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T. Malcolm Charlton, Placido Cicala

On the history of suspension bridge theory

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Meccanica. — *On the history of suspension bridge theory.* Nota di T. MALCOLM CHARLTON e PLACIDO CICALA, presentata (*) dal Socio P. CICALA.

ABSTRACT. — A linearized formulation of the elastic theory of suspension bridges is confronted with early investigations in the field. For decades, the structure was schematized as a beam (deck or girder) relieved by a one parameter distribution of forces exerted by the cable, disregarding the influence of beam deflection on that distribution as given by the linearized approach. An anonymous note presented the essential conclusions of this theory anticipating results of investigations following the methods started by Cotterill and Castigliano. Cotterill's approach is applied to a hyperstatic scheme.

KEY WORDS: Suspension bridges; Structural analysis history; Elastic structures.

RIASSUNTO. — *Sulla storia della teoria dei ponti sospesi.* Una formulazione linearizzata della teoria elastica dei ponti sospesi è confrontata con le prime indagini nel campo. Per decenni la struttura fu schematizzata in una trave sostenuta da una uniparametrica distribuzione di forze trasmesse dal cavo, trascurando l'influenza della deformata della trave su quella distribuzione come risulta dall'esame linearizzato. Una nota anonima presentò le conclusioni essenziali di quella teoria, in anticipo sulle ricerche che furono sviluppate seguendo i metodi fondati da Cotterill e Castigliano. Il procedimento di Cotterill è applicato allo schema con trave iperstatica.

1. About mid nineteenth century, construction of the first major suspension bridges confronted structural analysts with a problem of remarkable difficulty due to the presence of an element (chain or cable) capable of large deflections leading to geometric nonlinearity. In order to examine the simplified formulations, the linear theory will be sketched.

2. The structure is formed by a horizontal beam (girder) of flexural stiffness EI connected to a rod (cable) of zero flexural stiffness by vertical suspenders: these are assumed to constitute a continuous system of rigid bars. Starting from an initial configuration of dead loading, the straight girder is subjected to a live load of intensity p per unit length. This is assumed to be so small to allow disregarding powers of its effects in presence of linear terms.

The position vector $\boldsymbol{x}$ leading to the cable axis is written

$$\boldsymbol{x} = (x+\xi)\,\boldsymbol{i} + (y-\eta)\,\boldsymbol{j} \tag{1}$$

where $\boldsymbol{i}, \boldsymbol{j}$ are orthogonal unit vectors, with $\boldsymbol{j}$ vertical upward; x, y are the coordinates of the axis in the initial configuration; ξ, η are displacement components due to the load p. The coordinate x is taken as independent variable: derivatives d/dx are denoted by apex. Thus the unit tangent vector to the cable axis $\boldsymbol{t}$ is given by the relation

$$\boldsymbol{x}' = r\boldsymbol{t} = (1+\xi')\,\boldsymbol{i} + (y'-\eta')\,\boldsymbol{j}. \tag{2}$$

(*) Nella seduta del 14 giugno 1991.

The modulus $r = ds/dx$ derivative of the arch length s, is given by

(3) $$r^2 = (1+\xi')^2 + (y'-\eta')^2 .$$

Prior to loading, $\boldsymbol{t}$ and r have the values $\boldsymbol{t}_0, r_0$ given by (2), (3) for $\xi = \eta = 0$. If T_0 is the tension T in the cable under dead load, the equilibrium of the cable element yields

(4) $$d(\boldsymbol{t}_0\, T_0) = \boldsymbol{j}\, q_0\, dx, \qquad d(\boldsymbol{t}T) = \boldsymbol{j}(q_0 + q)\, dx .$$

Integrating the scalar products of (4) by $\boldsymbol{i}$ leads to

(5) $$T(1+\xi')/r = H + H_0, \qquad H_0 = T_0/r_0$$

giving the (constant) horizontal components of cable tension. From $\boldsymbol{j}$ components of (4) we obtain

(6) $$q = \{(y'-\eta')\, T/r - T_0 y'/r_0\}' .$$

Denoting by ε the thermic and/or elastic dilatation of the cable and linearizing the definition $r^2 = r_0^2(1+\varepsilon)^2$ yields the relation

(7) $$\xi = \int_0 (y'\eta' + \varepsilon r_0^2)\, dx .$$

Integration is started from $x = 0$, abscissa of the cable anchorage, assumed to coincide with the beam end point. The integral extended over the span equals zero. Hence and from (5) the linearised form of (6) is obtained

(8) $$q = Hy'' - H_0(r_0^2(\eta' + \varepsilon y'))' .$$

For the girder, linear elasticity gives the equation

(9) $$(EI\eta'')'' = p - q$$

completed by 4 boundary conditions at girder end points. Eliminating q from (8), (9) yields a fourth order equation for η containing the unknown constant H to be determined from the terminal value $\xi = 0$ of (7) with ε as given by cable dilatation by use of $(5)_1$.

3. The above formulation implies an approximation in eq. $(4)_2$: it neglects the rotation of the suspenders due to the displacements ξ which in the linearised theory, introduce a component due to q_0 in $(4)_2$. The approximation may be justified by assuming that $y \gg d$ along the cable, whose dip is d. A further approximation is usually adopted: the H_0 term in (8) is disregarded. Thus the problem reduces to ordinary beam theory with the unknown H calculated from (7). This will be termed I-theory, denoting as II-theory the analysis based on the simplified resultant equation

(10) $$EI\eta'''' - H_0\,\eta'' = p - Hy'' .$$

With the particular solution $\eta = \sin(\pi x/L)$, the ratio of the second summand to the first term is H_0/P with $P = \pi^2 EI/L^2$: this measures the ratio of the stiffness due to cable reaction and the stiffness due to beam reaction and evaluates the order of magnitude of the deviation of I- from II-theory. For both, the compatibility condition is written as follows.

Denoting by η_q upward beam deflections due to the cable loading q and by η_p the deflections due to live load for the beam not sustained by the cable, by product integration we obtain from (7)

$$\int \Delta y\, \eta_q''\, dx = \int \Delta y\, \eta_p''\, dx - \int \varepsilon\, r_0^2\, dx \tag{11}$$

with integrations ranging to the whole span. Here Δy is the difference of the current ordinate and that of the anchorage. For a parabolic symmetric cable $\Delta y = (x^2 - lx)\,4d/l^2$, $q = 8\,Hd/l^2$.

4. Let the girder be subjected to a force $-jF$ at the station $x = a = l - b$. For the beam of constant stiffness, with built-in ends, ordinary theory gives for $0 < x < a$

$$-EI\,\eta_p'' = F\,xb/l + F\,ab(xb - xa - bl)/l^3\,. \tag{12}$$

The expression reduces to its first summand for the simply supported beam. In this case we obtain

$$EI\int_0^a \eta_p''\,(x^2 - lx)\,dx = F\,ab(4\,a^2\,l - 3a^3)/12\,. \tag{13}$$

The related expression for the segment $a < x < l$ is obtained by interchanging a, b. Denoting by I_p the sum of the two expressions

$$I_p = F\,ab(a^2 + 3ab + b^2)/12\,. \tag{14}$$

For the built-in beam under uniform loading we know that

$$-EI\,\eta_q'' = q(lx - x^2)/2 - ql^2/12\,. \tag{15}$$

For the simply supported beam the last summand is missing. Hence

$$I_q = q\int (x^2 - lx)^2\,dx/2 = ql^5/60\,. \tag{16}$$

By virtue of eq. (11), with $\varepsilon = 0$ we have $ql = 5F\,ab(a^2 + 3ab + b^2)/l^4$.

Analogous computations for the built-in beam give

$$I_p = F\,a^2\,b^2/12\,, \qquad I_q = ql^5/360\,. \tag{17}$$

From (11), in the case of uniform elongation ε of the cable, the simply supported girder without external loading suffers a decrease in the cable support measured by

$$q = -(1 + 16d^2/3\,l^2)\,15\,EI\,\varepsilon/l^2\,d\,. \tag{18}$$

5. Rankine [1] was probably the first to address the problem of the stiffened suspension bridge. He assumed that the (parabolic) cable contributes a uniformly distributed reaction to bending of the girder, via the suspenders, of intensity given by the total live load divided by the span: according to (11), this holds true only in the case of uniform live loading. The essential results of the I-theory were first presented in an anonymous note [2] published in an engineering magazine in 1860. Interest in this paper resides primarily in the derivation and application of the compatibility condi-

tion written in the form $\int y'(d\eta_p - d\eta_q) = 0$. Hence the results expressed in (14), (16) were obtained. Also eq. (18) was found from the last summand in (11). The paper contains an accurate examination of the experimental results found by Barlow [3] and a clear discussion on the deviations arising from simplifications in the I-theory. An interesting remark in [2] deserves mention: for $\varepsilon = 0$, uniform loading on half the girder gives rise to an antisymmetric stress distribution. Later, the I-theory was dealt with, along the same lines but more extensively, by Lévy [4].

6. For the suspension bridge with simply supported girder subjected to a mid-span load the solution was obtained by Cotterill [5] from the moment form of I-theory. Cotterill's work started the statical indeterminacy analysis of elastic structures. His method is recalled here, considering only beam bending, for brevity.

Let V_α, with $\alpha = 1, \dots, N$, be the parameters appropriate to define the moment distribution. For instance, for the built-in girder subjected to the force F at $x = a$, these are the bending moments M_1 at $x = 0$, M_2 at $x = a$, M_3 at $x = l$ and the supporting action q, defining the curvature of the parabolic arcs giving intermediate moments. The parameters V_α are subjected to a number n of equilibrium conditions, written in the form

$$F_\beta = F_\beta(V_\alpha), \qquad \beta = 1, \dots, n \tag{19}$$

with F_β as prescribed values. Hence the variation equations are derived $(\partial F_\beta / \partial V_\alpha)\, \delta V_\alpha = 0$. The elastic energy U of the structure is written in terms of the parameters V_α: the extremum condition for U is added to the system in the form $(\partial U / \partial V_\alpha)\, \delta V_\alpha = 0$. Compatibility of the system for increments is ensured by the N equations [1]

$$\lambda_\beta (\partial F_\beta / \partial V_\alpha) = \partial U / \partial V_\alpha \,. \tag{20}$$

In linear theory, eqs. (19), (20) constitute a linear system for the N parameters V_α and n Lagrange multipliers λ_β. Extension of the incremental form to geometric nonlinearity is immediate.

This method is used here to check the solution obtained for the I-theory, with built-in girder. We may use the expression of U given by Cotterill, here restricted to the segment a of the girder

$$2EIU(a) = (M_1^2 + M_1 M_2 + M_2^2)\, a/3 - (M_1 + M_2)\, qa^3/12 + q^2 a^5/120 \,.$$

This is to be summed to the expression where M_1, a are replaced by M_3, b respectively. The equilibrium equation at the load point yields

$$F = (M_2 - M_1)/a + (M_2 - M_3)/b - ql/2 \,. \tag{21}$$

(1) Explicit use of the Lagrange multiplier method was made by Cotterill when dealing with problems where the unknowns are functions of the coordinates and (19) are differential equations (*Further application of the principle of least action*, Philosoph. Magaz., 1965, 430-436). In the case under consideration, he adopted elimination of increment ratios. Castigliano proved the minimum complementary energy theorem for trusses by means of the multiplier method (*Dissertazione ... Laurea in Ingegneria*, 1873).

In the present case ($N = 4, n = 1$) the compatibility conditions write $\lambda \partial F/\partial V_\alpha = = \partial U/\partial V_\alpha$ with $\alpha = 1, 2, 3, 4$.

With the moments given in (12) and (15) and the value of q giving $I_p = I_q$ in (17) we find the resultant moments

(22) $$M_1 = (3a - 2b)\, Fab^2/2\,l^3, \qquad M_2 = (9a^2 - 12ab + 9b^2)\, Fa^2 b^2/2\,l^5$$

and consequent M_3. The compatibility conditions are satisfied with

(23) $$\lambda = (4a^2 - 7ab + 4b^2)\, Fa^3 b^3/6\,l^5 .$$

7. In engineering textbooks the «Castigliano approach» is usually adopted starting from the statically determinate scheme, where $N - n$ stress parameters are nullified or are assigned to take given values. Thus $N - n$ self-equilibrated configurations are constructed: these are introduced with factors X_γ ($\gamma = 1, \ldots, N - n$) in combination with a stress configuration equilibrating the applied loads. The «elasticity equations» from which the factors X_γ are calculated, in particular

(24) $$\int (M/EI)(\partial M/\partial X_\gamma)\, dx = 0$$

emerged from energy considerations in Menabrea's and Castigliano's works, almost contemporary to Cotterill's contributions. They acquire their wide significance through the interpretation of virtual work: $\partial M/\partial X_\gamma$ is an arbitrary stress configuration in equilibrium without working forces. In the specific case, Δy is one such moment distribution in equilibrium with a cable tension change of unit horizontal component: thus (11) attains an alternative interpretation, maintaining its ample value. In fact, for the structure with 3 statical indeterminacies it has reduced the solution to one unknown. This consideration may help when the girder is stiffened by additional cables. For eqs. (20), the virtual stress configurations contain working loads represented by $\partial F_\beta/\partial V_\alpha$: the multipliers are the related displacement parameters. In the example, λ/EI is the deflection of the loaded point. The elimination of multipliers which reduces the system (19), (20) to eqs. (24) may lead to more complicated expressions. As in the above application, the «Cotterill approach» may turn out to be more expeditious than recourse to eqs. (24).

8. The above concepts offered a variety of methods to deal with suspension bridge problems. By introduction in (24) of additional terms due to axial deformations of elongation of the cable can be accounted for in agreement with the results furnished by (11). As in the I-theory the internal loading is independent of deflections η, also elongations of the suspenders are readily considered. Thus the theory was variously developed by W. Ritter, Müller-Breslau, Fränkel [6-9]: see [10]. Introduction of the H_0 term in eq. (8) seems to have been not immediate. The problem was felt with regard to early suspension bridges with very flexible girders. In an anonymous note [11], obviously by the same author of [2], the analysis was started under the extreme assumption $EI = 0$ for the girder, maintaining the remaining assumptions. This implies the solution of the equation $p = q$ with the aid of the expression (8). The

Anon. did not find this expression but proceeded through simplifications suggested by the smallness of slopes y'. This analysis, leading to an angular point on the cable at the point where a concentrated load is applied is to be considered as an introductive investigation. The same author made an attempt toward a second approximation approach [12], but this analysis was not completed.

9. The evolution of the theory of suspension bridges has received an excellent presentation by Pugsley [13]. Some additional remarks suggested by certain different points of view are here in order.

Chapt. 3 deals with the theory of [12]; Chapt. 4 is devoted to the Rankine theory: as this stems from an unjustified assumption we have left it aside. Chapt. 5 develops the I-theory («elastic» in the usual denomination: it could be better termed «linear»). No reference is made to the note [2], mentioned elsewhere to state that Rankine had taken cognizance of it. In fact, that paper has been overlooked by most subsequent investigations. Even in [4] no reference is made to it. Applications of (11) in eqs. (12)-(18) evidence its fundamental role: the equivalent forms of the magnitude $\int y' \eta' \, dx$, namely $-\int \Delta y \, \eta'' \, dx$, and $-y'' \int \Delta \eta \, dx$ (for the parabolic cable) establish important connections between the various approaches. As above noted, in [2] the settlement of the girder due to cable elongation was calculated: the expression (18) conforms exactly with the assumptions of the theory, whereas the corresponding eq. (44) in [13, p. 63] is not equally satisfactory.

Chapts. 6 to 9 give a lucid account of the II-theory (deflection theory) as governed by eq. (10) complemented by (11), considered in different form. These developments are beyond the scope of the present paper. We remark that derivation through linearised theory indicates for the second summand in (10) the form $H_0 (r_0^2 \eta')'$ and yields an additional summand due to cable dilatation. Further complications arise if the influence of horizontal displacements of the cable axis on the direction of the suspenders is taken into account: then the horizontal component of the cable tension cannot be considered as constant along the span. In case these refinements were to be undertaken, recourse to discretized computation should be preferred.

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T. M. Charlton: University of Aberdeen
Aberdeen, AB9 1FX (Gran Bretagna)

P. Cicala: Dipartimento di Ingegneria Strutturale
Politecnico di Torino
Corso Duca degli Abruzzi, 24 - 10129 Torino