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A note on central automorphisms of groups

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Teoria dei gruppi. — *A note on central automorphisms of groups.* Nota di GIOVANNI CUTOLO, presentata (*) dal Socio G. Zappa.

ABSTRACT. — A characterization of central automorphisms of groups is given. As an application, we obtain a new proof of the centrality of power automorphisms.

KEY WORDS: Groups; Central automorphisms; Power automorphisms.

RIASSUNTO. — *Una nota sugli automorfismi centrali dei gruppi.* Si fornisce una caratterizzazione degli automorfismi centrali dei gruppi. Si ottiene, come applicazione, una nuova dimostrazione della centralità degli automorfismi potenza.

A central automorphism of a group G is an automorphism acting trivially on the central factor group $G/Z(G)$. Equivalently an automorphism of G is central if and only if it centralizes (in the full automorphism group $\text{Aut } G$ of G) the group $\text{Inn } G$ of inner automorphisms. Aim of this paper is to prove the following characterization of the central automorphisms.

THEOREM. *Let G be a group and let θ be an automorphism of G . Then the following conditions are equivalent:*

(a) θ is a central automorphism of G .

(b) The following hold:

(i) $[g, g^\theta] = 1$ for all $g \in G$;

(ii) $\langle \theta \rangle$ is normalized by $\text{Inn } G$.

(c) The following hold:

(i) $[g, g^\theta] = 1$ for all $g \in G$;

(ii) $\langle [g, \theta] \rangle \triangleleft G$ for all $g \in G$.

Throughout the paper, if g is an element of the group G , \bar{g} will denote the inner automorphism of G determined by g .

PROOF OF THE THEOREM

Since (b) and (c) are obvious consequences of (a), we have only to prove the sufficiency of both conditions to ensure the centrality of θ .

(*) Nella seduta del 16 novembre 1991.

LEMMA 1. Let θ be an automorphism of a group G and assume that $[g, g^\theta] = 1$ for all $g \in G$. Then:

- (i) $[g, g^{\theta^n}] = 1$ for all $g \in G$ and $n \in \mathbb{Z}$.
- (ii) $[g, \theta, b]^{-1} = [b^{-1}, \theta, g^{-1}]$ for all $g, b \in G$.

PROOF. (i) Let $g \in G$. Since $[g, g^{\theta^{-1}}] = [g^\theta, g]^{\theta^{-1}} = 1$, the statement holds for $n = -1$ and it is enough to prove that it holds for all positive n . Apply induction on n and assume the property is verified for all $m \leq n$. Then $1 = [gg^\theta, (gg^\theta)^{\theta^n}] = [gg^\theta, g^{\theta^n} g^{\theta^{n+1}}] = [g, g^{\theta^{n+1}}]$ so that (i) is proved.

(ii) Since $[x, x^\theta] = 1$ for $x = g, b^{-1}, gb$, we get:

$$[g, \theta, b][b^{-1}, \theta, g^{-1}] = gg^{-\theta} b^{-1} g^{-1} g^\theta b b^{-1} b^\theta g b b^{-\theta} g^{-1} = 1. \quad \square$$

We are now in position to prove the first part of the Theorem.

Sufficiency of (b).

Assume first θ has infinite order and is not central. Let g, b be elements of G such that $[g, \theta, b] \neq 1$. Then clearly $[\bar{g}, \theta] \neq 1$ and, by Lemma 1 (ii), $[\bar{b}, \theta] \neq 1$. Since $\langle \theta \rangle$ is normalized by $\text{Inn } G$, this implies $[\bar{g}, \theta] = [\bar{b}, \theta] = \theta^2$. Hence $[g, \theta, b] = [\bar{g}, \theta, b] = [\theta^2, b] = [b, \theta, b] = 1$, a contradiction.

Suppose now θ is periodic. By Lemma 1 (i) we may assume without loss of generality that the order of θ is a power of a prime, say p^t . Let $t = 1$ and assume θ is not central. Then there exists $g \in G$ such that $[\bar{g}, \theta] \neq 1$. Hence $\langle [\bar{g}, \theta] \rangle = \langle \theta \rangle$. But $[\bar{g}, \theta, g] = [g, \theta, g] = 1$, so that $[\theta, g] = 1$, again a contradiction. Thus we may assume $t > 1$. Let g be an element of G such that $[g, \theta^{p^{t-1}}] \neq 1$ and let $[\bar{g}, \theta] = \theta^n$. Since $[g, \theta, g] = 1$, then θ^n fixes g and so $\theta^n = 1$. Thus $[\bar{g}, \theta]$ is the trivial automorphism and $[g, \theta] \in Z(G)$. Therefore G is the union of the subgroups $C = C_G(\theta^{p^{t-1}})$ and $D = \{x \in G \mid [x, \theta] \in Z(G)\}$. Since C is a proper subgroup, $G = D$, which means that θ is central.

We have proved that (b) implies (a). To complete the proof of the Theorem we need another Lemma.

LEMMA 2. Let the automorphism θ verify the condition (c) of the Theorem. Then $[g, \theta, b, b] = 1$ and $[g, \theta, b]^{-1} = [b, \theta, g]$ for all $g, b \in G$.

PROOF. By hypothesis and by Lemma 1 (ii), $[g, \theta, b] \in \langle [b, \theta] \rangle$; hence $[g, \theta, b, b] = 1$. It follows, again by Lemma 1 (ii), $[g, \theta, b]^{-1} = [b, \theta, g]$. \square

Sufficiency of (c).

Let g be an element of G . We shall prove that $[g, \theta] \in Z(G)$. Suppose first $[g, \theta]$ is not periodic. Then, for any $b \in G$, it holds $[g, \theta, b]^2 = [g, \theta, b^2] = 1$, as $\langle [g, \theta] \rangle \triangleleft G$. But $[g, \theta, b] \in \langle [g, \theta] \rangle$, hence $[g, \theta, b] = 1$. Thus $[g, \theta] \in Z(G)$ in this case.

Assume then $[g, \theta]$ is periodic of minimal order subject to $[g, \theta] \notin Z(G)$. It is clear that $[g, \theta]$ has order a power of a prime p . By Lemma 2 and the previous paragraph, there exists an element $b \in G$ such that $c = [g, \theta, b] \neq 1$ and $[b, \theta]$ has prime-power

order. Since $c \in \langle [g, \theta] \rangle \cap \langle [b, \theta] \rangle$, then $[b, \theta]$ is a p -element. Without loss of generality, we may assume $c = [b, \theta]^{p^i} = [g, \theta]^{p^k}$. By the choice of g , it holds $k \leq i$. Moreover, it follows from the hypothesis that the group $\langle [g, \theta], [b, \theta] \rangle$ is nilpotent of class at most 2. Therefore we get:

$$\begin{aligned} [b^{-p^{i-k}}g, \theta]^{p^k} &= (([b, \theta]^{-p^{i-k}})^g [g, \theta])^{p^k} = (([b, \theta]^{-p^{i-k}} [g, \theta])^{p^k})^g = \\ &= ([b, \theta]^{-p^i} [g, \theta]^{p^k} [[g, \theta], [b, \theta]^{-p^{i-k}}]^{p^k(p^k-1)/2})^g = \\ &= (c^{-1} c [[g, \theta], [b, \theta]]^{p^{i(1-p^k)/2}})^g = [[g, \theta], [b, \theta]]^{p^{i(1-p^k)/2}}. \end{aligned}$$

By Lemma 2, $[b^{-p^{i-k}}g, \theta, b] = c \neq 1$, so that $[b^{-p^{i-k}}g, \theta] \notin Z(G)$. By the minimality of the order of $[g, \theta]$, it follows $[[g, \theta], [b, \theta]]^{p^{i(1-p^k)/2}} \neq 1$. By the same reason, $[g, \theta]^{p^i} \in Z(G)$ and $c^{p^i} = 1$. Therefore $p = 2$, $i = k = 1$ and $[g, \theta]$ has order 4. Since $\langle [g, \theta] \rangle$ is normal in G , it holds $[g, \theta]^b = [g, \theta]^{[b, \theta]} = [g, \theta]^{-1}$, which implies $[g, \theta, b^\theta] = 1$. We have proved that any element of G not belonging to $C = C_G([g, \theta])$ is mapped by θ in C . Hence $G = C \cup C^{\theta^{-1}}$, a contradiction since C is a proper subgroup of G . \square

Let G be the infinite dihedral group. If $\langle x \rangle$ is an infinite cyclic subgroup of G , and $\theta = \bar{x}$, then $\langle \theta \rangle \triangleleft \text{Aut } G$ and $\langle [g, \theta] \rangle \triangleleft G$ for all $g \in G$ but θ is not central.

As an application of the above Theorem, we give an alternative proof of the well-known theorem by Cooper stating that power automorphisms (*i.e.* automorphisms fixing all subgroups) of a group are central (see [1]). This is a consequence of the implication $(c) \Rightarrow (a)$ in the Theorem and of the following Lemma.

LEMMA 3. *Let G be a group and let θ be an automorphism of G such that $[x, x^\theta] = 1$ for all $x \in G$. Let $g \in G$. If θ fixes all conjugates of $\langle g \rangle$ in G , then $\langle [g, \theta] \rangle \triangleleft G$.*

PROOF. Let b be any element of G . Since the group of the automorphisms of G fixing all conjugates of $\langle g \rangle$ in G is normalized by $\text{Inn } G$, it holds $[g, \theta, b] = [b^{-1}, \theta, g^{-1}]^{-1} \in \langle g \rangle$. Then $[g, \theta]^b \in \langle g \rangle$, as $[g, \theta] \in \langle g \rangle$. We conclude that $\langle [g, \theta] \rangle$ is contained in the normal core of $\langle g \rangle$ and so is normal. \square

As a matter of fact the above proof of Cooper's theorem may be shortened, since the group of power automorphisms is always abelian. In fact, if Γ is an abelian group of automorphisms of a group G normalized by $\text{Inn } G$ and such that $[g, g^\theta] = 1$ for all $g \in G$ and $\theta \in \Gamma$, then $[G, \Gamma]$ is abelian. This information clearly simplifies the proof of the implication $(c) \Rightarrow (a)$ of the Theorem. To prove that $[G, \Gamma]$ is abelian (where Γ is defined as above), let $g, b \in G$ and $\alpha, \beta \in \Gamma$. Then $[\overline{[g, \alpha, \beta]}] = 1$, as $[\overline{[g, \alpha]}] \in \Gamma$, hence $[g, \alpha, \beta; b] = 1$ and, by Lemma 1, $[[b, \beta], [g, \alpha]] = 1$, as we wanted to show.

As a final remark, we point out that there exist groups G and abelian normal subgroups A of $\text{Aut } G$ such that $[g, g^\alpha] = 1$ for all $g \in G$ and $\alpha \in A$ but not all elements of A are central. An example is given by the group A of quasi-power automorphisms of certain infinite groups (see [2]).

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