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## A sufficient condition for a polynomial centre to be global

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**Equazioni differenziali ordinarie.** — *A sufficient condition for a polynomial centre to be global.* Nota di Marco Sabatini, presentata (*) dal Corrisp. R. Conti.

Abstract. — A sufficient condition is given in order that a centre of a polynomial planar autonomous system be a global centre.

Key words: Planar autonomous systems; Singular points; Centres.

Riassunto. — *Una condizione sufficiente perché un centro polinomiale sia globale.* Per il sistema autonomo differenziale $(S)$ del testo si danno condizioni sufficienti affinché l'origine $O$ sia un centro globale.

## Introduction

Let us consider an autonomous differential system in the plane

$$(S) \qquad \begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases}$$

with $(x,y) \in \boldsymbol{R}^2$, and $P, Q$ polynomials. An isolated critical point $O$ of $(S)$ is said to be a centre if every orbit in a neighbourhood of $O$ is a cycle. It is said to be a global centre if every orbit different from $O$ is a nontrivial cycle.

The problem of determining whether a critical point is a centre or not has been studied by several authors (see [3, 9, 12]). A general method was developed by Poincaré and Liapunov, for analytic systems of the type

$$(S_L) \qquad \begin{cases} \dot{x} = y + p(x,y) \\ \dot{y} = -x + q(x,y) \end{cases}$$

where $p(x,y), q(x,y) = o\left(\sqrt{x^2+y^2}\right)$. Other ways to prove that $O$ is a centre, without assuming that the linear part of the field is nondegenerate, rely on the possibility to find a first integral (when, for instance, the divergence of the field is zero), or on symmetry arguments.

The related question of determining the extension of the region covered by cycles surrounding the centre has been less studied. Depending on what method has been used to prove that $O$ is a centre, this can lead to estimate the radius of convergence of a power series (Poincaré-Liapunov method), or to determine what solutions rotate around $O$. Some additional qualitative information can be given for some classes of systems. For instance, it is known [5] that polynomial systems of even degree cannot have global centres.

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In this paper, we give a sufficient condition for a centre to be global. We show that if Poincaré's extension of $(S)$ to the two-dimensional sphere has no critical points but $O$ and its antipodal, then $O$ is a global centre. Verifying the given condition amounts to prove that the algebraic curves $P(x,y)=0$ and $Q(x,y)=0$ do not meet at a point different from $O$, and that $xQ(x,y)-yP(x,y)=0$ has no real points at infinity. In some cases the latter condition absorbs the former one, as in the case of cubic systems without quadratic terms, whose centres are known.

## Global centres

In what follows, we consider autonomous differential systems in the plane

$$(S)\qquad \begin{cases} \dot{x}=P(x,y)\\ \dot{y}=Q(x,y)\end{cases}$$

where $P$ and $Q$ are real polynomials of degree $\leqslant n$:

$$P(x,y):=\sum_{j=0}^{n} p_j(x,y)=\sum_{j=0}^{n}\sum_{h=0}^{j} a_{hj}\,x^h\,y^{j-h}\,;\qquad Q(x,y):=\sum_{j=0}^{n} q_j(x,y)=\sum_{j=0}^{n}\sum_{h=0}^{j} b_{hj}\,x^h\,y^{j-h}\,.$$

Here, $p_j$ and $q_j$ are homogeneous polynomials of degree j. We assume that $p_n^2+q_n^2\not\equiv 0$, and that $P$ and $Q$ have no non-constant common factors. We also assume that $O$ is a centre of $(S)$. We call $N_O$ the largest connected region covered with nontrivial cycles surrounding $O$.

As usual, in order to study the behaviour of the solutions of $(S)$ at infinity, we associate to $(S)$ a homogeneous polynomial system in $\boldsymbol{R}^3$, obtained by means of a radial projection and a reparametrization:

$$(S_3)\qquad \begin{cases} \dot{u}=(v^2+z^2)\,\tilde{P}(u,v,z)-uv\tilde{Q}(u,v,z)\,,\\ \dot{v}=-uv\tilde{P}(u,v,z)+(u^2+z^2)\,\tilde{Q}(u,v,z)\,,\\ \dot{z}=-z(u\tilde{P}(u,v,z)+v\tilde{Q}(u,v,z))\,.\end{cases}$$

Here, $\tilde{P}(u,v,z):=z^n P(u/z,v/z)$, $\tilde{Q}(u,v,z):=z^n Q(u/z,v/z)$. The variables $x,y$ and $u,v,z$ are related by the equalities $u=xz$, $v=yz$. The function $u^2+v^2+z^2$ is a first integral of $(S_3)$, hence any sphere centered at $(0,0,0)$ is an invariant set for $(S_3)$. The restriction $(\tilde{S})$ of $(S_3)$ to one of such spheres is called Poincaré extension of $(S)$.

We set $\Sigma:=\{(u,v,z)\in\boldsymbol{R}^3: u^2+v^2+z^2=1\}$; $\Sigma^+:=\{(u,v,z)\in\Sigma: z\geqslant 0\}$; $\Sigma^-:=\{(u,v,z)\in\Sigma: z\leqslant 0\}$; $\Sigma_\infty:=\Sigma^+\cap\Sigma^-$.

Any point $P$ of the plane is associated to a couple of antipodal points $P^+$, $P^-$, on $\Sigma^+,\Sigma^-$, respectively. Points on $\Sigma_\infty$ are called points at infinity of $(S)$, and $\Sigma_\infty$ the line at infinity. If $P\in\Sigma_\infty$ is a critical point of $(\tilde{S})$, we say that it is a critical point at infinity of $(S)$. Critical points at infinity of $(S)$ have coordinates $(x_\infty, y_\infty, 0)$ where $(x_\infty, y_\infty)$ is a zero of the homogeneous polynomial $H_n:=xq_n-yp_n$.

THEOREM. *If $O$ is a centre of $(S)$, and $(\tilde{S})$ has no critical points other than $O^+$ and $O^-$, then $O$ is a global centre.*

PROOF. The line at infinity is a nontrivial cycle of $(\tilde{S})$. The associated Poincaré map is analytic, so that only two possibilities are allowed:

i) $\Sigma_\infty$ is a limit cycle;

ii) $\Sigma_\infty$ has a neighbourhod $U_\infty$ filled with cycles.

In case *i*), $(S)$ has a negatively (positively) bounded orbit $\gamma$. Its negative limit set $\alpha(\gamma)$ is nonempty. It cannot contain the origin, since $O$ is a centre. Hence $\alpha(\gamma)$ is a cycle $\Gamma$, containing the (unique) critical point $O$ in its interior. Hence $N_O$ and its boundary $\partial N_O$ are bounded. This implies (see [10, §3]) that $\partial N_O$ contains a critical point, contradicting the hypothesis.

Also in case *ii*) $\partial N_O$ should be bounded, contained in the interior of all the orbits in $U_\infty$, unless $\partial N_O = \emptyset$. This gives the thesis. □

REMARK 1. The above condition is not necessary. The origin is a global centre for the system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x^3 \end{cases}$$

that has critical points at infinity at $(0,1,0)$ and $(0,-1,0)$ (cfr. [3]).

A necessary condition for the existence of a global centre is that $H_n$ does not change sign (see [5]). Something more can be said, comparing the rotation of orbits close to the critical point to that of orbits close to $(\Sigma_\infty)$.

We say that two polynomials $p(x,y)$, $q(x,y)$ have the same sign if there are no points $(x_p, y_p)$, $(x_q, y_q)$ such that $p(x_p, y_p)\, q(x_q, y_q) < 0$. Let us set $H_j := xq_j - yp_j$. Let $m$ be the least $j$ for which $H_j \not\equiv 0$ and $M$ be the largest $j$ such that $H_j \not\equiv 0$. If $O$ is a global centre, both have to be odd, and $M = n$ (see [5, Th. 2.9]).

LEMMA. *If $O$ is a global centre of $(S_n)$, then $H_m$ and $H_M$ have the same sign.*

PROOF. $H_m$ cannot change sign, otherwise $O$ would be the limit set of a nontrivial orbit of $(S_n)$ (see [6], Ch. 4). The same holds for $H_M$, otherwise $\Sigma_\infty$ would contain a critical point $P_\infty$, limit set of an orbit of $(\tilde{S})$ (see [5]).

By contradiction, let us assume that $H_m \leqslant 0$, $H_M \geqslant 0$. Then there exists a neighbourhood $U_O$ of $O$, such that any orbit $\gamma \subset U_O$ rotates clockwise. Also, there exists a compact $K$ such that any orbit $\Gamma$ not intersecting $K$ rotates counterclockwise. The annular region bounded by $\gamma$ and $\Gamma$ is positively invariant for the orthogonal system

$$(S^\perp) \qquad \begin{cases} \dot{x} = -Q(x,y)\,, \\ \dot{y} = P(x,y)\,. \end{cases}$$

Any orbit $\delta$ of $(S^{\perp})$ starting at a point of $\gamma$ has non-empty $\omega$-limit set $\omega(\delta)$, that has to be a cycle, since $O$ is the unique critical point of $(S)$ and $(S^{\perp})$. But a cycle of $(S^{\perp})$ cannot intersect a cycle of $(S)$, hence $O$ is not a global centre. $\square$

Nonhomogeneous polynomial systems of least degree having global centres are cubic ones. The above result can be applied to a class of cubic systems whose centres have been studied by several authors (see [2] for a review of the results on this subject). We assume that the quadratic part of the vector field is identically zero, and that $p_1^2 + q_1^2 \not\equiv 0$, $p_3^2 + q_3^2 \not\equiv 0$:

$$(S_3) \qquad \begin{cases} \dot{x} = P(x,y) := p_1(x,y) + p_3(x,y), \\ \dot{y} = Q(x,y) := q_1(x,y) + q_3(x,y). \end{cases}$$

If such a system has a centre at the origin, it can be transformed into a system of the type

$$(S_\varepsilon) \qquad \begin{cases} \dot{x} = y + Ax^3 + Bx^2y + Cxy^2 + Dy^3 \\ \dot{y} = -\varepsilon x + Kx^3 + Lx^2y + Mxy^2 + Ny^3 \end{cases}$$

with $\varepsilon = 1$, if the linear map $(x,y) \to (p_1(x,y), q_1(x,y))$ is nondegenerate, $\varepsilon = 0$, if it is degenerate. Systems of this type having a centre at the origin have been characterized by means of algebraic relations in the coefficients of $p_3$, $q_3$ (see [1,2,7,8,11]).

COROLLARY 1. *If $O$ is a centre of $(S_\varepsilon)$ and the polynomial $H_3$ is definite negative, then $O$ is a global centre. If $H_3$ has a positive value, then $O$ is not a global centre.*

PROOF. Let $H_3$ be negative. Since $(S_\varepsilon)$ has no critical points at infinity, it is sufficient to prove that $(S_\varepsilon)$ cannot have critical points. We have $W(x,y) := (x,y) \wedge (\dot{x},\dot{y}) = -y^2 - \varepsilon x^2 + H_3(x,y)$. Then $W(x,y)$ vanishes only at the origin, so that $(S_\varepsilon)$ has a unique critical point.

If $H_3$ has a positive value, by the previous lemma $O$ cannot be a global centre. $\square$

REMARK 2. If $\varepsilon = 0$, and $H_3$ has no zeroes, then it has to be negative, since one of the conditions for the existence of a centre at $O$ is $K<0$ (see [2, case $(l_2)$, $(q_1)$, $(c)$]).

If $\varepsilon = 1$, the conditions for $O$ to be a centre are homogeneous in the coefficients of $p_3$, $q_3$. Hence the systems

$$(S_\pm) \qquad \begin{cases} \dot{x} = P_\pm(x,y) := y \pm p_3(x,y), \\ \dot{y} = Q_\pm(x,y) := -x \pm q_3(x,y), \end{cases}$$

have both a centre at the origin. By the previous Corollary, at least one of them is not a global centre.

The above Corollary can be easily extended to a class of polynomial systems of odd degree.

COROLLARY 2. *If O is a centre, the polynomials* $H_j, j=1,\ldots,n$ *have the same sign, and* $H_n$ *has no zeroes, then O is a global centre.*

It is easy to see that in this case $H_{2h} \equiv 0$ for any $h>0$.

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