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Extension of a regularity result concerning the dam problem

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Equazioni a derivate parziali. — *Extension of a regularity result concerning the dam problem.* Nota di Gianni Gilardi e Stephan Luckhaus, presentata (*) dal Socio E. Magenes.

Abstract. — One proves, in the case of piecewise smooth coefficients, that the time derivative of the solution of the so called dam problem is a measure, extending the result proved by the same authors in the case of Lipschitz continuous coefficients.

Key words: Regularity; Porous media; Non negative subharmonic functions.

Riassunto. — *Estensione di un risultato di regolarità sul problema della diga.* Si dimostra, nel caso di coefficienti regolari a tratti, che la derivata rispetto alla variabile temporale della soluzione del cosiddetto problema della diga è una misura, estendendo il risultato che gli stessi autori hanno già dimostrato nel caso di coefficienti lipschitziani.

Introduction

We deal with the solution to the dam problem, that is

$$u \in L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \chi \in L^\infty(Q); \tag{0.1}$$

$$u \geqslant 0, \ \ 0 \leqslant \chi \leqslant 1, \ \ u(1-\chi) = 0 \quad \text{in } Q; \tag{0.2}$$

$$u = g \quad \text{on } \Sigma_D; \tag{0.3}$$

$$\int_Q (-\chi \partial_t v + a(\nabla u + \chi e) \cdot \nabla v) \leqslant 0 \tag{0.4}$$

for every $v \in H^1(Q)$ such that

$$v \geqslant 0 \text{ on } \Sigma_D, \qquad v = 0 \text{ on } \Sigma_D \cap \{g > 0\}, \qquad v(\cdot, 0) = v(\cdot, T) = 0 \text{ in } \Omega; \tag{0.5}$$

$$\chi(\cdot, 0) = \chi^0 \quad \text{in } \Omega. \quad \square \tag{0.6}$$

Here Ω is a connected bounded open set in $\boldsymbol{R}^n$ with Lipschitz boundary and represents the porous medium. The boundary $\partial\Omega$ consists of the pervious part Γ_D and the impervious part Γ_N, whose closures are $C^{1,1}$ manifolds intersecting in a smooth $(n-2)$-dimensional submanifold of $\partial\Omega$. $\boldsymbol{a} = (a_{ij})$ is the permeability matrix and $\boldsymbol{e} \in \boldsymbol{R}^n$ is a given unit vector, which takes gravity into account in the physical model. The function u represents the unknown pressure and the velocity is given by $-\boldsymbol{a}(\nabla u + \chi \boldsymbol{e})$ by Darcy's law. Finally the following notations have been used

$$Q = \Omega \times]0, T[\quad \text{and} \quad \Sigma_D = \Gamma_D \times]0, T[. \tag{0.7}$$

We assume the following regularity for the data

$$g \in C^{0,1}(\boldsymbol{R}^{n+1}), \qquad g \geqslant 0, \qquad \chi^0 \in L^\infty(\Omega), \qquad 0 \leqslant \chi^0 \leqslant 1. \quad \square \tag{0.8}$$

This problem has been studied by several authors from different points of view:

(*) Nella seduta del 20 aprile 1991.

results deal mainly with the stationary case, but the evolution problem is also considered (see [1, 3-6, 8-11, 13, 14]; see also the lists of references of [2, 5, 7, 12]).

In [11] it is proved that $\partial_t u$ is a measure whose negative part is a function, under the assumption that $\boldsymbol{a}$ is a uniformly elliptic matrix with Lipschitz continuous coefficients. Here we extend that result to the case of piecewise smooth coefficients, in order to cover the case of layered materials. More precisely we assume

$$a_{ij} \in L^\infty(\Omega), \qquad |a_{ij}| \leq 1, \qquad \sum_{ij} a_{ij}(x)\,\xi_i\,\xi_j \geq \lambda|\xi|^2 \quad \text{a.e. in } \Omega, \ \forall \xi \in \boldsymbol{R}^n, \tag{0.9}$$

$$\operatorname{div}(\boldsymbol{ae}) \geq 0 \ \text{in } \Omega \qquad \text{and} \qquad \boldsymbol{ae}\cdot\boldsymbol{v} < 0 \ \text{on } \Gamma_N, \tag{0.10}$$

where $\lambda > 0$ is fixed, and $\mathbf{v}$ is the outer unit normal vector to $\partial\Omega$. Moreover we assume the following regularity condition: there exists a smooth manifold with boundary, whose interior Γ_0 lies in Ω, such that $\partial\Gamma_0 \subset \partial\Omega$ and

$$a_{ij} \in C^{0,1}(\overline{\Omega_k}) \ \forall i,j,k \qquad \text{and} \qquad \boldsymbol{ae}\cdot\boldsymbol{v} \neq 0 \ \text{on } \overline{\Gamma_0} \tag{0.11}$$

where Ω_k are the connected components of $\Omega \setminus \Gamma_0$, which we assume to be $C^{0,1}$.

Notice that no more than two $\overline{\Omega_k}$'s can have nonempty intersection. For technical reasons we assume that $\boldsymbol{a}$ is continuous in a neighbourhood of Γ_N.

1. Known results for smooth $\boldsymbol{a}$

In the paper [11] the idea of the proof was to use an inequality from below on the term $\operatorname{div}(\boldsymbol{a}(\nabla u + \boldsymbol{e}))$ in terms of the measure of the set where $u = 0$. We give a few of the inequalities from this paper without proof.

We work on sets of the type described in the following definition.

1.1. Definition. *Assume $\Lambda > 0$ and $\rho > 0$. A set $\mathcal{O} \subset \boldsymbol{R}^n$ is said to be a Λ, ρ-set if there exists a connected set $\mathcal{U}$ such that*

$$\operatorname{diam} \mathcal{U} \leq \Lambda\rho; \tag{1.1}$$

$$\mathcal{O} = B_\rho(\mathcal{U}), \qquad i.e.\ \ \mathcal{O} = \bigcup_{x\in\mathcal{U}} B_\rho(x). \tag{1.2}$$

A connected set $\mathcal{U}$ satisfying (1.1) *and* (1.2) *is called a core of $\mathcal{O}$.* □

Assuming that $\boldsymbol{a}$ is a uniformly elliptic matrix with L^∞ coefficients, for these sets the Harnack inequality holds. Moreover, if $\boldsymbol{a}$ is globally $C^{0,1}$, the Hopf maximum principle holds too.

The first Lemma holds for L^∞ coefficients (see [11], Lemma 1.5):

1.2. Lemma. *Let $\mathcal{O}$ to be a Λ, ρ-set with core $\mathcal{U}$. Suppose that $u \in H^1(\mathcal{O})$ is nonnegative and satisfies*

$$\operatorname{div}(\boldsymbol{a}(\nabla u + \boldsymbol{e})) \geq 0 \quad in\ \mathcal{O}, \tag{1.3}$$

and that $w \in u + H^1(\mathcal{O})$ satisfies

$$\operatorname{div}(\boldsymbol{a}(\nabla w + \boldsymbol{e})) = 0 \quad in\ \mathcal{O}. \tag{1.4}$$

Then for any $\beta \geqslant 0$ *we have*

$$\int_{\mathcal{O}} \operatorname{div}(\boldsymbol{a}(\nabla u + \boldsymbol{e})) \geqslant c_1\, \beta |\{w - u \geqslant \beta\} \cap \mathcal{O}|^{1-2/n} \tag{1.5}$$

where c_1 *depends only on* λ.

Furthermore, if for some $x \in \mathcal{U}$

$$\int_{B_{\rho/2}(x)} u \leqslant \frac{1}{2} \int_{B_{\rho/2}(x)} w \tag{1.6}$$

then, for any $y \in \mathcal{U}$ *and every* $\rho' \leqslant \rho$, *we have the estimate*

$$\int_{\mathcal{O}} \operatorname{div}(\boldsymbol{a}(\nabla u + \boldsymbol{e})) \geqslant c_2 \left(\fint_{B_{\rho'/2}(y)} u - c_3 \rho \right) \rho^{n-2}, \tag{1.7}$$

where c_2 *and* c_3 *depend only on* λ *and* Λ. □

Using 1.2. and also the Hopf maximum principle, the main results for «subharmonic» functions are (see [11], Lemmas 1.9 and 1.12):

1.3. Main Lemma. *Assume* $\mathcal{O}$ *to be a* Λ, ρ*-set, with* $\rho \leqslant 1$ *and core* $\mathcal{U}$, *and* $\boldsymbol{a}$ *to be Lipschitz continuous. If* $u \in H^1(\mathcal{O})$ *is nonnegative and satisfies* (1.3), *then, for every* $x \in \mathcal{U}$ *and* $\rho' \leqslant \rho$, *we have the estimate*

$$\int_{\mathcal{O}} \operatorname{div}(\boldsymbol{a}(\nabla u + \boldsymbol{e})) \geqslant c_1 \left(\frac{1}{\rho} \fint_{B_{\rho'/2}(x)} u - c_2 \right)^+ |\{u = 0\} \cap \mathcal{O}|^{1-1/n} \tag{1.8}$$

where c_1 *and* c_2 *depend only on* λ, Λ, *and the norm of* $\boldsymbol{a}$ *in* $C^{0,1}$. □

1.4. Main Lemma (for Neumann boundary points). *Assume that* $\boldsymbol{a}$ *is Lipschitz continuous and* $x_0 \in \Gamma_N$, *and let* $\mathcal{O}$ *be a* Λ, ρ*-set with core* $\mathcal{U}$ *such that*

$$B_\rho(x_0) \cap \Omega \subseteq \mathcal{O} \subseteq B_{\Lambda\rho}(x_0) \cap \Omega; \tag{1.9}$$

$$\partial\mathcal{O} \cap \partial\Omega \subseteq \Gamma_N. \tag{1.10}$$

Moreover assume that $u \in H^1(\Omega)$ *is non negative and satisfies* (1.3) *and* (*in a suitable sense, see* [11])

$$|\boldsymbol{a}\nabla u \cdot \boldsymbol{\nu}| \leqslant K \quad \text{on } \partial\mathcal{O} \cap \partial\Omega. \tag{1.11}$$

Then for any $x \in B_{\rho/2}(x_0) \cap \Omega$ *and any* $\rho' < \rho/2$ *we have*

$$\int_{\mathcal{O}} \operatorname{div}(\boldsymbol{a}(\nabla u + \boldsymbol{e})) \geqslant c_1 \left(\frac{1}{\rho} \fint_{B_{\rho'}(x)} u - c_2 (1 + K) \right) \cdot |\{u = 0\} \cap \mathcal{O}|^{1-1/n} \tag{1.12}$$

where c_1 *and* c_2 *depend only on* λ, Λ, Ω, *and the norm of* $\boldsymbol{a}$ *in* $C^{0,1}$. □

From Lemma 1.10 and Remark 1.11 of [11] we derive the following:

1.5. MAIN LEMMA (for DIRICHLET BOUNDARY POINTS). *Assume that* $\boldsymbol{a}$ *is Lipschitz continuous and* $x_0 \in \Gamma_D$, *and let* $\tilde{\mathcal{O}}$ *be a* Λ, ρ*-set whose core* $\mathcal{U}$ *is contained in* Ω. *We set* $\mathcal{O} = \Omega \cap \tilde{\mathcal{O}}$ *and assume*

$$B_\rho(x_0) \cap \Omega \subseteq \mathcal{O} \subseteq B_{\Lambda\rho}(x_0) \cap \Omega; \tag{1.13}$$

$$|\boldsymbol{v}(x) - \boldsymbol{v}(x')| \leq \Lambda |x - x'|/\rho \qquad \text{for } x, x' \in \Omega \cap \partial\mathcal{O}. \tag{1.14}$$

Finally let $u \in H^1(\Omega)$ *be nonnegative and satisfy* (1.3). *Then*

$$\int_{\mathcal{O}} \operatorname{div}(\boldsymbol{a}(\nabla u + \boldsymbol{e})) \geq c_1 \left(\frac{1}{\rho} \int_{B_\rho(x_0) \cap \partial\Omega} u \, d\mathcal{H}^{n-1} - c_2 \right) \cdot |\{u = 0\} \cap \mathcal{O}|^{1-1/n}; \tag{1.15}$$

$$\int_{\mathcal{O}} \operatorname{div}(\boldsymbol{a}(\nabla u + \boldsymbol{e})) \geq c_3 \left(\frac{1}{\rho} \int_{B_\rho(x_0) \cap \partial\Omega} u \, d\mathcal{H}^{n-1} - c_4 \right) \cdot \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} |A_\varepsilon| \tag{1.16}$$

where $A_\varepsilon = \{d(x, \partial\mathcal{O}) < \varepsilon\} \cap \{u = 0\}$ *and* $c_1, \dots, c_4$ *depend only on* λ, Λ, *and the norm of* $\boldsymbol{a}$ *in* $C^{0,1}$. □

We can apply these results because the following statements hold for the solution (u, χ) of the dam problem with L^∞ coefficients (see [11], Proposition 2.2 and Lemma 2.4):

1.6. PROPOSITION. *The following inequalities hold*

$$\operatorname{div}(\boldsymbol{a}(\nabla u + \boldsymbol{e})) \geq 0 \qquad \text{and} \qquad \partial_t(1 - \chi) - \operatorname{div}((1 - \chi)\, \boldsymbol{a}\boldsymbol{e}) \leq 0. \quad \square \tag{1.17}$$

1.7. LEMMA. *Assume that* ψ *is the characteristic function of an open subset of* $\Omega \times \boldsymbol{R}$. *Assume moreover* $\psi \in BV(\boldsymbol{R}^{n+1})$ *and*

$$\partial_t \psi - \boldsymbol{a}\boldsymbol{e} \cdot \nabla\psi \leq 0 \qquad \text{and} \qquad 0 \leq \psi \leq \chi_{\Omega \times \boldsymbol{R}}. \tag{1.18}$$

Then we have

$$\partial_t \int_\Omega \psi(t)\,(1 - \chi(t)) \leq - \int_\Omega \operatorname{div}(\boldsymbol{a}(\nabla u(t) + \boldsymbol{e}))\, \psi(t). \tag{1.19}$$

In particular the left hand side of (1.19) *is nonpositive.* □

1.8. REMARK. The condition (1.18) can be replaced by the following one: (1.18) holds in an open set ω_1 of $\boldsymbol{R}^{n+1}$, $\chi = 1$ in an open set ω_2, and $\omega_1 \cup \omega_2 \supseteq \Omega \times \boldsymbol{R}$. □

2. The proof of the result

Here, as in [11], we use the following approximation of $\partial_t u$:

$$\partial_t^h u(x,t) = \frac{2}{3h}\left(u(x,t) - \frac{1}{h} \int_{t-2h}^{t-h} u(x,s)\, ds \right). \tag{2.1}$$

for $x \in \Omega$, $h > 0$, and $2h < t < T$.

In the whole section we use the assumptions and the notations of the introduction and denote by (u, χ) the solution of the dam problem (0.1),...,(0.6).

2.1. Theorem. *Suppose that $\mathcal{O}_t \subset \Omega$ is a Λ, ρ-set such that $a_{ij} \in C^{0,1}(\overline{\mathcal{O}}_t)$ and suppose that $\psi(\cdot, t) := \chi_{\mathcal{O}_t}$ satisfies (1.18). Assume $2h < t < T$. Then there exists a constant c_1, depending only on λ, Λ, $\|a\|_{C^{0,1}(\overline{\mathcal{O}}_t)}$, and ρ/h, such that either*

$$\fint_{B_{\rho'}(x)} \partial_t^h u(y,t)\, dy \geq -c_1 \quad \textit{for all} \quad B_{\rho'}(x) \subseteq B_{\rho/2}(x) \subseteq \bigcap_{t-2h<\tau<t} \mathcal{O}_\tau \quad \textit{or} \tag{2.2}$$

$$\chi(\cdot, \tau) = 1 \quad \textit{in } \mathcal{O}_\tau \quad \textit{for } t-h < \tau < t. \quad \square \tag{2.3}$$

Proof. The proof of a quite similar statement is given in [11]. Here we sketch the proof again. By 1.7 and 1.3 we get

$$\partial_s \int_{\mathcal{O}_s} (1 - \chi(s)) \leq - \int_{\mathcal{O}_s} \operatorname{div}\left(a(\nabla u(s) + e)\right) \leq$$

$$\leq -c_2 \left(\frac{1}{\rho} \fint_{B_{\rho'}(x)} u(y,s)\, dy - c_3 \right)^+ |\mathcal{O}_s \cap \{u(s) = 0\}|^{1-1/n}.$$

Let c_1 be any positive number and assume (2.2) to be false for some ball $B_{\rho'}(x)$: we prove that if c_1 is large enough then (2.3) holds. Our assumption implies

$$\fint_{B_{\rho'}(x)} \left(\int_{t-2h}^{t-h} u(y,s)\, ds \right) dy > 3\, c_1 h^2/2. \tag{2.4}$$

But setting $f(s) = \int_{\mathcal{O}_s} (1 - \chi(s))$, we have $f(s) \leq |\mathcal{O}_s \cap \{u(s) = 0\}|$; so the previous inequalities give

$$f'(s) \leq -c_2 \left(\frac{1}{\rho} \fint_{B_{\rho'}(x)} u(y,s)\, dy - c_3 \right)^+ (f(s))^{1-1/n}.$$

From this it follows that either $f(t-h) = 0$ or

$$f^{1/n}(t-2h) - f^{1/n}(t-h) \geq \frac{c_2}{n} \left(\frac{1}{\rho} \fint_{B_{\rho'}(x)} \left(\int_{t-2h}^{t-h} u(y,s)\, ds \right) dy - c_3 h \right).$$

Using (2.4) this gives $f^{1/n}(t-2h) \geq (c_2/n)\ [(3/2)\, c_1 h \rho^{-1} - c_3]\, h$. As $f^{1/n}(t-2h) \leq c_4 \rho$, this gives for c_1 an upper bound depending on ρ/h. So, if c_1 is large enough (depending on ρ/h), we have $f(t-2h)=0$ and the monotonicity of f gives the result. □

For points on Γ_0 we have to split the argument into two parts: «above» and «below» the interface.

2.2. Theorem. *Assume $2h<t<T$, $x_0 \in \Gamma_0$,* e.g. *$x_0 \in \overline{\Omega_1} \cap \overline{\Omega_2}$. Then there exists a constant $c \geq 1$ depending on the geometry such that, if $B_{c\rho}(x_0) \subset \overline{\Omega_1} \cap \overline{\Omega_2}$, then either*

$$\fint_{B_{\rho'}(x)} \partial_t^h u(y,t)\, dy \geq -c_1 \qquad \text{for all} \quad B_{\rho'}(x) \subset B_{\rho/2}(x_0) \ \text{or} \tag{2.5}$$

$$\chi(\cdot,\tau)=1 \quad \text{in} \quad B_\rho(x_0) \quad \text{for } t-h<\tau<t. \qquad \square$$

Proof. We can assume $\boldsymbol{a}(x_0)\, \boldsymbol{e}\cdot \boldsymbol{\nu} < 0$ where $\boldsymbol{\nu}$ is the outer normal at x_0 to $\partial\Omega_1$. For $t-2h<\tau<t$ let $\mathcal{O}(\tau)$ be a Λ,ρ-set whose core $\mathcal{U}(\tau)$ contains x_0 and satisfying (1.18) with $\psi(\cdot,\tau)=\chi_{\mathcal{O}(\tau)}$. Let also $\mathcal{O}_1(\tau)$ be a Λ,ρ-set whose core $\mathcal{U}_1(\tau)$ intersects $\mathcal{U}(\tau)$ and assume that also $\psi(\cdot,\tau)=\chi_{\mathcal{O}_1(\tau)}$ satisfies (1.18). Assume moreover that $\mathcal{O}_1(\tau)\subset\Omega_1$ for $t-2h<\tau<s$ and that $\mathcal{O}_1(\tau)$ contains a $\beta(\tau-s)$-neighbourhood of $\Gamma_0\cap\mathcal{O}(\tau)$ for $s<\tau<t-h$.

Here Λ depends essentially on n, on the norm of $\boldsymbol{a}$, and on the infimum of $\boldsymbol{ae}\cdot\boldsymbol{\nu}$ in a neighbourhood of x_0; the constant c must be larger than a uniform Λ; β obviously depends on $\inf \boldsymbol{ae}\cdot\boldsymbol{\nu}$; s will be chosen later.

Take $c_1>0$ and assume (2.5) to be false for some x and ρ' as in the statement. Thus (for some c_2 depending on Λ)

$$\int_{t-2h}^{t-h}\left(\fint_{\mathcal{O}(\tau)} u(y,\tau)\, dy\right) d\tau \geq c_2 c_1 h^2 .$$

Choose s such that

$$\int_{t-2h}^{s}\left(\fint_{\mathcal{O}(\tau)} u(y,\tau)\, dy\right) d\tau = c_2 c_1 h^2/2 . \tag{2.6}$$

Define $w(\cdot,\tau)\in u(\cdot,\tau)+H^1(\mathcal{O}(\tau))$ by $\operatorname{div}(\boldsymbol{a}(\nabla w(\cdot,\tau)+\boldsymbol{e}))=0$ in $\mathcal{O}(\tau)$, and call $\mathcal{T}$ the set of all $\tau\in]t-2h,t-h[$ such that

$$\left|\left\{x\in\mathcal{O}(\tau): w(x,\tau)-u(x,\tau)>\alpha \fint_{\mathcal{O}(\tau)} u(\tau)\right\}\right| > \alpha\rho^n$$

where α will be specified later. From (1.5) and (1.19) we get

$$c_3\,\alpha \int_{\mathcal{T}} \fint_{\mathcal{O}(\tau)} u(\tau)\cdot\alpha^{1-2/n}\rho^{n-2}\, d\tau \leq \int_{t-2h}^{t-h}\int_{\mathcal{O}(\tau)} \operatorname{div}(\boldsymbol{a}(\nabla u(\tau)+\boldsymbol{e}))\, d\tau \leq$$

$$\leq \int_{t-2h}^{t-h} \partial_\tau \int_{\mathcal{O}(\tau)} (1-\chi(\tau))\, d\tau \leq c_4\,\rho^n .$$

Hence

$$\int_{\mathcal{T}} \fint_{\mathcal{O}(\tau)} u(\tau)\, d\tau \leqslant c_5\, \alpha^{(2/n)-2} \rho^2$$

and, if $c_1 > (c_5/4c_2)\,\alpha^{(2/n)-2}(\rho/h)^2$, then

$$\int_{\mathcal{T}_i} \fint_{\mathcal{O}(\tau)} u(\tau)\, d\tau \geqslant \frac{c_2\, c_1}{4}\, h^2, \qquad i=1,2, \tag{2.7}$$

where $\mathcal{T}_1 =]t-2h, s[\setminus \mathcal{T}$ and $\mathcal{T}_2 =]s, t-h[\setminus \mathcal{T}$.

Suppose that y' is such that $B_{\rho/4}(y') \subset \mathcal{O}(\tau)$: then by Harnack's inequality

$$w(y',\tau) \geqslant c_6 \fint_{\mathcal{O}(\tau)} u(\tau) - c_7\, \rho\,.$$

Therefore, if $\alpha < c_6/2$, $\alpha < |B_{1/4}|$, $\tau \notin \mathcal{T}$, and $B_{\rho/2}(y) \subset \mathcal{O}(\tau)$, then there exists $y' \in B_{\rho/4}(y)$ such that

$$u(y',\tau) \geqslant w(y',\tau) - \alpha \fint_{\mathcal{O}(\tau)} u(\tau) \geqslant \frac{c_6}{2} \fint_{\mathcal{O}(\tau)} u(\tau) - c_7\, \rho\,.$$

Taking $y \in \mathcal{U}_1(\tau) \cap \mathcal{U}(\tau)$, by the Main Lemma 1.3 we obtain

$$\int_{\mathcal{O}_1(\tau)} \operatorname{div}\left(\boldsymbol{a}(\nabla u(\tau) + \boldsymbol{e})\right) \geqslant \left(\frac{c_8}{\rho} \fint_{\mathcal{O}(\tau)} u(\tau) - c_9\right)^+ |\{u(\tau)=0\} \cap \mathcal{O}_1(\tau)|^{1-1/n}\,.$$

As in the previous proof we derive the differential inequality

$$\partial_\tau \int_{\mathcal{O}_1(\tau)} (1-\chi(t)) \leqslant -\left(\frac{c_8}{\rho} \fint_{\mathcal{O}(\tau)} u(\tau) - c_9\right)^+ \left(\int_{\mathcal{O}_1(\tau)} (1-\chi(\tau))\right)^{1-1/n},$$

which gives as a consequence, after integration over $\mathcal{T}_1$, either

$$\frac{1}{n}|\mathcal{O}_1(t-2h)|^{1/n} \geqslant \int_{\mathcal{T}_1} \left(\frac{c_8}{\rho} \fint_{\mathcal{O}(\tau)} u(\tau) - c_9\right) d\tau \quad \text{or} \tag{2.8}$$

$$\int_{\mathcal{O}_1(\tau)} (1-\chi(\tau)) = 0 \qquad \text{for } s<\tau<t. \tag{2.9}$$

By (2.7), from (2.8) it follows that

$$c_{10}(\Lambda)\,\rho \geqslant c_{11}\, c_1\, h^2/\rho - c_9\, h \qquad i.e. \qquad c_1 \leqslant (\rho/h)(c_{12}\,\rho/h + c_{13})\,.$$

So, if c_1 is larger than the last number, then (2.9) holds, *i.e.* $\chi(\tau)=1$ in $\mathcal{O}_1(\tau)$ for $s<\tau<t$.

Now we construct $\mathcal{O}_2(\tau) \subset \Omega_2$ such that $\chi(\tau)=1$ in $\mathcal{O}_2(\tau)$ for $t-h<\tau<t$. A naive

construction of such an $\mathcal{O}_2(\tau)$ satisfying (1.18) will produce a gap between $\mathcal{O}_2(\tau)$ and $\mathcal{O}_1(\tau)$ for $\tau > s$, since β is small. This can be avoided by constructing $\mathcal{O}_2(\tau) \subset \Omega_2$ such that we have (1.18) outside of $\mathcal{O}_1(\tau)$ (with $\psi(\cdot,\tau) = \chi_{\mathcal{O}_2(\tau)}$).

There is a difficult in constructing the «upper» boundary of $\mathcal{O}_2(\tau)$: changing coordinates, fix $\boldsymbol{a}(x_0)\,\boldsymbol{e}$ to be parallel to the n-th axis and represent Γ_0 locally by $x_n = \varphi(x')$ (where $x = (x', x_n)$). Then give the «upper» boundary by $x_n = \Phi(x',\tau) := \varphi(x') - \rho^{-1}(|x' - x_0'| - c_{14}\,\rho + c_{15}\beta^{-1/2}(\rho(\tau - s))^{1/2})_+^2$. We have $\partial_\tau \Phi(x',\tau) \leqslant -c_{15}$ at those x' such that $\varphi(x') - \Phi(x',\tau) \geqslant \beta(\tau - s)$. So we choose c_{15} according to the velocity of the solutions of $\dot{x} = -\boldsymbol{a}(x)\,\boldsymbol{e}$, and c_{14} large enough, namely $c_{14} \geqslant c_{15}\beta^{-1/2}\cdot((2h/\rho)^{1/2} + 1)$. So, applying Remark 1.8, we conclude che proof. □

Now we deal with points on $\overline{\Gamma_0} \cap \Gamma_D$ and distinguish between Dirichlet boundary values which are larger or smaller than a number of order h.

2.3. Lemma. *Let $x \in \overline{\Gamma_0} \cap \Gamma_D$. Then for every $\alpha > 0$ there exists $\eta > 0$ such that if $u(x,t) > \eta h$ then $\chi = 1$ in $(\Omega \cap B_{\alpha h}(x)) \times]t - h, t + h[$ provided $2\alpha h < d(x, \Gamma_N)$.* □

Proof. Let Ω_1 and Ω_2 be the components of $\Omega \setminus \Gamma_0$ whose closures contain x and choose open subsets $\mathcal{O}_1 \subset \Omega_1$ and $\mathcal{O}_2 \subset \Omega_2$ satisfying the following conditions (where Λ will depend *e.g.* in the $C^{1,1}$ structure of Γ_0): $|\boldsymbol{v}(x) - \boldsymbol{v}(x')| \leqslant \Lambda(\alpha h)^{-1}|x - x'|$ $\forall x, x' \in \Omega_i \cap \partial\mathcal{O}_i$, $i = 1,2$; $B_{\alpha h}(x) \cap \Omega \subset \overline{\mathcal{O}_1} \cup \overline{\mathcal{O}_2} \subset B_{\Lambda\alpha h}(x)$; $\exists x_i \in \overline{\mathcal{O}_i} \cap \Gamma_D$: $B_{\alpha h/4}(x_i) \cap \Omega \subset \mathcal{O}_i$, $i = 1,2$. If η is large enough, $u(x_i,\tau) > \eta h/2$ for $|\tau - t| < 2h$. Then we can apply Lemma 1.5 with $\rho = \alpha h$ and deduce

$$\int_{\mathcal{O}_i} \operatorname{div}(\boldsymbol{a}(\nabla u(\tau) + \boldsymbol{e})) \geqslant c_1\left(\frac{1}{h}\,\underline{\inf}_{\mathcal{O}_i \cap \Gamma_D} u(\cdot,\tau) - c_2\right)|\{\chi(\cdot,\tau) < 1\} \cap \mathcal{O}_i|^{1-1/n}$$

$$\int_{\mathcal{O}_i} \operatorname{div}(\boldsymbol{a}(\nabla u(\tau) + \boldsymbol{e})) \geqslant c_3\left(\frac{1}{h}\,\underline{\inf}_{\mathcal{O}_i \cap \Gamma_D} u(\cdot,\tau) - c_4\right)\cdot \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon}\,|A_{i\varepsilon}(\tau)|$$

where $A_{i\varepsilon} = \{d(\cdot, \Omega \cap \partial\Omega_i) < \varepsilon\} \cap \{u(\tau) = 0\}$. Now we argue as in the smooth case and deduce

$$\partial_t \int_{\mathcal{O}_i} (1 - \chi) = -\int_{\mathcal{O}_i} \operatorname{div}(\boldsymbol{a}(\nabla u + \boldsymbol{e})) + \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\{d(\cdot, \Omega \cap \partial\mathcal{O}_i) < \varepsilon\}} \|\boldsymbol{a}\|_{L^\infty}\, \chi_{\{u = 0\}} \leqslant$$

$$\leqslant -c_5\left(\frac{1}{h}\,\underline{\inf}_{\mathcal{O}_i \cap \Gamma_D} u(\cdot,\tau) - c_6\right)\cdot\left(\int_{\mathcal{O}_i}(1-\chi)\right)^{1-1/n} \leqslant -c_7\left(\int_{\mathcal{O}_i}(1-\chi)\right)^{1-1/n}.$$

From this differential inequality we conclude the result as in [11]. □

As in [11] we can find a «parallel» boundary to $\partial\Omega \cap \{g < \beta h\}$ such that $\partial_t^h u$ is bounded there.

2.4. LEMMA. *Assume* $\alpha, \beta > 0$ *and define* $w \in H^1(\Omega)$ *by*

$$\operatorname{div}(\boldsymbol{a}(\nabla w + \boldsymbol{e})) = 0 \quad \text{in } \Omega; \qquad w = g \quad \text{on } \Gamma_D \quad \text{and} \quad \boldsymbol{a}(\nabla w + \boldsymbol{e}) \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_N.$$

Then there exists a smooth $\Omega_h \subset \Omega$ *such that* $d(\Omega_h, \{g \leqslant \beta h\}) > \alpha h$ *and*

$$\int_{\Omega \cap \partial \Omega_h} |w|^\varepsilon \leqslant ch^\varepsilon$$

for some $\varepsilon > 0$, *where* c *depends on* α *and* β. □

2.5. REMARK ON THE PROOF. The estimate follows from an estimate on $\partial_\nu w$. Near the points of $\overline{\Gamma_D} \cap \overline{\Gamma_0}$ $\partial_\nu w$ has a singularity which can be estimate by the ε-Hölder continuity of w. □

2.6. MAIN THEOREM. *Under the assumptions of the introduction, let* (u, χ) *be the solution to the dam problem. Then* $\partial_t u$ *is a measure on* $\Omega \times]0, +\infty[$ *whose negative part is a locally bounded function.*

More precisely, for every $T > 0$ *and* $\delta > 0$ *there exists a constant* c, *wich depends only on* T, δ, λ, *on the norm of* $\boldsymbol{a}$ *in* $C^{0,1}$ *of the components* Ω_k, *on the norm of* g *in* $C^{0,1}$, *and on the geometry, such that*

$$\partial_t u \geqslant -c \quad \text{in } Q(T, \delta), \quad \text{where} \tag{2.10}$$

$$Q(T, \delta) = \{(x, t) \in \Omega \times]0, T[: d(x, \Gamma_D \cap \{g < \delta\}) > \delta\}. \quad \Box \tag{2.11}$$

PROOF. Arguing as in [11], but using Theorems 2.1 and 2.2 in the interior, a similar statement for Γ_N (see [11] Lemma 2.9), and Lemma 2.3, we see that $w_h := (\partial_t^h u) \wedge (-c_1)$ fulfills

$$\begin{cases} \operatorname{div}(\boldsymbol{a} \nabla w_h) \leqslant 0 & \text{in } \Omega_h, \\ w_h = (\partial_t^h u) \wedge (-c_1) & \text{on } \Gamma_D \cap \partial \Omega_h(s), \\ \boldsymbol{a} \nabla w_h \cdot \boldsymbol{\nu} \geqslant 0 & \text{on } \Gamma_N, \end{cases}$$

where Ω_h is as in the Lemma 2.4. So one can estimate w_h from below by the solution of the corresponding equalities.

Since on $\Omega \cap \partial \Omega_h$ is estimated by $-w/h$ from below, with w given in Lemma 2.4, one finally gets the estimate

$$\int_{\Omega_h} (w_h^-)^\varepsilon \leqslant c_2$$

and, more precisely, w_h is bounded pointwise from below away from the set where $g \leqslant O(h)$. □.

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