Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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On motions with bursting characters for Lagrangian mechanical systems with a scalar control. I. Existence of a wide class of Lagrangian systems capable of motions with bursting characters

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **2** (1991), n.4, p. 339–343.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1991_9_2_4_339_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1991.

Rend. Mat. Acc. Lincei
s. 9, v. 2:339-343 (1991)

Meccanica. — *On motions with bursting characters for Lagrangian mechanical systems with a scalar control. I. Existence of a wide class of Lagrangian systems capable of motions with bursting characters.* Nota di ALDO BRESSAN e MARCO FAVRETTI, presentata (*) dal Corrisp. A. BRESSAN.

ABSTRACT. — In this *Note* (which will be followed by a second) we consider a Lagrangian system Σ (possibly without any Lagrangian function) referred to N + 1 coordinates $q_1, \ldots, q_N, u$, with u to be used as a control, and precisely to add to Σ a frictionless constraint of the type $u = u(t)$. Let Σ's (frictionless) constraints be represented by the manifold V_t generally moving in Hertz's space. We also consider an instant d (to be used for certain limit discontinuity-properties), a point $(\bar{q}, \bar{u})$ of V_d, a value $\bar{p}$ for Σ's momentum conjugate to q, and a continuous control $v(\cdot)$ with $v(d) = \bar{u}$. Furthermore zero is assumed not to equal a certain quantity determined by Σ's kinetic energy and Σ's applied forces, which forces are assumed to be at most linear in $\dot{u}$. A purely mathematical work of Favretti allows us to quickly show that (i) $v(\cdot)$ is the C^0-limit of a sequence $u_a(\cdot)$ of continuous controls that have a jump character in some interval $[d, d + \eta_a]$ and satisfy certain conditions including that both $\eta_a \to 0^+$ and $u_a(d + \eta_a) \to u_a(d) = v(d)$ as $a \to \infty$. Furthermore on the basis of that work we quickly prove that (ii) for every choice of the above sequence $u_a(\cdot)$, calling Σ_a the system Σ added with the frictionless constraint $u = u_a(t)$ and assuming $(\bar{q}, \bar{p})$ to be Σ_a's state at $t = d$, along Σ_a's subsequent motion we have that $q(t) \in B(\bar{q}, 1/a)$ $\forall t \in [d, d + \eta_a]$ and $\dot{q}(d + \eta_a) > a$. Thus, for values of $a (\in N)$ large enough, Σ_a's motion has bursting characters.

KEY WORDS: Lagrangian systems; Feedback theory; Bursts.

RIASSUNTO. — *Sui moti per sistemi Lagrangiani con controllo scalare, aventi caratteri di scoppio. I. Esistenza di una vasta classe di sistemi Lagrangiani capaci di moti con caratteri di scoppio.* In questa *Nota* (cui farà seguito una seconda) si considera un sistema Lagrangiano Σ (eventualmente privo di Lagrangiano) riferito a N + 1 coordinate $q_1, \ldots, q_N, u$, con u da usarsi come controllo e precisamente per aggiungere a Σ un vincolo liscio del tipo $u = u(t)$. I vincoli (lisci) di Σ sian rappresentati nello spazio di Hertz dalla varietà V_t (generalmente mobile). Si considera pure un istante d (da usarsi per certe «proprietà di discontinuità al limite»), un punto $(\bar{q}, \bar{u})$ di V_d, un valore $\bar{p}$ per il momento di Σ coniugato a q, e infine un controllo continuo $v(\cdot)$ con $v(d) = \bar{u}$. Inoltre si suppone $\neq 0$ una certa quantità determinata dall'energia cinetica e dalle forze attive di Σ, queste forze essendo supposte al più lineari in $\dot{u}$. Un lavoro puramente matematico di Favretti ci permette di mostrare rapidamente che (i) $v(\cdot)$ è il limite in C^0 di una sequenza $u_a(\cdot)$ di controlli continui che hanno carattere di salto e salto $j_a (= u_a(d + \eta_a) - \bar{u})$ in qualche intervallo $[d, d + \eta_a]$ e inoltre soddisfano certe condizioni, tra le quali che si abbia: $\eta_a \to 0^+$, $j_a \to 0$ e $u_a(d + \eta_a) \to u_a(d) = v(d)$ per $a \to \infty$. Inoltre sulla base di quel lavoro dimostriamo rapidamente che (ii) per ogni scelta della suddetta sequenza $u_a(\cdot)$, detto Σ_a il sistema Σ soggetto all'addizionale vincolo liscio $u = u_a(t)$ e supposto che a $t = d$ Σ_a sia nello stato $(\bar{q}, \bar{p})$, lungo il susseguente moto di Σ_a si ha che $q(t) \in B(\bar{q}, 1/a)$ $\forall t \in [d, d + \eta_a]$ e $\dot{q}(d + \eta_a) > a$. Così, per valori di $a (\in N)$ abbastanza alti, il moto di Σ_a ha carattere di scoppio.

1. INTRODUCTION

This paper is divided in Part I and Part II.

Let Σ be a Lagrangian (mechanical) system referred to the $N = \mathrm{N} + 1$ coordinates $(q_1, \ldots, q_N, u)$ and denote by $\Sigma_{u(\cdot)}$ the system obtained from Σ by *controllizing u, i.e.* by adding the frictionless constraint $u = u(t)$. In the present work one assumes that (i)

(*) Nella seduta del 14 giugno 1991.

$u \in KC^2$, so that $\Sigma_{u(\cdot)}$ can be discontinuous, and that (*ii*) $u(\cdot)$ is the L^1-limit of certain sequences $u_a = u_a(\cdot) \in C^0 \cap KC^2$ of (physically) implementable controls. The dynamical evolution of $\Sigma_{u(\cdot)}$ is described by a semi-Hamiltonian equation $SHE_{\Sigma, u(\cdot)}$, introduced in [2-4], by A. Bressan and used in [5] (also in a generalized version)

(α) for performing certain applications of feedback (or guidance) theory to Σ and

(β) for stating a general theory T of Σ's hyper-hympulsive motions in which, besides velocities, positions suffer first order discontinuities.

The $SHE_{\Sigma, u(\cdot)}$ together with typical initial conditions, has the form

$$\dot{z} = F[t, u(t), z, \dot{u}(t)], \qquad z(d) = (\bar{q}, \bar{p})(\in \mathbf{R}^{2N}), \tag{1.1}$$

where p_i is q_i's conjugate momentum ($i = 1, \ldots, N$). Let us add that two main results of [2-4] are, first, to state – see [2] – for which choices of Σ's coordinate system $(q_1, \ldots, q_N, u)$ (possibly with $u \in \mathbf{R}^M$ where $M \geqslant 1$) the applications referred to in (α) are possible; in more details one proves that *e.g.* Σ's coordinate u is *controllizable, i.e.* can be controllized in a certain satisfactory way *iff* $(1.1)_1$ *is linear in* $\dot{u}$. Second, *in case the Lagrangian components of* Σ*'s applied forces have a polynomial dependence on* $\dot{u}$*, equation* $(1.1)_1$ *is shown – see* [3] *– to be linear in* $\dot{u}$ *iff the hyper-hympulsive theory* T *referred to in* (β) *can be applied to* Σ *in a certain satisfactory way which requires the* (1^{st} *order) discontinuities of* $\Sigma_{u(\cdot)}$*'s positions* (*a*) *to be finite (absence of bursting phenomena) and* (*b*) *to depend on* $u(\cdot)$*'s discontinuities continuously.*

Furthermore in [6] it is shown that the hypotheses of the above italicized second main result are essential; and that (in particular) in some cases where $(1.1)_1$ is quadratic in $\dot{u}$, $\Sigma_{u(\cdot)}$'s motion has no bursting character, in spite of $u(\cdot)$'s discontinuity. On the other hand bursting phenomena are interesting and much studied.

In the situation above it is natural to try and determine a wide class of choices for the above system $\Sigma_{u(\cdot)}$, outside the theory T (constructed in [2-5]) and surely undergoing motions with bursting character (by certain initial conditions). Such a determination is just the first main result of this Part I. It is stated by Theor. 3.1 which is a straightforward mechanical application of Favretti's main result obtained in [7] from a general and purely mathematical point of wiew.

Let $u(\cdot)$ have a discontinuity at $t = d$, with $u(d) = \bar{u}$ and $j = u(d^+) - u(d^-)$; and let a certain weak condition ($\mathcal{C}$) – *i.e.* (3.1) – on the coefficients of Σ's kinetic energy and on the Lagrangian components of Σ's applied forces be satisfied at $t = d$ for at least one of Σ's configurations, say $(\bar{q}, \bar{u})$. Then, *equation* $(1.1)_1$ *is quadratic in* $\dot{u}$ and, as Theor. 3.1 asserts, any (physical) process $(q(t), u(t))$ along which $\Sigma_{u(\cdot)}$ is in the position $(\bar{q}, \bar{u})$ at $t = d$ has a bursting character. In more details, the jump $j = u(d^+) - u(d^-)$ of $u(\cdot)$ is approximated, as $\eta \to 0$, by a sequence $u_{j,\eta}(\cdot) \in C^0 \cap KC^2$ of controls; and for the corresponding solution $z_{j,\eta}(\cdot) = (q_{j,\eta}(\cdot), p_{j,\eta}(\cdot))$ of (1.1) one has $|p_{j,\eta}(d+\eta)| > Cj^2/\eta$ and $C > 0$. Hence, $|p_{j,\eta}(d+\eta)| \to +\infty$ as $\eta \to 0^+$, and, as Theor. 6.1 in Part II shows, the representative point P of Σ (in Hertz's space) reaches the border (possibly at infinity) of the region where Σ's constitutive functions are meaningful. All this affords the bursting character of Σ's motion.

In Part II a geodesic property is shown to hold for the above motions with bursting character, in case the applied forces $Q_h = Q_h(t, q, \dot{q}, u, \dot{u})$ are at most linear in $\dot{u}$, see Part II, §4 for more details.

2. A Lagrangian system Σ with an impulsive control $u(\cdot)$ and the semi-Hamiltonian equation $\dot{z} = F[t, u(t), z, \dot{u}(t)]$ quadratic in $\dot{u}$

Let us consider a (mechanical) system Σ, subject to constraints that are frictionless, holonomic, time-dependent, and locally described by a system $\chi = (q^1, \ldots, q^N, u)$ of $N = \mathrm{N} + 1$ independent Lagrangian coordinates. Let

$$T = \frac{1}{2} A_{RS}(t, \chi)\, \dot{\chi}^R \dot{\chi}^S + B_R(t, \chi)\, \dot{\chi}^R + C(t, \chi), \qquad (R, S = 1, \ldots, N), \tag{2.1}$$

be Σ's kinetic energy. We can always suppose that the coordinate line $u = \text{var.}$ is orthogonal to the lines $q_i = \text{var.}$ for $i = 1, \ldots, \mathrm{N}$ in a neighbourhood of an arbitrarily fixed point $(d, \bar{u}, \bar{q})$. Hence the coefficients A_{RS} and B_R in (2.1) have the form

$$a_{rs}(t, \chi) := A_{rs}(t, \chi), \quad A_{rN} \equiv 0, \quad b_r(t, \chi) := B_r(t, \chi), \qquad (r, s = 1, \ldots, \mathrm{N}). \tag{2.2}$$

Furthermore, assume that (*i*) the Lagrangian components of the applied forces have the form

$$\begin{cases} Q_h(t, \chi) = Q_{hkl}(t, \chi)\, \dot{\chi}^k \dot{\chi}^l + Q_{hk}(t, \chi)\, \dot{\chi}^k + Q_{0h}(t, \chi), & (h, k = 1, \ldots, \mathrm{N}), \\ Q_{hkl} = Q_{hlk}, \qquad Q_{hkN} = 0, & (h, k, l = 1, \ldots, \mathrm{N}); \end{cases} \tag{2.3}$$

and (*ii*) the coordinate u is identified with a function $u(t)$ which acts as a control for Σ's motion. Note that by (2.3) this coordinate fails to be *controllizable, i.e.* to be «satisfactorily» identifiable with a control $u(t)$, to be implemented by adding some frictionless constraints (see [2] defs. 2.1, def. 4.1); in fact this is equivalent to the linearity of $(1.1)_1$ in $\dot{u}$ (see [2] Theor. 6.1 and Corollary 6.1, and [3] Theor. 9.1) *i.e.* to conditions (4.11) in [5], Theor. 4.1.

Then (even in the absence of a Lagrangian function) the dynamic equation for Σ can be put in a semi-Hamiltonian form w.r.t. (with respect to) the variables $z = (q, p) \in \mathbf{R}^{2\mathrm{N}}$ where the p's are q's conjugate momenta. These equations (see [5] N. 11) read

$$\begin{cases} \dot{p}_h = -\dfrac{1}{2} p[a^{-1}_{,h} - 2Q^{(2)}_h] p + p[(a^{-1} b)_{,h} + Q^{(1)}_h] + \dfrac{1}{2} [A_{NN,h} + 2Q_{hNN}]\, \dot{u}^2 + \\ \qquad\qquad [B_N + Q_{hN}]\, \dot{u} + \dfrac{1}{2} [b^{-1} ab + 2C]_{,h} + Q_{0h}, \\ \dot{q}^h = a^{hk}(p_k - b_k), \qquad a^{hk} = (a_{hk})^{-1}, \end{cases} \tag{2.4}$$

where $Q^{(2)}_h$ is the $\mathrm{N} \times \mathrm{N}$ matrix $(Q_h)_{kl}$ and $Q^{(1)}_h$ is the vector Q_{hk} of $\mathbf{R}^{\mathrm{N}}$ defined by (2.3).
Now we consider the (Cauchy) problem

$$\dot{z} = F[t, u(t), z, \dot{u}(t)], \qquad z(d) = \bar{z}, \tag{2.5}$$

where $F(\dots)$ is the R.H.S. (right hand side) of (2.4); and we assume that

(2.6) $\quad W=\overset{\circ}{W}\subseteq \mathbf{R}^2\times\mathbf{R}^N, \quad V=\overset{\circ}{V}=W\times\mathbf{R}^N, \quad \mathrm{V}=V\times\mathbf{R}, \quad F\in C^2(\mathrm{V},\mathbf{R}^{2N}).$

Furthermore we fix $(d,\bar{u},\bar{q},\bar{p})\in V$ and $v(\cdot)\in C^2(\mathbf{R},\mathbf{R})$ with $v(d)=\bar{u}$; and for every $j\in\mathbf{R}\setminus\{0\}$ (with $|j|$ not too large) we consider the function

(2.7) $$v_j := v(t) \quad \forall t\leq d, \qquad v_j(t) := v(t)+j \quad \forall t>d.$$

We now fix j and set $u(\cdot)=v_j(\cdot)$. One can approximate v_j (locally in the L^1-norm) by *e.g.* the family $\{u_{j,\eta}(\cdot)\}_{\eta>0}$ of functions in $C^0\cap KC^2$, where

(2.8) $$u=u_{j,\eta}=\begin{cases} v_j(t) & (t<d), \\ \bar{u}+j(t-d)/\eta & (d\leq t\leq d+\eta=:T), \\ v_j(t) & (t>T). \end{cases}$$

Lastly, let Σ_η be the system Σ with u controllized by setting $u=u_{j,\eta}$, hence (2.5) – for $u(t)=u_{j,\eta}(t)$ – can be regarded as Σ_η's dynamic equation. We can reasonably say that $\Sigma_\eta\to\Sigma_{u(\cdot)}$ as $\eta\to 0^+$ (in the L^1-norm).

3. On the first phase of the motions for $\Sigma_{u(\cdot)}$ that have bursting character

The following Theorem shows that many Lagrangian mechanical systems are capable of motions having a bursting character.

Theorem 3.1. *Assume (i) $(d,\bar{u},\bar{q})\in W$, that the factor $A_{NN,h}+2Q_{hNN}$ in* O.D.E. *(2.4) satisfies the condition*

(3.1) $\quad A_{NN,h}(\bar{\zeta})+2Q_{hNN}(\bar{\zeta})\neq 0 \quad$ *for at least one* $h\in\{1,\dots,\mathrm{N}\}$ $\quad (\bar{\zeta}=(d,\bar{q},\bar{u}))$.

Furthermore fix some $r>0$ for which (ii) $\{(d,\bar{u})\}\times B(\bar{q},r)\subseteq W$. Then there are some constants $C^*>C_*>0$ and $J>0$ such that to any $j\in[-J,J]\setminus\{0\}$, one can associate some η_j for which, given $\eta\in(0,\eta_j]$ and denoting by $t\mapsto(q_{j,\eta}(t),p_{j,\eta}(t))$ the maximal solution $z_{j,\eta}(u_{j,\eta}(\cdot),\cdot)$ of problem (2.5) with $u(t)=u_{j,\eta}(t)$, one has that

(3.2) $\quad q_{j,\eta}(t)\in B(\bar{q},r) \quad \forall t\in[d,d+\eta] \quad (\subseteq \mathrm{Dom}\ z_{j,\eta}(\cdot)),$

(3.3) $\quad |p_{j,\eta}(t)|\leq C^*j^2/\eta \quad \forall t\in[d,d+\eta], \qquad C_*j^2/\eta\leq|p_{j,\eta}(d+\eta)| \quad (\leq C^*j^2/\eta).$

More precisely, C^* and C_* can be identified with arbitrary numbers (>0) for which $0<C_*<\bar{C}<C<C^*$ where C and $\bar{C}$ are defined in (3.5) below.

Proof. This Theorem is a straightforward consequence of Theor. 3.1. in [7]. To see this, in connection with (2.4) identify $-[a^{-1}_{,h}(\zeta)-2Q^{(2)}(\zeta)]/2$ with the quantity $A_h(\zeta)$ in [7, $(3.4)_2$] – *i.e.* in the formula $(3.4)_2$ of [7] – and $[(a^{-1}b)_{,h}+Q^{(1)}_h]$ with $B_h(\zeta)$ in [7; $(3.4)_2$]; in connection with (2.4) identify a^{-1}_{hk} with $a_{hk}(\zeta)$ in [7, $(3.4)_1$], and $a^{-1}_{hk}(\zeta)\,b_k(\zeta)$ with $b_k(\zeta)$ in [7, $(3.4)_1$]; furthermore, identify $[A_{NN,h}(\zeta)+2Q_{hNN}(\zeta)]/2$ with $\alpha_h(\zeta)$ in [7, $(3.2)_3$], $[B_N+Q_{hN}]$ with $\beta_h(\zeta)$ in [7, $(3.2)_3$] and $[b^{-1}ab+2C]_{,h}/2+$ $+Q_{0h}$ with $\gamma_h(\zeta)$ in [7, (3.2)]. Then (2.4) has the form of [7, (3.4)] and thus (2.5) has the form of [7, (3.5)]. Lastly, note that the controls $u_{j,\eta}$ used in Theor. 3.1 here and in

Theor. 3.1 in [7] – and defined by (2.8) and by [7, (3.7)] respectively – are the same and that the domain of $F(\ldots)$ used for (2.5) satisfies the conditions on it assumed in [7, Theor. 3.1]; furthermore, hypothesis (3.1) is hypothesis (3.8) in [7].

Now we can say that (*i*) the Cauchy problem (2.5) – that is (2.4) with $u = u_{j,\eta}(\cdot)$ – is an instance of Cauchy problem [7, (3.5)] – *i.e.* [7, (3.4)] with $u = u_{j,\eta}(\cdot)$ –, that (*ii*) for the Cauchy problem (2.5) all the hypotheses of Theor. [7, 3.1] are verified, and – since (3.2) and (3.3) are [7, (3.9)] and [7, (3.10)] respectively – that (*iii*) the theses of Theor. 3.1 and Theor. 3.1 in [7] are the same. Hence, by Theor. 3.1 in [7], Theor. 3.1 here holds. Q.E.D.

REMARK. By [7, (3.14)] and [7, (6.6)], the positive numbers C and $\overline{C}$ introduced below [7, (3.10)] can be calculated here by using their corresponding quantities in (2.4). By (2.6) and Theor. 3.1, we can choose t_0 and $\bar{J}$ in such a way that

$$U := [d, d + t_0] \times [\bar{u} - \bar{J}, \bar{u} + \bar{J}] \times B(\bar{q}, r) \subseteq W, \tag{3.4}$$

hence U is compact. By hypothesis (3.1), for t_0 and $\bar{J} > 0$ small enough, $A_{NN,h}(\bar{\zeta}) + 2Q_{hNN}(\bar{\zeta}) \neq 0$ for at least one $h \in \{1, \ldots, N\}$. Then one has

$$\begin{cases} C = \max\limits_{h \in \{1,\ldots,N\}} \sup\limits_{\zeta \in U} \{2^{-1} |A_{NN,h}(\zeta) + 2Q_{hNN}(\zeta)|\}, \\ \overline{C} = \inf\limits_{\zeta \in U} \{2^{-1} |A_{NN,h}(\bar{\zeta}) + 2Q_{hNN}(\bar{\zeta})|\} > 0. \end{cases} \tag{3.5}$$

This work has been prepared in the activity sphere of the group N. 3 of the Consiglio Nazionale delle Ricerche in the academic years 1988-89 and 1989-90.

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