5.1). *Furthermore these tensors can be chosen totally skew-symmetric in the sets of indexes* $J_\rho = \{\beta^{[1]}_\rho, \dots, \beta^{[k]}_\rho\}$, *for* $\rho \in \{1, \dots, \tau\}$.

THEOREM 5.1. *The three assertions* (5.A), (5.B) *and* (5.C) *are equivalent.*

PROOF. Let function $\hat{\mathbf{Q}}$ in (2.2) solve (5.1). Then, for each $(i,j) \in \{1, \dots, u\} \times \{1, \dots, v\}$, function $\hat{\mathbf{Q}}$ solves the simpler system given by equalities (5.1) for i and j fixed; hence, by Theorem 4.1, assertion (4.C) holds and each tensor $\boldsymbol{R}^{[k]}$ is totally skew-symmetric in the sets of indexes

$$I^{[k]}_{ij} := \{\alpha_{\mu_i}, \beta^{[1]}_{\eta_j} \dots, \beta^{[k]}_{\eta_j}\} \quad \text{and} \quad J_\rho \ \text{for}\ \rho \in \{1, \dots, \tau\} \ \ (|I^{[k]}_{ij}| = k+1).$$

By the arbitrariness of $j \in \{1, \dots, v\}$, as $I^{[k]}_{ij} \cap I^{[k]}_{ih} \neq \emptyset$, each tensor $\boldsymbol{R}^{[k]}$ is totally skew-symmetric in the set of indexes $I^{[k]}_i = \{\alpha_{\mu_i}, \beta^{[1]}_{\eta_j}, \dots, \beta^{[k]}_{\eta_j} \,|\, j = 1, \dots, v\}$ $(|I^{[k]}_i| = kv + 1)$. As $I^{[k]}_i \cap I^{[k]}_h \neq \emptyset$ for $i, h \in \{1, \dots, v\}$, by the arbitrariness of $i \in \{1, \dots, v\}$, it follows that each tensor $\boldsymbol{R}^{[k]}$ is also totally skew-symmetric in the set of indexes $I^{[k]}$ – see (5.C). The equality $|I^{[k]}| = kv + u$ yields that $\boldsymbol{R}^{[k]} = 0$ if and only if $kv + u > n$; hence, each k in (3.1) satisfies the condition $k \leqslant (n-u)/v$, *i.e.* $k \leqslant \mathrm{Int}[(n-u)/v]$. Q.E.D.

Lastly set $u = v$ and $i = j$ in the assertion involving (5.1). For the resulting system Theorem 5.1 still holds provided «$\mathrm{Int}[(n/u) - 1]$» be substituted for «$\mathrm{Int}[(n-u)/v]$» in (5.C).

6. HINTS AT APPLICATIONS TO MATHEMATICAL PHYSICS

The knowledge of the solutions to system (2.4) in the cases $n = 3$, $\upsilon, \tau \in \{1, 2\}$ have allowed the author to establish (*i*) *some uniqueness properties of the response functions for the stress, internal energy, and entropy in thermo-elastic bodies*, and (*ii*) *a characterization of the indetermination in the response function for the heat flux* (see [5, 4]). Knowing the general solution of those systems is also useful for studying the indeterminations of *e.g.* the response functions for the stress and couple-stress in certain non-simple bodies of grade two, *e.g.* those considered in [10] (see [6]). Therefore I think that solving systems (2.4), for suitable values of υ and τ, and other more complex similar systems, gives us the natural tools for studying the indeterminateness of consti-

(10) For $r \in R$ let $\mathrm{Int}[r]$ denote the greatest integer i with $i \leqslant r$.

tutive equations and for establishing uniqueness theorems for them, in both continuum mechanics and thermodynamics ([11]).

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([11]) To support this assertion, remark that these systems naturally arise by the steps quickly explained below with regard to a thermo-elastic body $\mathscr{B}_{\mathrm{el}}$ referred to a configuration $\mathscr{K}$. Let both $\hat{I} = \{\hat{\boldsymbol{P}}_{\mathscr{K}}, \hat{\boldsymbol{q}}_{\mathscr{K}}, \hat{e}_{\mathscr{K}}, \hat{\eta}_{\mathscr{K}}\}$ and $\tilde{I} = \{\tilde{\boldsymbol{P}}_{\mathscr{K}}, \tilde{\boldsymbol{q}}_{\mathscr{K}}, \tilde{e}_{\mathscr{K}}, \tilde{\eta}_{\mathscr{K}}\}$ be admissible sets of response functions for the first Piola stress, heat flux, internal energy, and entropy, connected with $(\mathscr{B}_{\mathrm{el}}, \mathscr{K})$; and set $\hat{S} = \hat{\boldsymbol{P}}_{\mathscr{K}} - \tilde{\boldsymbol{P}}_{\mathscr{K}}$, $\hat{Q} = \hat{\boldsymbol{q}}_{\mathscr{K}} - \tilde{\boldsymbol{q}}_{\mathscr{K}}$, and $\hat{E} = \hat{e}_{\mathscr{K}} - \tilde{e}_{\mathscr{K}}$. Subtract anyone of the equalities expressing the balance laws, written using the set $\hat{I}$, with the corresponding equalities written using the set $\tilde{I}$; then functions $\hat{S}$, $\hat{Q}$ and $\hat{E}$ solve the equations (1) $\operatorname{Div} S = 0$, (2) $SF^T = (SF^T)^T$, (3) $\rho_{\mathscr{K}} \dot{E} = \boldsymbol{S} \cdot \dot{\boldsymbol{F}} - \operatorname{Div} \boldsymbol{Q}$, and hence (4) $\operatorname{Div} \boldsymbol{Q} = 0$, for $\boldsymbol{S} = \hat{S}(\boldsymbol{F}, \theta, \boldsymbol{X})$, $\boldsymbol{Q} = \hat{Q}(\boldsymbol{F}, \theta, \boldsymbol{G}, \boldsymbol{X})$, $\boldsymbol{E} = \hat{E}(\boldsymbol{F}, \theta, \boldsymbol{X})$, $\boldsymbol{F} = \operatorname{Grad} \boldsymbol{x}$, $\boldsymbol{G} = \operatorname{Grad} \theta$, where $\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{X}, t)$ and $\theta = \hat{\theta}(\boldsymbol{X}, t)$ are the position and the temperature, respectively, at $(\boldsymbol{X}, t)$. Then (1) and (4) rewrite as (5) $(\partial S^{aH}/\partial F^i_L) F^i_{LH} + (\partial S^{aH}/\partial \theta) G_H + (\partial S^{aH}/\partial X_H) = 0$ $(a = 1,2,3)$ and (6) $(\partial \boldsymbol{Q}^H/\partial F^i_L) F^i_{LH} + (\partial \boldsymbol{Q}^H/\partial \theta) G_H + (\partial \boldsymbol{Q}^H/\partial G_L) G_{LH} + \partial \boldsymbol{Q}^H/\partial X_H = 0$. A physical possibility axiom assumes that certain local experiments can take place. It can be summarized as follows. *Choose any material point* $\boldsymbol{X}$ *of the body and any value* v *for the set* s *of state variables. Then there is a suitable open set* G *such that for each* $g \in G$ *there is a possible process of the body in which, at some time, the value of* s *at* $\boldsymbol{X}$ *equals* v *and the gradient of* s *equals* g. By this axiom equalities (5)-(6) yield (7) $(\partial S^{aH}/\partial F^i_L) F^i_{LH} = 0$, $(\partial Q^H/\partial F^i_L) F^i_{LH} = 0$, $(\partial Q^H/\partial G_L) G_{LH} = 0$; furthermore this axiom allows us the choice $F^i_{LH} = \varepsilon\delta^{ib}(\delta_{HA}\delta_{LB} + \delta_{HB}\delta_{LA})$, $G_{LH} = \varepsilon(\delta_{HA}\delta_{LB} + \delta_{HB}\delta_{LA})$ $(\varepsilon > 0)$; thus equalities (7) yield $\partial S^{a(A}/\partial F_{bB)} = 0$ $\partial Q^{(A}/\partial F_{bB)} = 0$ and $\partial Q^{(A}/\partial G_{B)} = 0$, respectively. Note that these systems are of the kind (2.4) for $n = 3$ and $\upsilon, \tau \in \{1,2\}$.

REFERENCES

[1] R. M. BOWEN - C.-C. WANG, *Introduction to vectors and tensors.* Plenum Press, New York - London 1976.

[2] A. BRESSAN - A. MONTANARO, *On the uniqueness of response stress-functionals for purely mechanical continuous media, from the Mach-Painlevé point of view.* Mem. Mat. Acc. Lincei, s. 9, vol. 1, fasc. 3, 1990, 59-94.

[3] D. G. B. EDELEN, *The null set of the Euler-Lagrange operator.* Arch. Rational Mech. Anal., vol. 11, 1962, 117-121.

[4] A. MONTANARO, *On heat flux in simple media.* Journal of Elasticity, to appear.

[5] A. MONTANARO, *On the response functions of a thermo-elastic body, from the Mach-Painlevé point of view.* Mem. Mat. Acc. Lincei, s. 9, vol. 1, fasc. 5, 1990, 123-146.

[6] A. MONTANARO, *On the response functions for the stress and couple-stress in nonsimple elastic materials of grade two without internal constraints.* Part 1 - *General smooth solution of certain threefold-symmetric systems of linear PDEs, for tensor functions, usefuld to study physically equivalent response stress-functions.* Part 2 - *On the indeterminations in the skew-symmetric parts of stress and couple-stress in nonsimple elastic materials of grade two.* To appear.

[7] A. MONTANARO - D. PIGOZZI, *On a large class of symmetric systems of linear PDEs for tensor functions, useful in mathematical physics.* Annali di Matematica Pura e Applicata, to appear.

[8] A. MONTANARO - D. PIGOZZI, *On the response function for the heat flux in bodies of the differential type.* To appear.

[9] A. MONTANARO - D. PIGOZZI, *On the physical indetermination of the response functions for general differential bodies.* Part 1 - *Physical solutions to some symmetric systems of linear PDEs*

in tensor functions. Part 2 - *On uniqueness properties of the response functions for the stress and internal energy in differential bodies*. To appear.

[10] R. A. Toupin, *Theories of Elasticity with Couple-stress*. Arch. Rational Mech. Anal., vol. 17, 1964, 7-112.

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