Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Zbginiew Slodkowski, Giuseppe Tomassini

Levi-equation in higher dimensions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **2** (1991), n.4, p. 277–279.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1991_9_2_4_277_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1991.

Rend. Mat. Acc. Lincei
s. 9, v. 2:277-279 (1991)

Funzioni di variabili complesse. — *Levi-equation in higher dimensions.* Nota di
ZBGINIEW SLODKOWSKI e GIUSEPPE TOMASSINI, presentata (*) dal Socio E. VESENTINI.

ABSTRACT. — We announce some results concerning the Dirichlet problem for the Levi-equation in $\mathbf{C}^n$. We consider for the sake of simplicity the case $n=3$.

KEY WORDS: Levi form; Monge-Ampère equation; Dirichlet problem.

RIASSUNTO. — *L'equazione di Levi in dimensioni superiori.* Si annunciano alcuni risultati ottenuti nello studio del problema di Dirichlet per l'equazione di Levi in $\mathbf{C}^n$ considerando per semplicità il caso $n=3$.

1. We consider the case $\mathbf{C}^3$. The proofs in the general case are based on the same techniques. Let $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, $z_3 = x_5 + ix_6$, be complex coordinates in $\mathbf{C}^3$, let Ω be a bounded domain contained in $x_6 = 0$ with a connected boundary, and let $x = (x_1, \dots, x_5)$ denote a generic point in Ω. For a given $u \in C^0(\Omega)$ let $\Gamma(u)$ be the graph of u in $\mathbf{C}^3$ and $\Gamma_+(u) = \{u - x_6 < 0\}$, $\Gamma_-(u) = \{u - x_6 > 0\}$. If $u \in C^2(\Omega)$ and if $T_z^{\mathbf{C}}$ denotes the complex tangent plane to $\Gamma(u)$ at z then, as is well known, $\Gamma_+(u)$ is a domain of holomorphy if and only if $\mathit{Levi}\,(u - x_6)|_{T_z^{\mathbf{C}}} \geq 0$ for every $z \in \Gamma(u)$; *i.e.*, $\Gamma(u)$ is *Levi-convex*. In terms of u this condition can be written:

$$\sum_{\alpha,\beta=1}^{2} A_{\alpha\bar\beta}(u)(x)\,\zeta_\alpha\,\overline{\zeta_\beta} \geq 0 \tag{1}$$

for every $x \in \Omega$. The $A_{\alpha\bar\beta}(u)$ are quasi-linear differential operators of second order, $A_{\alpha\bar\beta}(u) = \overline{A_{\beta\bar\alpha}}(u)$ and $A_{1\bar1}(u)$, $A_{2\bar2}(u)$ are degenerate elliptic. Let us set $LMA(u) = \det(A_{\alpha\bar\beta}(u))$; then (1) is equivalent to $LMA(u)(x) \geq 0$, $A_{\alpha\bar\alpha}(u)(x) \geq 0$, $\alpha = 1,2$. We say that $\Gamma(u)$ is *Levi-flat* at x^0 if $\Gamma(u)$ is Levi-convex at $(z_1^0, z_2^0, x_5^0 + iu(x^0))$ and $LMA(u)(x^0) = 0$. We also say that u is *Levi-convex* respectively *Levi-flat* whenever $\Gamma(u)$ satisfies the condition above at each point.

In the notation $LMA(u)$, L stands for «Levi» and MA for «Monge-Ampère», due to the fact that $LMA(u)$ is obtained as determinant of the hermitian form $\mathit{Levi}\,(u - x_6)|_{T_z^{\mathbf{C}}}$, $z \in \Gamma(u)$.

The problem we will be dealing with is the following:

$$LMA(u) = k(\cdot, u)(1 + |Du|^2)^2 \quad \text{in } \Omega, \tag{2}$$

$u = g$ on $\partial\Omega$, u is Levi-convex and $k \in C^0(\Omega \times \mathbf{R})$, $g \in C^0(\partial\Omega)$. It is the natural generalization in $\mathbf{C}^3$ of the Dirichlet problem for the Levi-equation in $\mathbf{C}^2$ [1,2,7,8]. Here k represents a sort of «total Levi-curvature» of $\Gamma(u)$ (as the boundary of $\Gamma_+(u)$ [8]). In the general case of a C^2 hypersurface $\mathcal{S}$ in $\mathbf{C}^3$ given by $\rho = 0$, we define the *total Levi-*

(*) Nella seduta dell'11 maggio 1991.

curvature of $\mathcal{S}$ (as the boundary of $\{\rho<0\}$) to be the function

$$k_{\text{Levi}}(\mathcal{S})(z)=\lambda(z)\left(\sum_{j=1}^{3}|\rho_{z_j}|^2\right)^{-2}, \qquad z\in\mathcal{S}$$

where $\lambda(z)$ is given by $\partial\rho\wedge\overline{\partial\rho}\wedge\partial\overline{\partial}\rho=\lambda(z)\,dz_1\wedge d\overline{z_1}\wedge dz_2\wedge d\overline{z_2}$.

Of particular interest is the case $k\equiv 0$. As has already been pointed out in [7], the «natural solutions» for this kind of problem seems to be the «weak ones» (*i.e.* in the sense of «viscosity»).

In our case we first must generalize the notion of Levi-convexity to a C^0 function u: we say that u is *Levi-convex* at x^0 if $LMA(\varphi)(x^0)\geq 0$, $A_{\alpha\bar\alpha}(\varphi)(x^0)\geq 0$, $\alpha=1,2$ whenever $u-\varphi$ has a local maximum at x^0 and $\varphi\in C^\infty(\Omega)$. Under these assumptions we say that $u\in C^0(\Omega)$ is a *weak subsolution* of (2) if $u=g$ on $\partial\Omega$ and $LMA(\varphi)(x^0)\geq k(x^0,u(x^0))\cdot(1+|D\varphi(x^0)|^2)^2$, $A_{\alpha\bar\alpha}(\varphi)(x^0)\geq 0$, $\alpha=1,2$, at any local maximum point x^0 of $u-\varphi$, for $\varphi\in C^\infty(\Omega)$; u is a *weak supersolution* if $u=g$ on $\partial\Omega$ and either $LMA(\varphi)(x^0)\leq k(x^0,u(x^0))\cdot(1+|D\varphi(x^0)|^2)^2$ or $LMA(\varphi)(x^0)>k(x^0,u(x^0))(1+|D\varphi(x^0)|^2)^2$ and $A_{\alpha\bar\alpha}(\varphi)(x^0)<0$, $\alpha=1,2$ at any local minimum point x^0 of $u-\varphi$, for $\varphi\in C^\infty(\Omega)$. Finally we say that u is a *weak solution* if it is both a weak subsolution and a weak supersolution.

PROPOSITION 1. *If $u\in C^0(\overline{\Omega})$ is a weak solution of* (2) *with $k=0$, then $\Gamma_+(u)$, $\Gamma_-(u)$ are not strictly pseudoconvex at any point of $\Gamma(u)$.*

We recall that a domain $D\neq\mathbf{C}^n$, $n\geq 2$ is called *strictly pseudoconvex* (s.p.c.) at a point $z^0\in\partial D$ if there exists a neighbourhood U of z^0 and $\Psi\in C^\infty(U)$ which is strictly plurisubharmonic on U, such that: $\Psi(z^0)=0$ and $\Psi<0$ on $U\cap D$.

2. Problem (2) can be reduced to a «Bellman problem» for a family of quasi-linear degenerate elliptic operators. Indeed if $u\in C^2(\Omega)$ is Levi-convex and $A(u)=(A_{\alpha\bar\beta}(u))$ then $\inf\limits_{B\in V}\operatorname{Tr}B\cdot A(u)=(LMA(u))^{1/2}$, where V is the space of hermitian, positive definite 2×2 matrices with $\det B=1/4$ (see [3]).

PROPOSITION 2. *If $u\in C^0(\Omega)$ is a weak solution of* $(2')$: $\inf\limits_{B\in V}\operatorname{Tr}B\cdot A(u)=h(\cdot,u)\cdot(1+|Du|^2)$ *in Ω, $u=g$ on $\partial\Omega$, $h=k^{1/2}$, then u is a weak solution of* (2).

In what follows we shall sketch the main steps of the proof of the existence theorem for $(2')$.

Fix a countable everywhere dense subset $\{B_m\}$ in V and set $L_m(u)=(1+|Du|^2)^{-1}\operatorname{Tr}B_m\cdot A(u)$, $F_m(u)=\inf(L_1(u),\dots,L_m(u))$. As F_m does not depend smoothly on the first and second derivatives u_i, u_{ij} of u, we provide a «good approximation» of $F_m(u)$ by an operator $F_{m,\eta}(u)$, which is C^∞ in u_i, u_{ij}, and then let $\eta\to 0^+$. So we consider the approximate problem:

$$F_{m,\eta}(u)+\varepsilon\Delta u=h(\cdot,u)\quad\text{in }\Omega, \tag{3}$$

$u = g$ on $\partial\Omega$ and (under suitable conditions for $\partial\Omega$, g, h) we prove for (3) the following «a priori» estimates:

(*A*) $\max_{\overline{\Omega}} |u|$, $\max_{\overline{\Omega}} |Du| <$ const uniformly with respect to ε, η, m (*i.e.* for ε, $\eta \to 0^+$, $m \to +\infty$);

(*B*) $|u|^*_{2,\alpha;\Omega} <$ const uniformly with respect to η, m.

Then starting from (*A*), (*B*) and using the «continuity methods» (see [4]) it is possible to prove for (2′) (and consequently for (2)) the existence of a weak solution u under the following conditions:

1) $\mathcal{S} = \partial\Omega \times \mathbf{R}$ is strictly Levi-convex, $k \in C^{1,\alpha}(\overline{\Omega} \times \mathbf{R})$, $g \in C^{2,\alpha}(\partial\Omega)$, $0 < \alpha < 1$, and $\sup_{t \in \mathbf{R}} k(\cdot, t) < k_{\mathrm{Levi}}(\mathcal{S})$ at each point of $\partial\Omega \times \{0\}$;
2) $\partial k/\partial t + |D_x k| \leq 0$, $D_x(\partial/\partial x_1, \ldots, \partial/\partial x_5)$;
3) $|k(x,t)| \leq \mu(|t|)$; $|k_x(x,t)|$, $|k_t(x,t)| \leq \tilde{\mu}(|t|)$; $|k_{xx}(x,t)|$, $|k_{x,t}(x,t)|$, $|k_{tt}(x,t)| \leq \tilde{\mu}(|t|)$

where $\mu, \tilde{\mu}$ are non decreasing (here $k_{xx}(x,t) = (k_{x_i x_j}(x,t))$, $k_{xt}(x,t) = (k_{x_i t}(x,t))$).

References

[1] E. Bedford - B. Gaveau, *Envelopes of holomorphy of certain 2-spheres in* $\mathbf{C}^2$. Amer. J. Math., 105, 1983, 975-1009.

[2] E. Bedford - W. Klingenberg, *On the envelope of holomorphy of a 2-sphere in* $\mathbf{C}^2$. To appear.

[3] B. Gaveau, *Méthodes de contrôle optimal en analyse complexe*. I. *Résolution d'équations de Monge-Ampère*. J. Funct. Anal., 25, 1977, 391-411.

[4] D. Gilbarg - N. S. Trudinger, *Elliptic partial differential equations of second order*. Grundlehren Math. Wiss., 224, Springer, 1983.

[5] G. Lupacciolu, *A Theorem on holomorphic extension of CR-functions*. Pacific J. Math., vol. 124, n. 1, 1986, 177-191.

[6] Z. Slodkowski, *Local maximum property and q-plurisubharmonic functions in uniform algebras*. J. Math. Anal. Appl., 115, 1986, 105-130.

[7] Z. Slodkowski - G. Tomassini, *Weak solutions for the Levi-equation and envelope of holomorphy*. J. Funct. Anal., to appear.

[8] G. Tomassini, *Geometric properties of solutions of the Levi equation*. Ann Mat. Pura Appl., (IV), vol. 152, 1988, 331-344.

Z. Slodkowski: University of Illinois at Chicago
Department of Mathematics
Chicago, Ill. (U.S.A.)

G. Tomassini: Scuola Normale Superiore
Piazza dei Cavalieri, 7 - 56126 Pisa