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Frame Indifference

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Meccanica dei continui. — *Frame Indifference.* Nota (*) del Socio RONALD RIVLIN.

ABSTRACT. — It is shown, in the context of the Thermomechanics of simple materials with memory, that frame indifference and, equivalently, rotation invariance are necessary consequences of the laws of classical Mechanics and the definition of the stress matrix and heat flux vector.

KEY WORDS: Constitutive equations; Frame indifference; Rotation invariance.

RIASSUNTO. — *Principio di indifferenza materiale.* Nel contesto della Termomeccanica dei materiali con memoria, si dimostra che il principio di indifferenza materiale e quello, equivalente, di invarianza rispetto alle rotazioni sono conseguenze necessarie delle leggi della Meccanica classica e delle definizioni di matrice di stress e di vettore flusso di calore.

1. INTRODUCTION

One of the basic elements of continuum mechanics is the concept of rotation invariance, or its equivalent frame indifference. It was used in the middle of the nineteenth century in the formulation of the Navier-Stokes equation for Newtonian fluids. Since then it has been widely employed to place restrictions on the possible forms which can be taken by the constitutive equations of continuum mechanics and thermomechanics. In 1957 Green and Rivlin [1] applied it to obtain restrictions on the constitutive functional for the Cauchy stress in a simple material with memory subjected to isothermal time-dependent deformations.

It is a commonly held belief that the requirement that frame indifference be satisfied by a constitutive equation is an axiom, independent of the laws of classical thermomechanics. Its validity has been questioned, notably by Müller [2], Edelen and McLennan [3], and Ryskin and Rallison [4]. Müller's concerns are based mainly on the observation that the second-order constitutive equations for the Cauchy stress and heat flux vector in a gas undergoing a thermomechanical process, derived from the kinetic theory by Burnett (see, for example, Chapman and Cowling [5]), are *not* frame indifferent. The concerns of Ryskin and Rallison stem from a micromechanical theory developed by Ryskin [6] for the flow of a suspension of hard spheres in a Newtonian fluid. Truesdell [7] is unconcerned that the Burnett equations do not satisfy frame indifference and maintains that this probably results from the approximations which are made in deriving them, without, however, identifying the point at which the error is introduced. Wang [8] appears to take a similar position. Woods [9] maintains that the nature of the approximations made in arriving at the Burnett equations are not such as could lead to the introduction of the non-invariant terms.

The present paper is a contribution to the debate concerning the validity of frame

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indifference. It is shown that within the context of thermomechanical processes in simple materials with memory, for which the constitutive assumptions in (2.4) below are appropriate, frame indifference must indeed be valid. It is seen that if it were violated, then the manner in which the applied surface tractions and body forces transform from one reference frame to another rotating relative to it would depend on the material to which the forces are applied.

We conclude that frame indifference is not an axiom, independent of the laws of thermomechanics, but is implied by them. This, of course, leaves unresolved the conflicts with the Burnett equations and with the results of Ryskin. These will be addressed in a later paper.

2. The constitutive assumption

We consider a body to undergo a thermomechanical process $\mathscr{D}$ in the time interval $[t_0, t_1]$. For times $\tau \leq t_0$, the body is in mechanical and thermal equilibrium and no forces are applied to it, nor do any temperature gradients exist in it, *i.e.* the body is in its *virgin state*. In the process $\mathscr{D}$ a generic particle $\mathscr{P}$ of the body has vector position $\boldsymbol{x}(\tau)$ at time τ, referred to an arbitrarily chosen rectangular cartesian coordinate system x, and temperature $\theta(\tau)$. Let t be some arbitrarily chosen time in the interval $[t_0, t_1]$. We adopt the abbreviations

$$(2.1) \qquad \boldsymbol{x}_0 = \boldsymbol{x}(t_0), \quad \boldsymbol{x} = \boldsymbol{x}(t), \quad \theta_0 = \theta(t_0), \quad \theta = \theta(t).$$

The thermomechanical process is described by specifying the dependence of $\boldsymbol{x}(\tau)$ and $\theta(\tau)$ on $\boldsymbol{x}_0$ and τ:

$$(2.2) \qquad \boldsymbol{x}(\tau) = \boldsymbol{\chi}(\boldsymbol{x}_0, \tau), \qquad \theta(\tau) = \psi(\boldsymbol{x}_0, \tau).$$

The deformation gradient matrix $\boldsymbol{g}(\tau)$ and temperature gradient $\boldsymbol{\gamma}(\tau)$ are defined by

$$(2.3) \qquad \boldsymbol{g}(\tau) = \partial \boldsymbol{x}(\tau)/\partial \boldsymbol{x}_0, \qquad \boldsymbol{\gamma}(\tau) = \partial \theta(\tau)/\partial \boldsymbol{x}_0.$$

The vector field $\boldsymbol{x}(\tau)$ and scalar field $\theta(\tau)$ together define the *configuration* $\varkappa(\tau)$ of the body at time τ. The configurations at times t_0 and t will be denoted by $\varkappa_0$ and $\varkappa$ respectively.

In this paper we are concerned with materials with memory and shall make the constitutive assumption that the Cauchy stress matrix $\boldsymbol{\sigma}$ and heat flux vector $\boldsymbol{q}$ at time t, referred to the coordinate system x, are functionals of the histories of $\boldsymbol{g}(\tau)$, $\boldsymbol{\gamma}(\tau)$ and $\theta(\tau)$ in the interval $[t_0, t]$. Then we may write

$$(2.4) \qquad \boldsymbol{\sigma} = \mathscr{F}\{\boldsymbol{g}(\tau), \boldsymbol{\gamma}(\tau), \theta(\tau)\}, \qquad \boldsymbol{q} = \mathscr{H}\{\boldsymbol{g}(\tau), \boldsymbol{\gamma}(\tau), \theta(\tau)\},$$

where $\mathscr{F}$ is a symmetric matrix-valued functional and $\mathscr{H}$ is a vector-valued functional.

3. The governing equations

We consider that in order to effect the process $\mathscr{D}$, body forces $\boldsymbol{\phi}(\tau)$ per unit mass and surface forces $\boldsymbol{f}(\tau)$, per unit area measured at time τ, must be applied to the body. Also, we suppose that heat flows out of the body through its surface at a rate $q(\tau)$, per unit area measured at time τ. Let $\rho(\tau)$ be the mass density at time τ. We introduce the abbreviations

$$\boldsymbol{\phi}=\boldsymbol{\phi}(t), \quad f=f(t), \quad q=q(t), \quad \rho=\rho(t). \tag{3.1}$$

Let $\mathscr{B}$ be the domain occupied by the body at time t and let $\partial\mathscr{B}$ be the boundary of this domain. Then, $\boldsymbol{\sigma}$ must satisfy the equation of motion (1)

$$(\boldsymbol{\nabla}\boldsymbol{\sigma})^{\dagger}+\rho\boldsymbol{\phi}=\rho\ddot{\boldsymbol{x}} \quad \text{in } \mathscr{B}, \tag{3.2}$$

and the force boundary condition

$$f=\boldsymbol{\sigma}\boldsymbol{n} \quad \text{on } \partial\mathscr{B}, \tag{3.3}$$

where $\boldsymbol{\nabla}=\partial/\partial\boldsymbol{x}$ and $\boldsymbol{n}$ denotes the outward-drawn unit normal to $\partial\mathscr{B}$. A dot over a symbol denotes its material time derivative.

Let $\boldsymbol{v}$ be the velocity of the particle $\mathscr{P}$ at time t and let $\boldsymbol{d}$ be the velocity gradient matrix:

$$\boldsymbol{v}=\dot{\boldsymbol{x}}, \quad \boldsymbol{d}=\partial\boldsymbol{v}/\partial\boldsymbol{x}. \tag{3.4}$$

Also, let U be the internal energy per unit mass at time t. Then, the energy balance equation may be written as

$$\rho\dot{U}=\operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{d})-\boldsymbol{\nabla}\boldsymbol{q} \quad \text{in } \mathscr{B} \tag{3.5}$$

and the heat flux boundary condition may be written as

$$q=\boldsymbol{q}^{\dagger}\boldsymbol{n} \quad \text{on } \partial\mathscr{B}. \tag{3.6}$$

4. Frame indifference

Let $\bar{x}$ be a rectangular Cartesian coordinate system which coincides with the system x at time t_0 and undergoes a rotation relative to it. We suppose that the origins of the systems x and $\bar{x}$ coincide at all times. The particle $\mathscr{P}$, which has vector position $\boldsymbol{x}(\tau)$ at time τ referred to the coordinate system x, has vector position $\bar{\boldsymbol{x}}(\tau)$ referred to the system $\bar{x}$, where

$$\bar{\boldsymbol{x}}(\tau)=\boldsymbol{R}(\tau)\,\boldsymbol{x}(\tau), \tag{4.1}$$

and $\boldsymbol{R}(\tau)$ is a proper orthogonal matrix-valued function of time such that $\boldsymbol{R}(t_0)=\boldsymbol{I}$. The deformation gradient matrix $\bar{\boldsymbol{g}}(\tau)$ and temperature gradient $\bar{\boldsymbol{\gamma}}(\tau)$, referred to the system $\bar{x}$, are given by

$$\bar{\boldsymbol{g}}(\tau)=\partial\bar{\boldsymbol{x}}(\tau)/\partial\boldsymbol{x}_0=\boldsymbol{R}(\tau)\,\boldsymbol{g}(\tau), \qquad \bar{\boldsymbol{\gamma}}(\tau)=\boldsymbol{\gamma}(\tau). \tag{4.2}$$

We note here that the temperature $\theta(\tau)$ is the same in all reference systems.

The Cauchy stress matrix $\bar{\boldsymbol{\sigma}}$ and heat flux vector $\bar{\boldsymbol{q}}$ at time t, referred to the coordinate system $\bar{x}$, are given by expressions of the forms (cf. (2.4))

$$\bar{\boldsymbol{\sigma}}=\bar{\mathscr{F}}\{\bar{\boldsymbol{g}}(\tau),\bar{\boldsymbol{\gamma}}(\tau),\theta(\tau)\}, \qquad \bar{\boldsymbol{q}}=\bar{\mathscr{H}}\{\bar{\boldsymbol{g}}(\tau),\bar{\boldsymbol{\gamma}}(\tau),\theta(\tau)\}, \tag{4.3}$$

(1) Throughout this paper vectors will, in general, be regarded as column matrices and correspondingly the operator $\boldsymbol{\nabla}$ will be a row matrix. Here and throughout this paper a dagger denotes the transpose.

where $\overline{\mathscr{F}}$ is a symmetric matrix-valued functional and $\overline{\mathscr{H}}$ is a vector-valued functional.

The functionals $\overline{\mathscr{F}}$ and $\overline{\mathscr{H}}$ in (4.3) must necessarily be the same as the functionals $\mathscr{F}$ and $\mathscr{H}$ in (2.4); so that (4.3) may be replaced by

$$\bar{\boldsymbol{\sigma}} = \mathscr{F}\{\bar{g}(\tau), \bar{\gamma}(\tau), \overset{\circ}{\theta}(\tau)\}, \qquad \bar{q} = \mathscr{H}\{\bar{g}(\tau), \bar{\gamma}(\tau), \theta(\tau)\}. \tag{4.4}$$

This follows trivially from the fact that the laws of mechanics are independent of the reference frame in which we choose to express them; the outcome of an experiment which measures the dependence of the stress and heat flux vector on the histories of the deformation and temperature is independent of the rotation of the laboratory as a whole, absent any external influences such as gravitation.

We suppose further that $\bar{\boldsymbol{\sigma}}$, $\bar{q}$ are related to $\boldsymbol{\sigma}$, q by the relations

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^\dagger, \qquad \bar{q} = \boldsymbol{R}q, \tag{4.5}$$

where $\boldsymbol{R} = \boldsymbol{R}(t)$. If the relations (4.4) and (4.5) are satisfied, for all $\boldsymbol{R}(\tau)$ such that $\boldsymbol{R}(t_0) = \boldsymbol{I}$, the constitutive equations (2.4) are said to be *frame indifferent*. We have already seen that (4.4) is necessarily valid. We will show in §5 that this is also the case for the relations (4.5).

From (2.4), (4.2), (4.4) and (4.5) we see that the functionals $\mathscr{F}$ and $\mathscr{H}$ in (2.4) must satisfy the relations

$$\begin{cases} \mathscr{F}\{\boldsymbol{R}(\tau)\boldsymbol{g}(\tau), \gamma(\tau), \theta(\tau)\} = \boldsymbol{R}\mathscr{F}\{\boldsymbol{g}(\tau), \gamma(\tau), \theta(\tau)\}\boldsymbol{R}^\dagger, \\ \mathscr{H}\{\boldsymbol{R}(\tau)\boldsymbol{g}(\tau), \gamma(\tau), \theta(\tau)\} = \boldsymbol{R}\mathscr{H}\{\boldsymbol{g}(\tau), \gamma(\tau), \theta(\tau)\}, \end{cases} \tag{4.6}$$

for all proper orthogonal $\boldsymbol{R}(\tau)$ such that $\boldsymbol{R}(t_0) = \boldsymbol{I}$. It can be shown that the necessary and sufficient conditions for (4.6) to be satisfied is that $\mathscr{F}$ and $\mathscr{H}$ in (2.4) be expressible in the forms

$$\begin{cases} \mathscr{F}\{\boldsymbol{g}(\tau), \gamma(\tau), \theta(\tau)\} = \boldsymbol{g}(t)\, \mathscr{G}\{\boldsymbol{C}(\tau), \gamma(\tau), \theta(\tau)\}\, \boldsymbol{g}^\dagger(t), \\ \mathscr{H}\{\boldsymbol{g}(\tau), \gamma(\tau), \theta(\tau)\} = \boldsymbol{g}(t)\, \mathscr{K}\{\boldsymbol{C}(\tau), \gamma(\tau), \theta(\tau)\}, \end{cases} \tag{4.7}$$

where $\boldsymbol{C}(\tau)$ is the Cauchy strain matrix at time τ defined by

$$\boldsymbol{C}(\tau) = \boldsymbol{g}^\dagger(\tau)\boldsymbol{g}(\tau), \tag{4.8}$$

$\mathscr{G}$ is a symmetric matrix-valued functional and $\mathscr{K}$ is a vector-valued functional.

A number of derivations of (4.7) from (4.6) have been given. For a recent derivation of $(4.7)_1$ from $(4.6)_1$ and for some comments on earlier derivations see [10]. The result $(4.7)_2$ can be derived from $(4.6)_2$ in a similar manner.

5. Proof of frame indifference

The equations of motion, boundary conditions, and energy balance equation referred to the coordinate system $\bar{x}$ may be written as (cf. (3.2), (3.3), (3.5), (3.6)).

$$(\bar{\boldsymbol{\nabla}}\bar{\boldsymbol{\sigma}})^\dagger + \rho\bar{\boldsymbol{\phi}} = \rho\ddot{\bar{\boldsymbol{x}}} \quad \text{in } \mathscr{B}, \tag{5.1}$$

$$\bar{f} = \bar{\boldsymbol{\sigma}}\bar{\boldsymbol{n}} \quad \text{on } \partial\mathscr{B}, \tag{5.2}$$

$$\rho\dot{\bar{U}} = \operatorname{tr}(\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{d}}) - \bar{\boldsymbol{\nabla}}\bar{\boldsymbol{q}} \quad \text{in } \mathscr{B}, \tag{5.3}$$

$$\bar{q} = \bar{\boldsymbol{q}}^{\dagger}\,\bar{\boldsymbol{n}} \quad \text{on } \partial\mathscr{B}. \tag{5.4}$$

As in § 4, symbols with an overbar are defined with respect to the coordinate system $\bar{x}$ in the same way as those without an overbar are defined with respect to the system x.

We note that

$$\bar{\boldsymbol{n}} = \boldsymbol{R}\boldsymbol{n}. \tag{5.5}$$

It follows from (3.3), (5.2) and (5.5) that if $(4.5)_1$ is satisfied then

$$\bar{f} = \boldsymbol{R}\boldsymbol{f}. \tag{5.6}$$

Similarly, from (3.6), (5.4) and (5.5), if $(4.5)_2$ is satisfied then

$$\bar{q} = q. \tag{5.7}$$

Now, suppose that $(4.5)_1$ is not satisfied, *i.e.*

$$\bar{\boldsymbol{\sigma}} \neq \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^{\dagger}. \tag{5.8}$$

It then follows with (5.5) that, *in general,*

$$\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{n}} \neq \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{n} \tag{5.9}$$

and hence, from (3.3) and (5.2),

$$\bar{f} \neq \boldsymbol{R}\boldsymbol{f}. \tag{5.10}$$

Similarly, if $(4.5)_2$ is not satisfied, *i.e.*

$$\bar{\boldsymbol{q}} \neq \boldsymbol{R}\boldsymbol{q}, \tag{5.11}$$

it follows, with (3.6), (5.4) and (5.5), that, *in general,*

$$\bar{q} \neq q. \tag{5.12}$$

Now, for elastic solids and Newtonian fluids which satisfy Fourier's Law the relations (4.5) are satisfied and consequently

$$\bar{f} = \boldsymbol{R}\boldsymbol{f}, \qquad \bar{q} = q. \tag{5.13}$$

We accordingly conclude that unless $(4.5)_1$ is satisfied for *all* simple materials the manner in which forces applied to the surface of the body transform, from one coordinate system to another rotating relative to it, will depend on the nature of the material of which the body consists. Also unless $(4.5)_2$ is satisfied for all simple materials, the rate at which heat flows out of the body through its surface may depend on the reference coordinate system for some materials and not for others.

It follows that, at any rate on the surface $\partial\mathscr{B}$, the relations (4.5) must be satisfied and, consequently, the constitutive functionals $\mathscr{F}$ and $\mathscr{H}$ must be expressible in the forms given in (4.7). In the constitutive assumptions (2.4) it is assumed that the manner in which $\boldsymbol{\sigma}$ and $\boldsymbol{q}$ at a particle depend on the histories $\boldsymbol{g}(\tau)$, $\gamma(\tau)$, $\theta(\tau)$ is independent of whether the particle considered lies on the surface or in the interior of the body. Consequently, the stress matrix and heat flux vector must satisfy the relations (4.5) throughout the body and the constitutive functionals must be expressible in the forms given in (4.7) throughout the body.

6. Transformation of body forces and internal energy

If $(4.5)_1$ is satisfied, (5.1) yields with (4.1)

$$R(\nabla\sigma)^\dagger + \rho\bar{\phi} = \rho(R\ddot{x} + 2\dot{R}\dot{x} + \ddot{R}x). \tag{6.1}$$

By comparing (6.1) with (3.2) we obtain

$$\bar{\phi} = R\phi + 2\dot{R}\dot{x} + \ddot{R}x. \tag{6.2}$$

If on the other hand $(4.5)_1$ is not satisfied we obtain from (5.1), (3.2) and (4.1)

$$[\nabla(R^\dagger\bar{\sigma} - \sigma R^\dagger)]^\dagger + \rho(\bar{\phi} - R\phi) = \rho(2\dot{R}\dot{x} + \ddot{R}x). \tag{6.3}$$

Since $(4.5)_1$ is satisfied for an elastic solid and a Newtonian fluid it follows that the assumption that it is not satisfied for *all* simple materials leads to the unacceptable conclusion that the transformation law for the body force must depend, in general, on the constitutive equation for the material to which the force is applied.

We now suppose that (4.5) are satisfied. Then (5.3) yields, with (4.5) and (4.1),

$$\rho\dot{\bar{U}} = \mathrm{tr}\,(\sigma d) - \nabla q. \tag{6.4}$$

(To see this we note from (4.1) that

$$\bar{d} = RdR^\dagger + \dot{R}R^\dagger. \tag{6.5}$$

Since R is orthogonal, $\dot{R}R^\dagger$ is a skew-symmetric matrix. It then follows, since $\bar{\sigma}$ is a symmetric matrix, that $\mathrm{tr}\,\bar{\sigma}\dot{R}R^\dagger = 0$.) By comparing (6.4) and (3.5) we find that

$$\dot{\bar{U}} = \dot{U}. \tag{6.6}$$

If, on the other hand $(4.5)_1$ is satisfied but $(4.5)_2$ is not, then from (5.3) and (3.5)

$$\rho(\dot{\bar{U}} - \dot{U}) = \nabla\,(q - R^\dagger\bar{q}); \tag{6.7}$$

the manner in which the internal energy transforms depends, in general, on the constitutive equation for the material considered.

7. Rotation invariance

Instead of discussing a single thermomechanical process which is described in two coordinate systems in relative rotation, we may consider two processes $\mathscr{D}$ and $\bar{\mathscr{D}}$ taking place in the time interval $[t_0, t_1]$, which differ only by a superposed rigid rotation and are described in the *same* coordinate system, x say. We employ the notation in §§2 and 3 for quantities pertaining to the process $\mathscr{D}$ and we use an overbar to denote the corresponding quantities pertaining to the process $\bar{\mathscr{D}}$. We suppose that the process $\mathscr{D}$ is described by (2.2). The process $\bar{\mathscr{D}}$ is then described by

$$\bar{x}(\tau) = R(\tau)\,x(\tau), \qquad \bar{\theta}(\tau) = \theta(\tau), \tag{7.1}$$

where $R(\tau)$ is a proper orthogonal matrix and $R(t_0) = I$.

We note that in order to effect the process $\bar{\mathscr{D}}$ we must apply, in addition to the forces appropriate to the process $\mathscr{D}$, body forces which annul the inertial and Coriolis forces associated with the superposed rotation.

Let $\hat{x}$ be a rectangular Cartesian coordinate system, whose origin coincides with that of the system x, and which rotates relative to it with the rotation by which the process $\overline{\mathscr{D}}$ differs from the process $\mathscr{D}$. We use an overhat to denote quantities appropriate to the process $\overline{\mathscr{D}}$ referred to the system $\hat{x}$. Thus $\hat{\boldsymbol{x}}(\tau)$ and $\hat{\theta}(\tau)$ denote the vector position and temperature respectively, in the process $\overline{\mathscr{D}}$, of the particle $\mathscr{P}$ at time τ, referred to the coordinate system $\hat{x}$. Then, with (7.1),

$$\hat{\boldsymbol{x}}(\tau) = \boldsymbol{R}^{\dagger}(\tau)\bar{\boldsymbol{x}}(\tau) = \boldsymbol{x}(\tau), \qquad \hat{\theta}(\tau) = \bar{\theta}(\tau) = \theta(\tau). \tag{7.2}$$

It follows that

$$\hat{\boldsymbol{g}}(\tau) = \boldsymbol{g}(\tau), \qquad \hat{\boldsymbol{\gamma}}(\tau) = \boldsymbol{\gamma}(\tau). \tag{7.3}$$

If $\boldsymbol{\sigma}$ and $\boldsymbol{q}$ are given, for the process $\mathscr{D}$, by (2.4), then $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{q}}$ are given, for the process $\overline{\mathscr{D}}$, by

$$\hat{\boldsymbol{\sigma}} = \mathscr{F}\{\hat{\boldsymbol{g}}(\tau), \hat{\boldsymbol{\gamma}}(\tau), \hat{\theta}(\tau)\}, \qquad \hat{\boldsymbol{q}} = \mathscr{H}\{\hat{\boldsymbol{g}}(\tau), \hat{\boldsymbol{\gamma}}(\tau), \hat{\theta}(\tau)\}. \tag{7.4}$$

This arises in the same manner as (4.4). With (7.2), (7.3) and (2.4)

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}, \qquad \hat{\boldsymbol{q}} = \boldsymbol{q}. \tag{7.5}$$

If, in addition,

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^{\dagger}, \qquad \bar{\boldsymbol{q}} = \boldsymbol{R}\boldsymbol{q}, \tag{7.6}$$

where $\boldsymbol{R} = \boldsymbol{R}(t)$, the constitutive equations (2.4) are said to be *rotation invariant* (cf. (4.5))

Applying (2.4) to the process $\overline{\mathscr{D}}$ we obtain

$$\bar{\boldsymbol{\sigma}} = \mathscr{F}\{\bar{\boldsymbol{g}}(\tau), \bar{\boldsymbol{\gamma}}(\tau), \bar{\theta}(\tau)\}, \qquad \bar{\boldsymbol{q}} = \mathscr{H}\{\bar{\boldsymbol{g}}(\tau), \bar{\boldsymbol{\gamma}}(\tau), \bar{\theta}(\tau)\}. \tag{7.7}$$

$\bar{\boldsymbol{x}}(\tau)$, $\bar{\theta}(\tau)$ and $\hat{\boldsymbol{x}}(\tau)$, $\hat{\theta}(\tau)$ describe the *same* process $\overline{\mathscr{D}}$ in two coordinate systems, x and $\hat{x}$ respectively, which are in relative rotation. From the conclusions of §§5 and 6 it follows, with (7.2), that (cf. (4.5))

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{R}\hat{\boldsymbol{\sigma}}\boldsymbol{R}^{\dagger}, \qquad \bar{\boldsymbol{q}} = \boldsymbol{R}\hat{\boldsymbol{q}}. \tag{7.8}$$

With (7.5) we obtain

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^{\dagger}, \qquad \bar{\boldsymbol{q}} = \boldsymbol{R}\boldsymbol{q}; \tag{7.9}$$

the constitutive equations (2.4) are necessarily rotation invariant.

8. Some further remarks on frame indifference

We note that, as far as the stress is concerned, the argument advanced in §5 depends essentially on the definition of the stress matrix at the particle $\mathscr{P}$ as the matrix whose columns are the stress vectors on three orthogonal planar elements at $\mathscr{P}$. Accordingly, if the constitutive equation for the stress is not frame indifferent these stress vectors may transform in a manner which depends on the material considered.

We now suppose that the body considered consists of two different materials 1 and 2 which at time t occupy the sub-domains $\mathscr{B}_1$, $\mathscr{B}_2$ of $\mathscr{B}$. Let $\partial\mathscr{B}_{12}$ be the interface between $\mathscr{B}_1$ and $\mathscr{B}_2$. We suppose that at time t the only external forces acting on the body are body forces acting throughout $\mathscr{B}$ and surface forces on $\partial\mathscr{B}$; no external surface forces are applied to the interface $\partial\mathscr{B}_{12}$.

Let $\mathscr{P}$ be a generic particle which lies on $\partial\mathscr{B}_{12}$ at time t. Let $\boldsymbol{n}$ be the unit normal to $\partial\mathscr{B}_{12}$ at $\mathscr{P}$, outward-drawn from $\mathscr{B}_1$. Then if $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the Cauchy stresses in $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively at $\mathscr{P}$ at time t, we have

(8.1) $$f_{12} = \boldsymbol{\sigma}_1 \boldsymbol{n}, \qquad f_{21} = -\boldsymbol{\sigma}_2 \boldsymbol{n},$$

where f_{12} (f_{21}) is the surface force at $\mathscr{P}$ per unit area of $\partial\mathscr{B}_{12}$ exerted by the material in $\mathscr{B}_2$ ($\mathscr{B}_1$) on that in $\mathscr{B}_1$ ($\mathscr{B}_2$). All the symbols in (8.1) are referred to the coordinate system x. According to Newton's Third Law

(8.2) $$f_{12} = -f_{21}.$$

We now suppose that the material 2 is a material which satisfies frame indifference while the material 1 does not. Then, denoting the surface forces on $\partial\mathscr{B}_{12}$ referred to the coordinate system $\bar{x}$ by $\bar{f}_{12}$ and $\bar{f}_{21}$, we have

(8.3) $$\bar{f}_{21} = \boldsymbol{R} f_{21},$$

and, in general,

(8.4) $$\bar{f}_{12} \neq \boldsymbol{R} f_{12}.$$

(The inequality (8.4) must necessarily be true for some value of $\boldsymbol{n}$ and we suppose that the interface is such that this is the case). It follows from (8.3) and (8.4) that

(8.5) $$\bar{f}_{12} \neq -\bar{f}_{21};$$

Newton's Third Law is violated. This paradoxical situation can be avoided only if we suppose that the manner in which the constitutive equation for the stress transforms is the same for all choices of the constitutive functional $\mathscr{F}$ in $(2.4)_1$.

9. Some remarks on the constitutive assumption

In discussing frame indifference we have considered thermomechanical processes taking place in a time interval $[t_0, t]$, the material being in its virgin state for $\tau \leq t_0$. The deformation was described by (cf. (2.2))

(9.1) $$\boldsymbol{x}(\tau) = \boldsymbol{\chi}(\boldsymbol{x}_0, \tau),$$

where $\boldsymbol{x}_0$ denotes the vector position of the particle considered at time t_0. Correspondingly, for purely mechanical processes the Cauchy stress matrix at time t is given by (cf. $(2.4)_1$)

(9.2) $$\boldsymbol{\sigma} = \mathscr{F}\{\boldsymbol{g}(\tau)\}, \qquad \boldsymbol{g}(\tau) = \partial\boldsymbol{x}(\tau)/\partial\boldsymbol{x}_0.$$

Plainly, by using the chain rule we can replace $\boldsymbol{g}(\tau)$ by the deformation gradient matrix relative to the configuration at any specified time, $\tilde{\tau}$ say, in the interval $[t_0, t]$. We have

(9.3) $$\boldsymbol{g}(\tau) = \tilde{\boldsymbol{g}}(\tau)\tilde{\boldsymbol{g}}^{-1}(t_0), \qquad \tilde{\boldsymbol{g}}(\tau) = \partial\boldsymbol{x}(\tau)/\partial\boldsymbol{x}(\tilde{\tau}).$$

Hence, from (9.2)

(9.4) $$\boldsymbol{\sigma} = \mathscr{F}\{\tilde{\boldsymbol{g}}(\tau)\tilde{\boldsymbol{g}}^{-1}(t_0)\} = \tilde{\mathscr{F}}\{\tilde{\boldsymbol{g}}(\tau)\} \text{ say}.$$

Instead of describing the deformation by (9.1), Truesdell and Noll [7, §21] describe the deformation by

(9.5) $$x(\tau) = \chi^*(X, \tau),$$

where X is the vector position of the particle $\mathscr{P}$ considered in a configuration X which «may be, but need not be, one actually occupied by the body in the course of its motion». They consider deformations taking place in the time interval $(-\infty, t]$ for which the body is not necessarily in its virgin state at any time. Truesdell and Noll adopt the constitutive assumption

(9.6) $$\boldsymbol{\sigma} = \mathscr{F}^*\{g^*(\tau)\} \qquad \tau = (-\infty, t]$$

where

(9.7) $$g^*(\tau) = \partial x(\tau)/\partial X.$$

Unless the functional $\mathscr{F}^*$ is expressible in the form

(9.8) $$\mathscr{F}^*\{g^*(\tau)\} = \tilde{\mathscr{F}}\{\tilde{g}(\tau)\},$$

where $\tilde{g}(\tau)$ is defined in (9.3) and $\tilde{\tau}$ is some specified time in the interval $(-\infty, t]$, it does not contain the kinematic information required for the determination of $\boldsymbol{\sigma}$. Moreover, even if $\mathscr{F}^*$ is expressible in the form (9.8) the argument $\tilde{g}(\tau)$ does not, in general, contain the necessary kinematic information unless the material is in its virgin state at some time during the interval $(-\infty, t]$.

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