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On the domain of applicability of the Mori-Tanaka effective medium theory

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Meccanica dei solidi. — *On the domain of applicability of the Mori-Tanaka Effective Medium Theory.* Nota di MAURO FERRARI, presentata (*) dal Corrisp. G. MAIER.

ABSTRACT. — The Mori-Tanaka effective stiffness tensor is shown to be asymmetric in general. This tensor is proven to be symmetric for composites with isotropic inclusions, or with spherical reinforcements. Symmetry is also proven for the case of unidirectional fibers, of any shape and material. The Mori-Tanaka theory is shown to yield physically unacceptable predictions at the high concentration limit.

KEY WORDS: Homogenization; Elasticity; Symmetry.

RIASSUNTO. — *Sul dominio di applicabilità della teoria di Mori-Tanaka.* Esibita l'asimmetria generale del tensore elastico equivalente associato all'ipotesi di Mori-Tanaka, si prova che, per compositi bifase, l'isotropia, la sfericità, l'allineamento e la distribuzione isotropa delle inclusioni sono condizioni sufficienti per la simmetria di questo tensore. Si evidenziano previsioni inaccettabili al limite superiore per la concentrazione della fase dispersa.

1. INTRODUCTION

Some restrictions on the applicability of the Mori-Tanaka approach [1] to estimating the effective elastic properties of composites are deduced. This is achieved by studying the material and distributional domains within which the Mori-Tanaka stiffness tensor is symmetric, and by analyzing the Mori-Tanaka predictions at very high fiber concentrations. A review of the Mori-Tanaka effective medium theory for macroscopically isotropic composites is given in [2], while [3] extends the theory to cases of arbitrary texture of the inclusions.

2. THE MORI-TANAKA EFFECTIVE MEDIUM THEORY

Under the «homogeneous displacement boundary conditions»

$$\boldsymbol{u} = \boldsymbol{\varepsilon}_0 \boldsymbol{x}, \tag{1}$$

the effective stiffness tensor $\boldsymbol{C}$ of a composite is defined as the tensor that maps the applied homogeneous strain $\boldsymbol{\varepsilon}_0$ into the average stress $\bar{\boldsymbol{\tau}}$ [4]:

$$\bar{\boldsymbol{\tau}} = \boldsymbol{C}\boldsymbol{\varepsilon}_0 . \tag{2}$$

Mori and Tanaka [1] provided the following approximate expression for the average strain in the inclusion:

$$\bar{\boldsymbol{\varepsilon}}^f = \boldsymbol{T}[(1-\alpha)\boldsymbol{I} + \alpha\langle \boldsymbol{T}\rangle]^{-1}\boldsymbol{\varepsilon}_0 \tag{3}$$

where

$$\boldsymbol{T} \equiv [\boldsymbol{I} + \boldsymbol{P}(\boldsymbol{C}^f - \boldsymbol{C}^m)]^{-1} . \tag{4}$$

(*) Nella seduta del 20 aprile 1991.

In these relations, the polarization tensor $\boldsymbol{P} \equiv \boldsymbol{E}(\boldsymbol{C}^m)^{-1}$ was introduced, in terms of Eshelby's tensor $\boldsymbol{E}$, overbars denote volumetric averaging, α is the volume fraction occupied by the fibers, and superscripts m and f denote the matrix and fiber phase, respectively. When the fibers are distributed according to an orientation probability density function $f(\varphi_1, \varphi_2, \phi)$, then pointed brackets denote f-weighted orientational averaging. Throughout this work, φ_1, φ_2 and ϕ denote Euler angles, defined according to the convention of [3]. The case of a random orientation distribution corresponds to $f(\cdot) = 1$. Under the assumption (3), the effective stiffness tensor $\boldsymbol{C}$ may be expressed as [5]:

$$\boldsymbol{C}^{MT} = \boldsymbol{C}^m + \alpha\langle(\boldsymbol{C}^f - \boldsymbol{C}^m)\,\boldsymbol{T}\rangle[(1-\alpha)\boldsymbol{I} + \alpha\langle \boldsymbol{T}\rangle]^{-1}. \tag{5}$$

3. The high concentration limit

An unacceptable characteristic of the Mori-Tanaka scheme is made evident by taking the limit of its predicted effective stiffness for α approaching unity. By (5), it is deduced that this limit is

$$\boldsymbol{C}^{MT} = \langle \boldsymbol{C}^f \boldsymbol{T}\rangle\langle \boldsymbol{T}\rangle^{-1}. \tag{6}$$

If the fiber material is isotropic or the fibers are aligned, then $\boldsymbol{C}^{MT} = \boldsymbol{C}^f$. However, for the general case, $\boldsymbol{C}^{MT}$ depends on the *matrix* moduli, through $\boldsymbol{T}$, even for unitary fiber concentrations.

4. Symmetry of the Mori-Tanaka stiffness tensor: general results

For the case of a macroscopically isotropic distribution of the fibers, the orientational averaging $\langle\cdot\rangle$ yields a tensor that is isotropic, and hence diagonally symmetric. Thus, the symmetry problem must be studied within the domain of composites exhibiting texture.

The Mori-Tanaka stiffness tensor (5) *is non-symmetric in general.* Consider for instance a composite with inclusion moduli $(c_{11}^f, c_{12}^f, c_{13}^f, c_{33}^f, c_{44}^f) = (30, 20, 10, 6, 50)$ GPa, matrix moduli $(\lambda, \mu) = (1, 3)$ GPa, and only non-vanishing texture coefficients $(C_0^{00}, C_2^{00}, C_4^{00}) = (1, 1.91, -1.92)$, corresponding to a transversely isotropic bipolar orientation distribution. The details of this texture description and of the matricial representation of the elastic tensor can be found in [3, 6]. The inclusion geometry is penny-shaped, with axes ratios $a1/a3 = a2/a3 = 0.1$. For this case, $(C_{13}^{MT}, C_{31}^{MT}) = (2.88, 3.57)$ GPa. The computations leading to this result were performed on a symbolic manipulation package. Thus, the differences in the moduli are not of numerical nature.

Even though the effective stiffness tensor is not symmetric in general, important subcases exist, for which symmetry is granted. In order to study these, four preliminary theorems are introduced:

Theorem 1. The polarization tensor $\boldsymbol{P}$ and the contraction $(\boldsymbol{C}^f - \boldsymbol{C}^m)\,\boldsymbol{T}$ are both symmetric [7].

THEOREM 2. The tensor $P^{-1}T$ is symmetric.

PROOF. By (4), it may be shown that $P^{-1}T = \{P + P(C^f - C^m)P\}^{-1}$. This implies the Theorem, since both of the terms in curly brackets are symmetric.

THEOREM 3. The effective stiffness tensor C^{MT} is symmetric if and only if the tensor $R \equiv \langle (C^f - C^m)T \rangle \langle T \rangle$ is symmetric.

PROOF. Upon inverting the condition $C^{MT} = (C^{MT})^t$, where a superscript t denotes the diagonal transpose, and using Theorem 1, the symmetry condition is reduced to

$$\langle T \rangle^t \langle (C^f - C^m)T \rangle = \langle (C^f - C^m)T \rangle \langle T \rangle. \tag{7}$$

By Theorem 1 this concludes the proof.

THEOREM 4. R is symmetric if $\langle P^{-1}T \rangle \langle T \rangle$ and $\langle P^{-1} \rangle \langle T \rangle$ are symmetric.

PROOF. Equation (4) implies that $C^f - C^m = P^{-1}(T^{-1} - I)$, so that the tensor $R = -\langle P^{-1}T \rangle \langle T \rangle + \langle P^{-1} \rangle \langle T \rangle$.

5. SOME CLASSES OF BIPHASE COMPOSITES WITH SYMMETRIC MORI-TANAKA STIFFNESS

For C^{MT} to be symmetric, it suffices that the matrix be isotropic, and the inclusions be ellipsoidal, and *i*) isotropic, or *ii*) equiaxed, or *iii*) aligned. This is shown next.

Case i). It is first noted that

$$R^t = [\langle (C^f - C^m)T \rangle \langle T \rangle]^t = \langle T^t \rangle \langle (C^f - C^m)T \rangle \tag{8a, b}$$

$$= \langle T^t (C^f - C^m) \rangle \langle T \rangle \tag{8c}$$

$$= \langle (C^f - C^m)T \rangle^t \langle T \rangle. \tag{8d}$$

In (8*b*, *d*), Theorem 1 was employed, while (8*c*) followed from the isotropy of the fiber and matrix materials. By Theorem 1 again, it now follows that $R^t = R$, which, by Theorem 3, guarantees the symmetry of C^{MT}.

Case ii). If the inclusions are spherical, Eshelby's tensor E is isotropic. Thus the polarization tensor P is isotropic as well. This implies that $\langle P^{-1} \rangle \langle T \rangle = \langle P^{-1}T \rangle$, which is symmetric, in view of Theorem 2. Furthermore, for isotropic P, the tensor $\langle P^{-1}T \rangle \langle T \rangle$ is symmetric if and only if $\langle T \rangle^t = \langle P^{-1}TP \rangle$. However,

$$\langle P^{-1}TP \rangle = \langle P^{-1}(P^{-1}T^{-1})^{-1} \rangle \tag{9a}$$

$$= \langle P^{-1}(P^{-1} + C^f - C^m)^{-1} \rangle \tag{9b}$$

$$= \langle [(P^{-1} + C^f - C^m)P]^{-1} \rangle \tag{9c}$$

$$= \langle [(T^{-1})^t]^{-1} \rangle \tag{9d}$$

$$= \langle T^t \rangle = \langle T \rangle^t, \tag{9e, f}$$

where (4) was used in (9*b*, *d*). By Theorem 4, this concludes the proof.

Case iii). For aligned fibers, the orientational averagings may be deleted, so that the proof of the symmetry of $\langle \boldsymbol{P}^{-1} \rangle \langle \boldsymbol{T} \rangle$ and $\langle \boldsymbol{P}^{-1} \boldsymbol{T} \rangle \langle \boldsymbol{T} \rangle$, given for case *ii*), applies to this case as well, upon substituting the pointed brackets with regular brackets.

The symmetry of $\boldsymbol{C}^{MT}$ in the cases *i*) and *iii*) was known to Y. Benveniste (private communication, 1988). G. J. Dvorak (private communication, 1990) noted that $\langle \boldsymbol{P}^{-1} \boldsymbol{T} \rangle \langle \boldsymbol{T} \rangle$ may be proven to be symmetric, by the arguments of case *ii*), any time the tensor $\boldsymbol{P}$ is orientation independent. This may be realized by non-ellipsoidal inclusions in an anisotropic matrix.

6. Discussion and conclusions

As shown in Sect. 4, the effective stiffness tensor, obtained under the assumption of Mori and Tanaka, is non-symmetric in general. Thus, the corresponding effective medium does not possess a strain energy function, and may hence have non-vanishing dissipation in a closed cycle of deformation in the elastic range. On these grounds, the use of the Mori-Tanaka approximation (5) appears justifiable only in the cases, for which symmetry is granted. The following conditions were proven to ensure the symmetry of the Mori-Tanaka stiffness tensor:

1) Isotropic inclusions, for any orientation distribution and morphology of the inclusions;

2) Spherical inclusions, for any texture and material combination;

3) Perfect alignment of the inclusions, independently of their material and morphological characteristics.

It was noted that symmetry is trivially obtained for any composite with a random orientation distribution of the inclusions. Further counterexamples prove that neither the disk-like nor the cylindrical geometry suffice to ensure symmetry in general.

Of the symmetry-ensuring conditions listed above, only 1) and 3) are sufficient, for the Mori-Tanaka stiffness not to exhibit dependance on the matrix material properties at the unitary fiber concentration limit. This appears to discourage the use of the Mori-Tanaka effective medium theory for composite materials with an high concentration of anisotropic fibers, even if these are spherical or randomly oriented.

On the basis of the analysis of the symmetry and of the high-concentration limit, no counter-indications were presently found to the use of the Mori-Tanaka theory for composites with isotropic inclusions or with perfectly aligned fibers of any material symmetry.

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