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Multiple periodic solutions for Hamiltonian systems with singular potential

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Analisi matematica. — *Multiple periodic solutions for Hamiltonian systems with singular potential.* Nota di ADDOLORATA SALVATORE, presentata (*) dal Corrisp. A. Ambrosetti.

ABSTRACT. — In this *Note* we prove the existence of infinitely many periodic solutions of prescribed period for a Hamiltonian system with a singular potential.

KEY WORDS: Singular Hamiltonian systems; Periodic solutions; Critical points.

RIASSUNTO. — *Molteplicità di soluzioni periodiche per sistemi Hamiltoniani con potenziale singolare.* In questa *Nota* si stabilisce l'esistenza di infinite soluzioni periodiche di periodo assegnato per un sistema Hamiltoniano con potenziale singolare.

1. INTRODUCTION

In this paper we look for periodic solutions of prescribed period of the following Hamiltonian system

(HS) $$\ddot{x} + ax + V'(x) = 0$$

Here $x = (x_1, \dots, x_N)$, $N \geq 3$, $a \in \boldsymbol{R}$, $V: \boldsymbol{R}^N - S \to \boldsymbol{R}$ is singular on S, *i.e.* $V(x) \to -\infty$ as $x \to \partial S$.

This problem has been studied by many authors (see *e.g.* [1-3], [8-10], [12-14] and their references).

Without loss of generality, we can assume that S is a single point, for example the origin. Slight modifications of our methods permit to deal with more general compact sets S.

Let us denote by $|\cdot|$ the euclidean norm and by $(\cdot,\cdot)$ the usual inner product of $\boldsymbol{R}^N$.

Assume that V satisfies the following hypotheses:

(V_1) $V \in C^2(\boldsymbol{R}^N - \{0\}, \boldsymbol{R})$, $N \geq 3$;

(V_2) there exist two real constants α and R, $R > 0$, such that

$$(V''(x)\,v, v) \leq \alpha |v|^2 \qquad \forall x \in \boldsymbol{R}^N - \{0\},\ |x| \geq R,\ \forall v \in \boldsymbol{R}^N;$$

(V_3) $\lim_{|x| \to 0} V(x) = -\infty$;

(V_4) there exist a function $U \in C^1(\boldsymbol{R}^N - \{0\}, \boldsymbol{R})$ and a neighborhood W of 0 in $\boldsymbol{R}^N$ such that

i) $\lim_{|x| \to 0} U(x) = -\infty$;

ii) $-V(x) \geq |U'(x)|^2 \qquad \forall x \in W - \{0\}$;

(V_5) $\lim_{|x| \to 0} V'(x)\,x/|x|^2 = +\infty$.

(*) Nella seduta dell'8 febbraio 1992.

The «strong force condition» (V_4) has been introduced by Gordon in [12]. Let us observe that $V(x) \simeq -|x|^{-\gamma}$ (near 0) satisfies (V_4) when $\gamma \geqslant 2$, but it does not satisfy (V_4) when $\gamma < 2$.

In particular the gravitational potential $-|x|^{-1}$ does not verify the strong force condition (V_4).

The following Theorem holds:

THEOREM 1.1. If V(x) satisfies assumptions (V_1)-(V_4), then for any $T > 0$, (HS) has infinitely many distinct T-periodic solutions, whose Morse index goes to infinity. Moreover, if $V(x)$ satisfies (V_5) too, then the solutions found are nonconstant.

Now, assume that $V = V(t, x)$ depends on t in a T-periodic way; the strong version of (HS) becomes

$$\text{(FHS)} \qquad \ddot{x} + ax + V'(t, x) = 0$$

where $V'(t, x)$ denotes the gradient of V with respect to x.

Let (V_1'), (V_3'), (V_4') and (V_5') be the natural extension of the hypotheses (V_1), (V_3), (V_4) and (V_5) to a T-periodic potential $V(t, x)$.

Moreover, we strengthen assumption (V_2) as follows:

(V'_2) there exists $\alpha \in \boldsymbol{R}$ with $a + \alpha < (\pi/T)^2$ and there exist some positive constants $b, R, \theta < 2$ such that for any $t \in \boldsymbol{R}$ and for $|x| \geqslant R$: $V(t, x) \leqslant b|x|^\theta$, $(V''(t, x)\, v, v) \leqslant \alpha |v|^2 \ \forall v \in \boldsymbol{R}^N$.

The following result holds:

THEOREM 1.2. If $V(t, x)$ is a T-periodic potential satisfying (V_1')-(V_4'), then (FHS) possesses infinitely many distinct T-periodic solutions whose Morse index goes to infinity. Moreover, if $V(t, x)$ satisfies (V_5') too, then the solutions found are nonconstant.

REMARK 1.3. Analogous results have been stated by Majer in [14] by different methods and under slightly different assumptions on V (instead of (V_1) and (V_2) he assumes V of class C^1 and $V'(x)\,x - 2V(x) \leqslant a_1 |x|^\theta$ for $|x| \geqslant R$ with $\theta < 2$).

Let us point out that, if $V(x)$ is bounded, that assumption implies that $V'(x)\,x \leqslant a_2 |x|^\theta$ while by (V_2) it follows $V'(x)\,x \leqslant a_3 |x|^2$.

If condition (V_4) (or (V_4')) is dropped, a solution of (HS) (or (FHS)) can vanish and therefore such «collision orbit» cannot be a classical solution. Using the definition of generalized T-periodic solution of (HS) (or (FHS)) introduced in [3], the following results will be stated:

THEOREM 1.4. If $V(x)$ satisfies (V_1), (V_2), (V_3) and (V_5), then, for any $T > 0$, (HS) has infinitely many distinct nonconstant generalized T-periodic solutions.

THEOREM 1.5. If $V(t,x)$ is T-periodic in t and satisfies (V_1'), (V_2'), (V_3') and (V_5'), then (FHS) has at least one nonconstant generalized T-periodic solution.

REMARK 1.6. The existence of generalized solutions has been stated by Bahri and Rabinowitz in [3] (cf. also [13]) in the case where $a=0$, $V<0$, $V\to 0$ and $V'\to 0$ as $|x|\to +\infty$. On the other hand, the existence of a T-periodic «non collision orbit» has been proved in [2, 9] and [10] when $V(x)\simeq -|x|^{-\gamma}$ (near 0) with $\gamma<2$.

REMARK 1.7. Let us point out that we can always assume, in the autonomous case also, that $a+\alpha<(\pi/T)^2$. Infact, if that is not true, it is sufficient to look for T/k-periodic solutions of (HS) with k so large that $a+\alpha<(k\pi/T)^2$.

Moreover, without loss of generality, we can suppose that there exists $\beta\in \boldsymbol{R}$ such that

$$(V_6) \qquad V(x)\leq\beta \quad \forall x\in \boldsymbol{R}^N-\{0\}.$$

Indeed, if we take $\bar a\in]a+\alpha,(\pi/T)^2[$, problem (HS) can be written as

$$\ddot x+\bar a x+\bar V'(x)=0$$

where $\bar V(x)=V(x)-1/2(\bar a-a)|x|^2$ satisfies assumptions (V_1)-(V_6).

Now, we introduce some notations which will be used in the following. For any $T>0$, let $|\cdot|_2$ and $|\cdot|_\infty$ the usual norms in $L^2=L^2([0,T],\boldsymbol{R}^N)$ and $C([0,T],\boldsymbol{R}^N)$; moreover let $H^1=H^1([0,T],\boldsymbol{R}^N)$ be the Sobolev space obtained by the closure of the C^∞ T-periodic $\boldsymbol{R}^N$-valued functions $x=x(t)$ equipped with the norm

$$\|x\|_{H^1}=\left[\int_0^T(|\dot x(t)|^2+|x(t)|^2)\,dt\right]^{1/2}.$$

In the sequel we shall consider the following equivalent norm in H^1

$$\|x\|=\left[|x(0)|^2+\int_0^T|\dot x(t)|^2\,dt\right]^{1/2}.$$

Then, clearly,

$$H^1=\boldsymbol{R}^N+H_0^1 \tag{1.8}$$

where

$$H_0^1=\{x\in H^1\,|\,x(0)=0\}.$$

Given $x\in H^1$, if we set $\bar x(t)=x(t)-x(0)$, it follows that $\bar x\in H_0^1$ and therefore

$$|\bar x|_2\leq T/\pi|\dot x|_2. \tag{1.9}$$

Finally, let us recall the following inequality

$$|x|_2\leq T/\pi|\dot x|_2+\sqrt{T}\rho(x) \tag{1.10}$$

where $\rho(x)=\min_t|x(t)|$.

2. The strong force case

From now on, Ω will be denote the open set $\boldsymbol{R}^N - \{0\}$ and $\Lambda^1\Omega$ the loop space on $\boldsymbol{R}^N - \{0\}$, *i.e.* $\Lambda^1\Omega = \{x \in H^1 \,|\, x(t) \neq 0 \text{ for any } t \in [0,T]\}$.

Let us consider the functional

$$I(x) = 1/2 \int_0^T |\dot{x}(t)|^2\,dt - a/2 \int_0^T |x(t)|^2\,dt - \int_0^T V(x(t))\,dt \tag{2.1}$$

defined on the open subset $\Lambda^1\Omega$. It is easy to verify that $f \in C^2(\Lambda^1\Omega, \boldsymbol{R})$ and its critical points are T-periodic classical solutions of system (HS).

Unfortunately, the action functional $I(x)$ is not bounded from below and does not satisfy the Palais-Smale compactness condition. Then, arguing as in [6] (cf. also [5]), we shall use a penalized functional.

Let $U \in C^2(\boldsymbol{R}^N, \boldsymbol{R}_+)$ be a function with the property

$$U(x)/|x|^2 \to +\infty \qquad \text{as } |x| \to +\infty. \tag{2.2}$$

Moreover, for every $\sigma > 0$ let $\psi_\sigma \in C^2(\boldsymbol{R}_+, \boldsymbol{R}_+)$ such that

$$\begin{cases} \psi_\sigma(t) = 0 & \text{if } t \leqslant \sigma \\ \psi_\sigma(t) = t & \text{if } t \geqslant \sigma + 1 \end{cases} \tag{2.3}$$

and

$$U_\sigma(x) = \psi_\sigma(U(x)). \tag{2.4}$$

Finally, we define the C^2-functional $I_\sigma : \Lambda^1\Omega \to \boldsymbol{R}$ as follows:

$$I_\sigma(x) = I(x) + U_\sigma(x(0)) \tag{2.5}$$

where I is the functional defined in (2.1).

In the sequel we shall denote by c_i some positive constants.

The following Lemmas will be needed:

Lemma 2.6. Let V satisfy (V_1) and (V_4). If $\{x_n\} \subset \Lambda^1\Omega$ and $\{x_n\}$ converges weakly in H^1 to $x \in \Lambda^1\Omega$, then

$$\int_0^T V(x_n(t))\,dt \to -\infty$$

(and therefore $I_\sigma(x_n) \to +\infty$ as $n \to +\infty$).

Proof. See Theorem 0.1 in [8].

Lemma 2.7. For any $\sigma > 0$, I_σ is bounded from below.

Proof. By (V_6) and (1.10) we have that, for any $\varepsilon > 0$,

$$\begin{aligned} I_\sigma(x) &\geqslant 1/2|\dot{x}|_2^2 - a/2|\bar{x}(t) + x(0)|_2^2 - \beta T + U_\sigma(x(0)) \geqslant \\ &\geqslant 1/2(\pi/T)^2|\bar{x}|_2^2 - a/2|\bar{x}|_2^2 - aT/2|x(0)|^2 - a\sqrt{T}\,|\bar{x}|_2\,|x(0)| - \beta T + U_\sigma(x(0)) \geqslant \\ &\geqslant 1/2\{[(\pi/T)^2 - a - \varepsilon^2]|\bar{x}|_2^2 - a(T + aT/\varepsilon^2)|x(0)|^2\} - \beta T + U_\sigma(x(0)). \end{aligned}$$

Choosing ε small enough, by (2.2) and (2.4) it follows that

(2.8) $$I_\sigma(x) \geq c_1 |\bar{x}|_2^2 + c_2 |x(0)|^2 - c_3$$

and therefore I_σ is bounded from below.

LEMMA 2.9. For any $\sigma > 0$, I_σ verifies the Palais-Smale condition on $\Lambda^1\Omega$, *i.e.* every sequence $\{x_n\} \subset \Lambda^1\Omega$ such that

(2.10) $$\{I_\sigma(x_n)\} \quad \text{is bounded}$$

and

(2.11) $$\{I'_\sigma(x_n)\} \quad \text{converges to } 0 \text{ as } n \to +\infty,$$

possesses a subsequence convergent to an element of $\Lambda^1\Omega$.

PROOF. Let $\{x_n\}$ be a sequence in $\Lambda^1\Omega$ satisfying (2.10) and (2.11). By (2.8) it follows that the sequences $\{|\bar{x}_n|_2\}$ and $\{|x_n(0)|\}$ are bounded. Then

(2.12) $$\{|x_n|_2\} \quad \text{is bounded.}$$

Moreover, by (2.10) and (2.12) also $\{|\dot{x}_n|_2\}$ is bounded, and therefore there exists a positive constant M_σ such that

(2.13) $$\|x_n\| \leq M_\sigma \quad \text{for any } n \in \mathbf{N}.$$

Then there exist a subsequence of $\{x_n\}$, still denoted $\{x_n\}$, and $x \in H^1$ such that $x_n \to x$ weakly in H^1 and uniformly in $[0, T]$.

By (2.10) and by Lemma 2.6 we have that $x \in \Lambda^1\Omega$.

Finally, using standard arguments, it can be shown that $\{x_n\}$ strongly converges to x in H^1 (cf. *e.g.* [7]).

REMARK. 2.14. From now on, we shall consider the singular homology with coefficients in a field G such that $H_q(\Lambda^1\Omega, G) \neq 0$ for infinitely many $q \in \mathbf{N}$ (cf. [11], Prop. 3.6).

Moreover, let us denote by $m(x)$ (respectively by $m^*(x)$) the strict Morse index (resp. the large Morse index) of a critical point x of f, *i.e.* $m(x)$ (resp. $m^*(x)$) is the dimension of the maximal subspace of H where the second derivative $f''(x)$ is negative definite (resp. negative semidefinite).

By Lemmas 2.6, 2.7 and 2.9 and by an abstract critical point theorem proved in [6] (cf. also [5], [4] and [15]) we get the following

PROPOSITION 2.15. For any $\sigma > 0$, the functional I_σ has a critical point x_σ in $\Lambda^1\Omega$ corresponding to the critical value

(2.16) $$c_\sigma = \inf_{A \in \Gamma_q} \sup_{x \in A} I_\sigma(x)$$

where $\Gamma_q = \{A \subset \Lambda^1\Omega \,|\, i^*(H_q(A, G) \neq 0)\}$ and $i: A \to \Lambda^1\Omega$ is the inclusion map.

Moreover

(2.17) $$m(x_\sigma) \leq q \leq m^*(x_\sigma)$$

where $q \in \mathbf{N}$ and q is independent of σ.

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1. Let $q \in \boldsymbol{N}$, $q > N$, such that $H_q(\Lambda^1 \Omega, G) \neq 0$. For any $\sigma > 0$ let x_σ be a critical point of I_σ satisfying (2.16) and (2.17). In order to prove Theorem 1.1, it suffices to show that there exist two positive constants $\bar{\sigma}$ and c_4 such that

$$|x_\sigma|_\infty \leqslant c_4 \qquad \text{for any } \sigma > \bar{\sigma}. \tag{2.18}$$

Indeed, if (2.18) holds, for any $\sigma > \max\{\bar{\sigma}, c_4\}$ it results $U_\sigma(x_\sigma(0)) = 0$, then $I_\sigma(x_\sigma) = I(x_\sigma)$ and x_σ is a critical point of I.

Assume by contradiction that (2.18) does not hold, *i.e.* there exist $\{\sigma_n\} \to +\infty$ and $\{x_n\} \subset \Lambda^1 \Omega$ such that x_n is a critical point of $I_n = I_{\sigma_n}$ satisfying (2.16), (2.17) and

$$|x_n|_\infty \to +\infty \qquad \text{as } n \to +\infty. \tag{2.19}$$

As the singular homology has compact support, by (2.16) it follows that there exists c_5 such that

$$I_n(x_n) \leqslant c_5 \qquad \text{for any } n \in \boldsymbol{N}. \tag{2.20}$$

By (1.10) and (V_6), as $U_\sigma(x(0)) \geqslant 0$, it follows that

$$\begin{aligned} I_n(x_n) &\geqslant 1/2|\dot{x}_n|_2^2 - a/2[(T/\pi)|\dot{x}|_2 + \sqrt{T}\rho(x_n)]^2 \geqslant \\ &\geqslant 1/2[(1 - a(T/\pi)^2)|\dot{x}|_2^2 - aT\rho^2(x_n)] - (a\sqrt{T^3}\rho(x_n)/\pi)|\dot{x}_n|_2 . \end{aligned} \tag{2.21}$$

By (2.19), (2.20) and (2.21) we have that

$$\min_t |x_n(t)| \to +\infty \qquad \text{as } n \to +\infty,$$

then there exists $\nu \in \boldsymbol{N}$ such that

$$|x_n(t)| \geqslant R \qquad \text{for any } n > \nu, \qquad \text{for } t \in [0, T], \tag{2.22}$$

where R is the constant introduced in assumption (V_2).

So, by (2.22) and (V_2) (cf. Remark 1.7 also) it follows that, for any $n > \nu$,

$$\begin{aligned} I_n''(x_n)[v, v] &= \int_0^T [|\dot{v}|^2 - a|v|^2 - (V''(x_n)v, v)]\,dt + (U_n''(x_n(0))v(0), v(0)) \geqslant \\ &\geqslant [(\pi/T)^2 - a - \alpha]|v|_2^2 > 0 \qquad \text{for any } v \in H_0^1 . \end{aligned} \tag{2.23}$$

By virtue of (1.8) and (2.23) we can deduce that $m^*(x_n) \leqslant N$ for any $n > \nu$, in contradiction with (2.17).

We conclude that (2.18) holds, and therefore the action functional I has infinitely many distinct critical points whose corresponding Morse index increases to infinity.

Finally, let us prove that the solutions found are nonconstant, if the potential V satisfies the additional condition (V_5).

Indeed, by (V_5), there exists $\delta > 0$ such that

$$|x| \geqslant \delta \qquad \text{for any } x \in \boldsymbol{R}^N - \{0\} \qquad \text{with } V'(x) + ax = 0.$$

Let x be a constant critical point of I (*i.e.* a constant solution of the equation

$V'(x) + ax = 0$); there are two possibilities:

$$i)\ |x| > R \qquad ii)\ \delta \leqslant |x| \leqslant R.$$

In the first case, arguing as in (2.23), it results $m^*(x) \leqslant N$; in the second case, as V'' is bounded on the set $\{x \in \boldsymbol{R}^N \,|\, \delta \leqslant |x| \leqslant R\}$, there exists a constant c_6, independent of x, such that $m^*(x) \leqslant c_6$. Choosing q large enough, the conclusion follows by (2.17).

Proof of Theorem 1.2. See proof of Theorem 1.1.

3. The weak force case

In this section we will state the existence of T-periodic solutions of (HS) (or (FHS)) when assumption (V_4) is dropped. It is known that, if the potential V satisfies the «weak force» condition (V_3), there exist $x \in H^1$ such that $x(t) = 0$ for some t and $I(x) < +\infty$, then the critical points of I may enter the singularity, *i.e.* may be «collision orbits». So, we shall use the following definition of a generalized solution introduced in [3]:

Definition 3.1. We say that $x \in H^1$ is a generalized T-periodic solution of (HS) (or (FHS)) iff

(i) The set $D = \{t \in \boldsymbol{R} \,|\, x(t) = 0\}$ has zero measure;

(ii) $x = x(t)$ is C^2 and solves (HS) (or (FHS)) on $\boldsymbol{R} - D$.

As the sublevels $I^b = \{x \in H^1 \,|\, I(x) \leqslant b\}$ are not complete, we modify V by setting

$$V_\varepsilon(x) = V(x) - \varepsilon/|x|^2 \quad \text{for any } \varepsilon > 0. \tag{3.2}$$

System (HS) becomes

$$\ddot{x} + ax + V'(x) + 2\varepsilon x/|x|^4 = 0 \tag{HS$_\varepsilon$}$$

with energy functional

$$I_\varepsilon(x) = \int_0^T [1/2(|\dot{x}|^2 - a|x|^2) - V(x)]\, dt + \int_0^T \varepsilon/|x|^2\, dt. \tag{3.3}$$

Proof of Theorem 1.3. Let $q \in \boldsymbol{N}$, $q > N$, such that $H_q(\Lambda^1\Omega, G) \neq 0$. Since ($V_\varepsilon$) verifies conditions ($V_1$)-($V_4$), Theorem 1.1 implies that for any $\varepsilon > 0$ the functional I_ε has a critical point x_ε in $\Lambda^1\Omega$ such that

$$I_\varepsilon(x_\varepsilon) = \inf_{A \in \Gamma_q} \sup_{x \in A} I_\varepsilon(x) \tag{3.4}$$

and

$$m(x_\varepsilon) \leqslant q \leqslant m^*(x_\varepsilon). \tag{3.5}$$

As the singular homology has compact support, by (3.4) there exists a positive constant c_7 such that

$$(3.6) \qquad I_\varepsilon(x_\varepsilon) \leqslant c_7 \quad \text{for any } \varepsilon \in]0,1].$$

Arguing as in the proof of Theorem 1.1, by (3.5), (3.6), and (V_2) it follows that there exists a positive constant c_8 such that $|x_\varepsilon|_\infty \leqslant c_8$ for any $\varepsilon \in]0,1]$.

Then there exists a sequence $\{x_n\}$, $x_n = x_{\varepsilon_n}$, with $\varepsilon_n \to 0$, such that $x_n \to x$ weakly in H^1 and uniformly in $[0,T]$.

By (3.6) and (V_3), the set D where x vanishes has measure 0. Since x_n solves $(\mathrm{HS})_{\varepsilon_n}$, it is easy to prove that x is a classical solution of (HS) on $\boldsymbol{R} - D$, and therefore x is a generalized solution of (HS).

Finally, we prove that x is nonconstant. This is obvious if $D \neq \emptyset$; assume $D = \emptyset$, *i.e.* $x \in \Lambda^1\Omega$; by (3.5) it results

$$(3.7) \qquad m^*(x) \geqslant q.$$

On the other hand, by (V_5) and arguing as in the proof of Theorem 1.1, there exists a positive constant c_9 such that

$$(3.8) \qquad m^*(\bar{x}) \leqslant c_9 \quad \text{for any } \bar{x} \in \boldsymbol{R}^N - \{0\} \quad \text{with } V'(\bar{x}) + a\bar{x} = 0.$$

By (3.7) and (3.8), choosing q large enough, we can conclude that the solution found is nonconstant. Then for any $T > 0$ (HS) has a nonconstant generalized T-periodic solution. Now, for each $k \in \boldsymbol{N}$, let $x_k(t)$ be a nonconstant T/k periodic solution of (HS); say $x_1(t)$ has minimal period T/k_1. Then for $k > k_1$, $x_k(t)$ is distinct from $x_1(t)$. Arguing similarly, it can shown that infinitely many of the functions $x_k(t)$ are distinct.

Proof of Theorem 1.5. See proof of Theorem 1.3.

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