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The nappe profile of a free overfall

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ABSTRACT. — The phenomenon of the free overfall at the sharp drop of a channel bed has been deeply investigated experimentally since the pioneering work of Rouse (1933). Its behaviour is well known at least in the usual case of a wide rectangular channel. However, no complete theoretical solution has yet been obtained. Assuming the steady flow to be two-dimensional, irrotational and frictionless, an analytical solution for the flow field is obtained accounting for the presence of two free boundaries. By applying the conservation laws we then derive an equation for the lower nappe profile which is found to fit the observed data satisfactorily.

KEY WORDS: Open-channel; Free overfall; Nappe.

RIASSUNTO. — *Il profilo della vena fluente in una caduta libera.* Il fenomeno della caduta libera di una corrente nel salto brusco di fondo di un canale è stato esaminato con cura sperimentalmente fin dall'iniziale lavoro di Rouse del 1933. Il suo comportamento è ben noto almeno nella situazione consueta di canale rettangolare largo. Tuttavia, una completa soluzione teorica non è stata ancora ottenuta. Assumendo che il moto permanente sia bidimensionale, irrotazionale e senza resistenze, si ricava una soluzione analitica per il campo di moto mettendo in conto la presenza di due contorni liberi. Applicando le leggi di conservazione, si deduce quindi un'equazione del profilo inferiore della vena libera che risulta in accordo con i rilievi sperimentali noti.

NOTATIONS

ρ = liquid density;
g = acceleration of gravity;
q^* = rate of discharge per unit span;
$Y_c^* = (q^{*2}/g)^{1/3}$ = critical depth.

Dimensionless variables:

$x = x^*/Y_c^*$ = horizontal co-ordinate;
$y = y^*/Y_c^*$ = vertical co-ordinate;
$h = h^*/Y_c^*$ = vertical co-ordinate of the upper profile;
$z = z^*/Y_c^*$ = vertical co-ordinate of the lower profile;
$Y = h - z$ = stream depth or vertical dimension of the nappe;
$Y_0 = Y_0^*/Y_c^*$ = brink depth (depth at the cross section $x = 0$);
$Y_u = Y_u^*/Y_c^*$ = depth for uniform supercritical flow ($x \to -\infty$);
$Y_L = Y_L^*/Y_c^*$ = vertical dimension of the nappe at the limit section ($x \to \infty$);
$v_x = v_x^*/(g\,Y_c^*)^{1/2}$ = horizontal velocity component;
$v_y = v_y^*/(g\,Y_c^*)^{1/2}$ = vertical velocity component;
$\Psi = \Psi^*/q^*$ = stream function;
$p = p^*/\rho g Y_c^*$ = pressure (over the atmospheric value);
$H = H^*/Y_c^*$ = total head;
$M = M^*/\rho g Y_c^{*2}$ = total momentum per unit span.

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1. Introduction

The open-channel flow at the sharp drop of a wide rectangular channel – the so-called free overfall (see [7, 8] and fig. 1) – is studied below, under steady conditions, as a *frictionless, irrotational* and *two-dimensional flow*. Reference is made to an orthogonal system x^*, y^* with x^* the horizontal axis and the origin located at the edge of the channel. Denoting by $y^* = h^*(x^*)$ and $y^* = z^*(x^*)$ the equations of the upper and lower profiles of the flowing stream, v_x^* and v_y^* the velocity components, p^* the pressure, ρ the constant density and g the gravity, in any vertical section we write:

$$Y^* = h^* - z^*, \tag{1.1}$$

$$H^* = y^* + p^*/\rho g + (v_x^{*2} + v_y^{*2})/2g, \tag{1.2}$$

$$M^* = \int_{z^*}^{h^*} (p^* + \rho v_x^{*2})\, dy^*, \tag{1.3}$$

where Y^* is the depth of the stream or the vertical dimension of the nappe, H^* is total head referred to the x^*-axis and M^* is the «total» momentum, *i.e.* the momentum flow rate corrected for changes in the horizontal pressure force.

With reference to a discharge q^* per unit span, the following scaling is introduced:

a) lengths are scaled by the critical depth Y_c^* defined as

$$Y_c^* = (q^{*2}/g)^{1/3}; \tag{1.4}$$

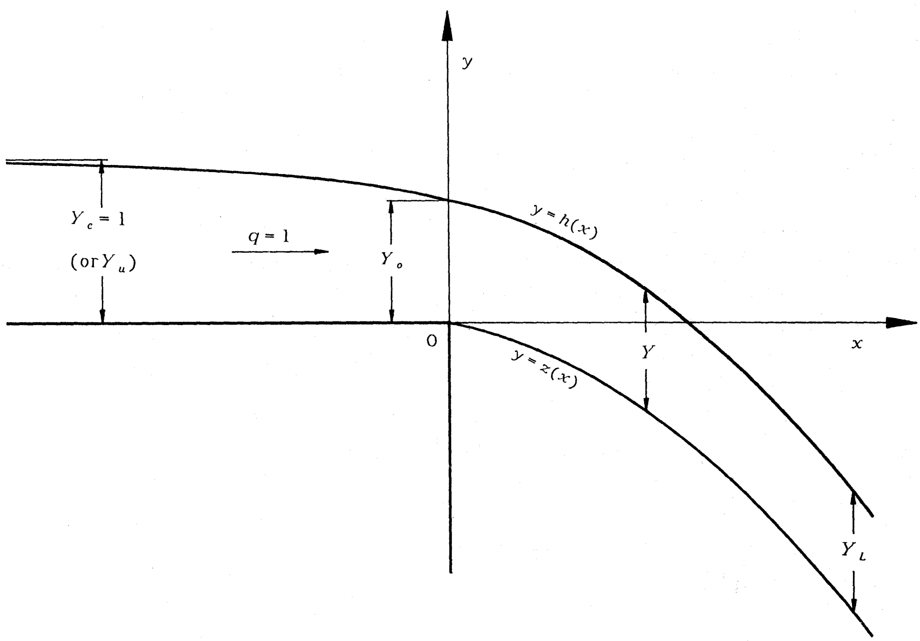

Fig. 1. – Definition sketch of the free overfall.

b) velocities are scaled by the critical speed $(gY_c^*)^{1/2}$;

c) pressures are scaled by the pressure $\rho g Y_c^*$;

d) the stream function Ψ^* is scaled by the discarge q^*.

Each dimensionless variable will be written using the same symbol as for the dimensional one dropping the star: $x = x^*/Y_c^*$, etc.

2. Equations of motion

The dimensionless stream function $\Psi(x, y)$ is assumed to be zero at $y = z$ and, consequently, $\Psi = 1$ at $y = h$. It is expanded in a power series of $(y - z)$, truncated at the third order. The procedure follows that proposed by Benjamin and Lighthill [1] to study cnoidal waves flowing over a flat horizontal bottom.

Since the function Ψ must be a harmonic function, it can be written in the form

$$\Psi(x,y) = (y-z)f(x) + \frac{1}{2}(y-z)^2(2z'f' + z''f) - \frac{1}{3!}(y-z)^3 f'' \tag{2.1}$$

where $f(x)$ is arbitrary and a prime denotes differentiation with respect to x.

By imposing the boundary condition

$$\psi(x,h) = 1 \tag{2.2}$$

and by using the dimensionless form of eq. (1.1)

$$Y = h - z \tag{2.3}$$

it follows

$$f(x) = 1/Y - (1/2)(2z'f' + z''f)Y + (1/6)f''Y^2 . \tag{2.4}$$

We then differentiate twice eq. (2.4) and neglect terms contributing to the fourth order in eq. (2.1) to obtain

$$f' = -(1/Y^2)(h' - z'), \tag{2.5}$$

$$f'' = (1/Y^3)[2(h' - z')^2 - Y(h'' - z'')]. \tag{2.6}$$

Substituting f' and f'' from (2.5) and (2.6) in eq. (2.1), we can write the expression of the stream function Ψ to the previous order of approximation and, consequently, the velocity components

$$v_x = \Psi_{,y} = \tag{2.7}$$

$$= \frac{1}{Y}\left\{1 + \left(\frac{y-z}{Y} - \frac{1}{2}\right)[Yz'' - 2z'(h'-z')] + \left[\left(\frac{y-z}{Y}\right)^2 - \frac{1}{3}\right]\left[\frac{Y}{2}(h''-z'') - (h'-z')^2\right]\right\},$$

$$v_y = -\Psi_{,x} = \frac{1}{Y}\left\{z' + \frac{y-z}{Y}(h'-z')\right\}. \tag{2.8}$$

The total head is then derived from (1.2) in the dimensionless form

$$H = H^*/Y_c^* = y + p + (1/2)(v_x^2 + v_y^2). \tag{2.9}$$

Now, in any steady, inviscid and irrotational flow the quantities H and M must take constant values; hence, we can calculate H with reference to the streamline $y = h = Y + z$ where the relative pressure vanishes

$$H = Y + z + \frac{1}{2Y^2}\left[1 + \frac{Y}{3}(2h'' + z'') - \frac{1}{3}h'^2 + \frac{2}{3}z'^2 + \frac{2}{3}h'z'\right]. \tag{2.10}$$

The total momentum M^*, given by eq. (1.3), is equivalent to

$$M^* = \int_{z^*}^{Y^*+z^*} \rho g\left[H^* - y^* + \frac{1}{2g}(v_x^{*2} - v_y^{*2})\right] dy^* \tag{2.11}$$

and, in non-dimensional form:

$$M = \frac{M^*}{\rho g Y_c^{*2}} = HY - \frac{Y^2}{2} - Yz + \frac{1}{2}\int_z^{Y+z}(v_x^2 - v_y^2)\,dy. \tag{2.12}$$

Substituting (2.7) and (2.8) in eq. (2.12), we obtain

$$h'^2 + z'^2 + h'z' = -3Y^3 + 6HY^2 - 6MY + 3 - 6Y^2 z \tag{2.13}$$

which reduces to the equation of Benjamin and Lighthill [1, p. 458] when z and z' vanish everywhere (flat horizontal bottom).

Eliminating H from eqs. (2.10) and (2.13) and solving for M we finally find

$$M = \frac{Y^2}{2} + \frac{1}{Y}\left[1 + \frac{Y}{6}(2h'' + z'') - \frac{h'^2}{3} + \frac{z'^2}{6} + \frac{h'z'}{6}\right]. \tag{2.14}$$

3. Pressure distribution

Recalling that the flow is steady and irrotational (hence $H = \text{const}$), from eq. (2.9) we obtain

$$\frac{\partial p}{\partial y} = -1 - v_x\frac{\partial v_y}{\partial x} - v_y\frac{\partial v_y}{\partial y}. \tag{3.1}$$

Differentiating eqs. (2.7) and (2.8) to the order of approximation stated before and substituting in eq. (3.1), we have

$$\frac{\partial p}{\partial y} = -1 - \frac{z''}{Y^2} + \frac{z'(h' - z')}{Y^3} + \frac{y - z}{Y^4}[(h' - z')^2 - Y(h'' - z'')] \tag{3.2}$$

from which the following pressure distribution is obtained

$$p(x, y) = -\int_y^h \frac{\partial p}{\partial y}dy = h - y + \frac{z''(h - y)}{Y^2} - \frac{z'(h' - z')(h - y)}{Y^3} - \left[\frac{h^2 - y^2}{2Y^4} - \frac{z(h - y)}{Y^4}\right][(h' - z')^2 - Y(h'' - z'')] \tag{3.3}$$

where $p(x, h) = 0$.

On the lower profile ($y=z$) the pressure reads

$$p(x,z) = Y + \frac{h'' + z''}{2Y} - \frac{h'^2 - z'^2}{2Y^2}. \tag{3.4}$$

Therefore, in order for the pressure to vanish on the lower profile of the nappe the following condition

$$2Y^3 + Yh'' + Yz'' - h'^2 + z'^2 = 0 \qquad (\text{for } x \geq 0) \tag{3.5}$$

must be satisfied.

4. Forms of free overfall

Free overfall is the flow which takes place at the drop of a flat channel when the bottom presents a discontinuity such that the stream detaches from the bottom (see, for example, [5]). The flow may originate upstream from a subcritical state or from a supercritical state depending on the channel slope, respectively smaller or greater than the critical slope for the given discharge. In the supercritical case, the upstream flow is assumed to take always the uniform depth.

Proceeding upstream from the *brink section* – the crosss section at the end of the channel ($x=0$) – it was experimentally observed that the flow reaches the critical depth Y_c^* (or the uniform depth Y_u^* in the supercritical case) at a distance not greater than $4Y_c^*$ (or $4Y_u^*$). Since this channel stretch – represented by the whole field of negative abscissae in the present model – is very short, we may neglect the energy dissipation and assume the channel bottom to be horizontal. In fact, physically, this assumption is equivalent to considering the slope of the total head line coincident with the slope of the channel bottom.

Therefore the above two configurations are characterized by the following conditions:

a) Upstream subcritical flow

$$[Y]_{x \to -\infty} = 1; \qquad H = 3/2; \qquad M = 3/2; \tag{4.1}$$

b) Upstream supercritical flow (uniform stream)

$$[Y]_{x \to -\infty} = Y_u; \qquad H = H_u = Y_u + 1/2Y_u^2; \qquad M = M_u = Y_u^2/2 + 1/Y_u. \tag{4.2}$$

5. The nappe

A number of approximate solutions have been proposed to obtain the nappe profile. They are based on relaxation methods [9], iterative methods [3] or electrical analogy [4]. Here we propose an analytical solution derived on the basis of the above mentioned hypothesis, namely that the total momentum M, expressed by eq. (2.14), maintains a constant value. Inserting eq. (3.5), valid for $x \geqslant 0$, in eq. (2.14) the result is

$$Yh'' - h'^2 + h'z' = -Y^3 + 6MY - 6 \tag{5.1}$$

and

$$Yz'' + z'^2 - h'z' = -Y^3 - 6MY + 6. \tag{5.2}$$

Subtracting (5.2) from (5.1) it follows

$$Y(h'' - z'') - (h' - z')^2 = 12MY - 12, \tag{5.3}$$

that is

$$YY'' - Y'^2 - 12MY + 12 = 0. \tag{5.4}$$

The integral of eq. (5.4) is

$$Y = Y_L + Ae^{-Bx} \tag{5.5}$$

where A and B are arbitrary constants and Y_L is the *limit* value of Y for $x \to \infty$ (where $Y' = Y'' = 0$). This gives

$$Y_L = 1/M. \tag{5.6}$$

In the case of *upstream subcritical flow* Y_L has the well known [8] value

$$Y_L = 2/3. \tag{5.7}$$

Otherwise, if the jet derives from an *upstream supercritical flow* with uniform depth Y_u, the value of Y_L is

$$Y_L = 1/M_u = 2Y_u(Y_u^3 + 2)^{-1} \tag{5.8}$$

as obtained from (4.2).

As regards the constant A, we refer to the section $x = 0$. Denoting by Y_0 the *brink depth*, it follows

$$A = Y_0 - Y_L. \tag{5.9}$$

In order to determine the constant B we differentiate twice eq. (5.5)

$$Y' = -ABe^{-Bx}; \qquad Y'' = AB^2 e^{-Bx}$$

and substitute these expressions in eq. (5.4)

$$YAB^2 e^{-Bx} - A^2 B^2 e^{-2Bx} - 12MY + 12 = 0. \tag{5.10}$$

Inserting (5.5) in the latter relationship, it becomes

$$Y_L B^2 - 12M = 0 \tag{5.11}$$

that is

$$B = \sqrt{12M/Y_L}. \tag{5.12}$$

Summarizing we find:

$$\text{upstream subcritical flow: } B = 3^{3/2}; \tag{5.13}$$

$$\text{upstream supercritical flow: } B = 3^{1/2} \cdot 2M_u. \tag{5.14}$$

6. The pressure distribution on the nappe

By applying eq. (3.2) to the nappe profiles – where eqs. (5.1) and (5.2) hold – we obtain

$$[\partial p/\partial y]_{y=z} = (6MY-6)/Y^3\,, \tag{6.1}$$

$$[\partial p/\partial y]_{y=h} = -(6MY-6)/Y^3\,. \tag{6.2}$$

These equations show that the y-component of the pressure gradient has the same absolute value on both profiles; along any vertical section of the jet the pressure distribution is symmetric.

In particular, at the *limit section* , where $h' = z'$ and $Y = Y_L = 1/M$, it follows from (5.1) and (5.2) that

$$h'' = z'' = -Y^2 \tag{6.3}$$

and hence, by means of eq. (3.2)

$$[\partial p/\partial y]_{Y=Y_L} = 0\,. \tag{6.4}$$

Thus we find the known result that the pressure p attains the atmospheric value in every point of the limit vertical section.

7. The lower nappe profile

The relative difference between the vertical dimension of the jet at an arbitrary section and the dimension of the limit section is given by eq. (5.5)

$$(Y-Y_L)/Y_L = (Y_0/Y_L - 1)\,e^{-Bx}\,. \tag{7.1}$$

When the upstream flow is *subcritical* we can assume the brink depth Y_0 to take the experimental mean value 0.716 found by Rouse [8]. Using the value $B = 3^{3/2}$ (see (5.13)), eq. (7.1) leads to results some of which are listed in table I.

Table I.

x	$(Y-Y_L)/Y_L$	x	$(Y-Y_L)/Y_L$
0	0.074	0.3	0.016
0.1	0.044	0.4	0.009
0.2	0.026	0.5	0.005

The relative difference given by eq. (7.1) has the maximum value of about 7% at $x = 0$ and becomes smaller than 1% if $x \geqslant 0.4$. In the case of an upstream *supercritical flow* this difference is even less, because Y_0/Y_L is smaller than 1.074 and B is larger than $3^{3/2}$.

On the basis of these observations, in order to calculate the lower profile of the jet, we may assume Y to be constant and equal to Y_L along the whole jet. Under this condition it follows that $h' = z'$, $h'' = z''$ and, by eq. (3.5),

$$z'' = -Y_L^2 \tag{7.2}$$

just as in the limit section.

The integral of eq. (7.2) is

$$z' = -Y_L^2 x - b \tag{7.3}$$

where b (assumed to be positive) can be derived from eq. (2.13) at $x = z = 0$, that is

$$3[z']_0^2 = -3Y_L^3 + 6HY_L^2 - 6MY_L + 3 \tag{7.4}$$

with the previous assumption that $[Y]_0 = Y_L$. Therefore, we have

$$b = (-Y_L^3 + 2HY_L^2 - 2MY_L + 1)^{1/2}. \tag{7.5}$$

Inserting (7.5) in eq. (7.3), it follows that

$$-z = \frac{1}{2} Y_L^2 x^2 + x(-Y_L^3 + 2HY_L^2 - 2MY_L + 1)^{1/2} \tag{7.6}$$

being $z = 0$ at $x = 0$.

For *upstream subcritical flow*, where $Y_L = 2/3$, eq. (7.6) becomes

$$-z = 0.222\,x^2 + 0.192\,x \tag{7.7}$$

which is valid for a gravitational jet flowing from a smooth channel and unaffected by surface tension [2, 6].

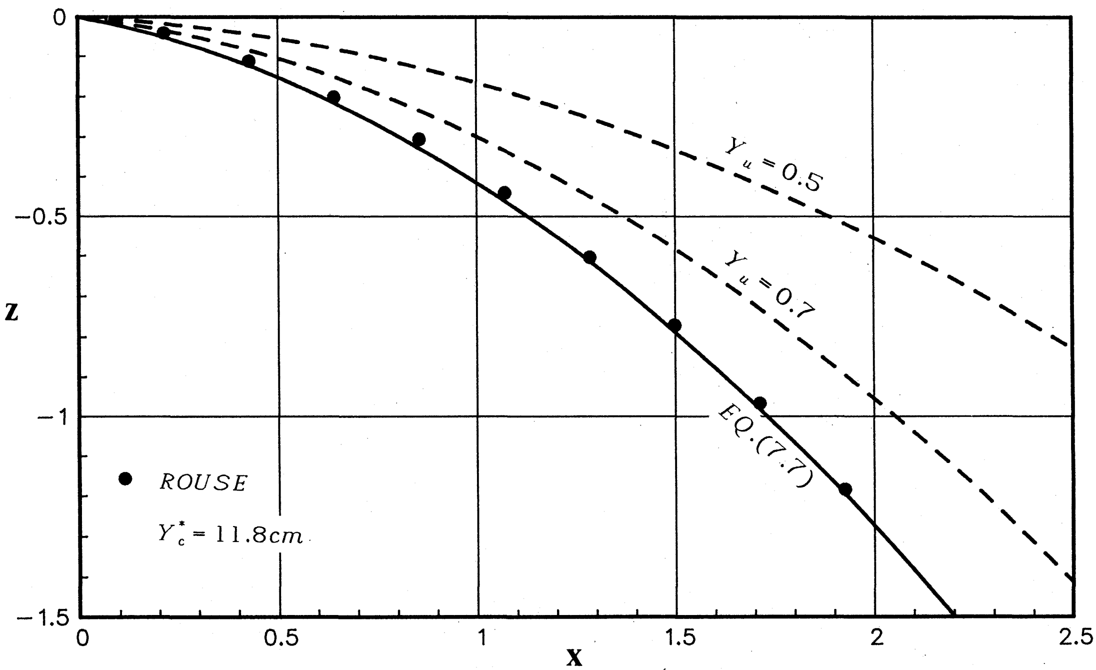

Fig. 2. – The lower profile (7.7) of the nappe – corresponding to upstream subritical flow – compared with the experimental data of Rouse [7]. The theoretical profiles corresponding to $Y_u = 0.5$ and $Y_u = 0.7$ are also plotted (dashed lines).

Recalling the expression (5.6) for Y_L, eq. (7.6) may be written in the form:

(7.8) $$-z = ax^2 + bx$$

with

(7.9) $$a = (1/2)M^{-2}$$

(7.10) $$b = (2HM^{-2} - M^{-3} - 1)^{1/2}.$$

Some values of a and b, as functions of Y_u, are given in table II.

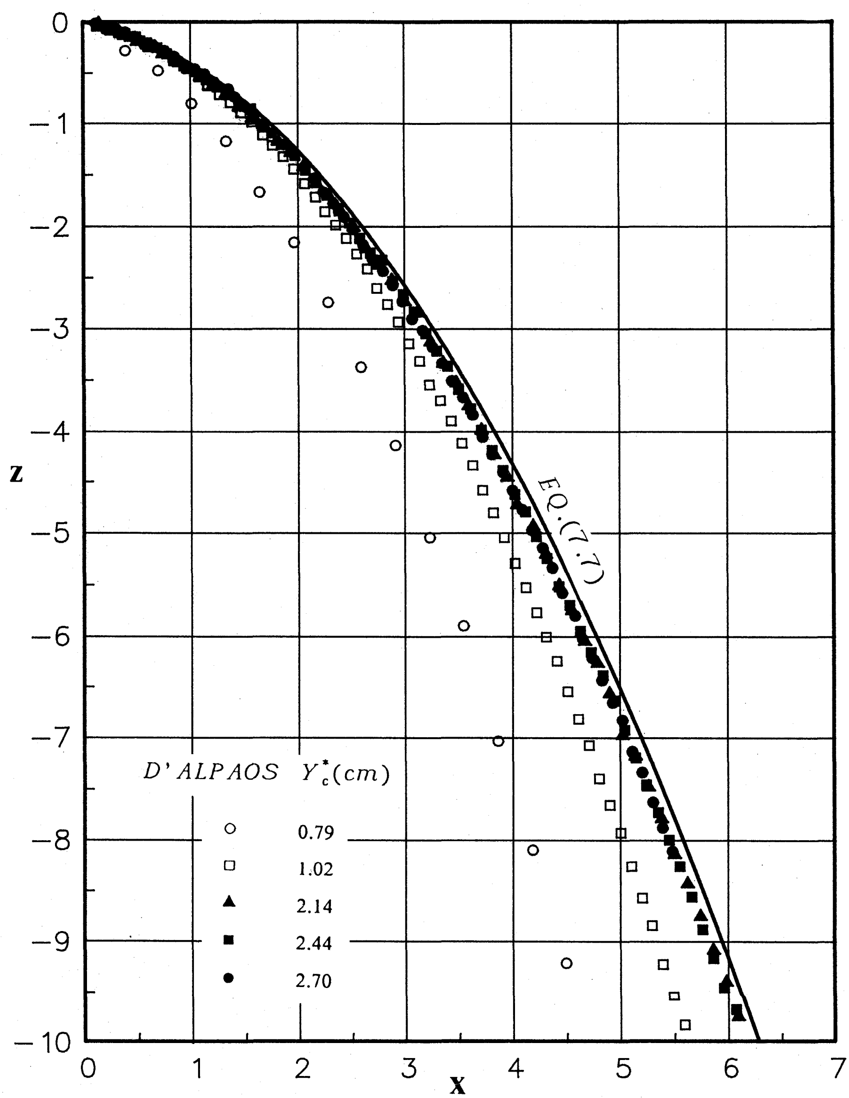

Fig. 3. – Comparison of the lower profile (7.7) of the nappe with the experimental data of D'Alpaos [2] referring to small values of the critical depth.

TABLE II.

$[Y]_{-\infty}$	H	M	Y_L	a	b
$Y_c = 1$	3/2	3/2	2/3	0.222	0.192
$Y_u = 0.9$	1.517	1.516	0.660	0.217	0.182
$Y_u = 0.8$	1.581	1.570	0.637	0.203	0.157
$Y_u = 0.7$	1.720	1.674	0.597	0.178	0.123
$Y_u = 0.6$	1.989	1.847	0.541	0.147	0.087
$Y_u = 0.5$	2.500	2.125	0.471	0.111	0.055

The nappe lower profile corresponding to $Y_c = 1$ (eq. (7.6)) is plotted in fig. 2 together with the experimental points of Rouse [7] referring to a horizontal smooth channel. The agreement appears to be satisfactory. In the same fig. 2 two profiles corresponding to the upstream condition $Y_u = 0.5$ and $Y_u = 0.7$ are also plotted.

In fig 3 we have plotted the experimental data of D'Alpaos [2], related to water streams characterized by small values of the critical depth ($0.79 \div 2.70$ cm), so as to point out the influence of surface tension on the nappe. As Y_c^* increases – hence the Weber number decreases – the effect of surface tension on the lower profile is found to vanish. We may observe that our curve (7.7), representing the theoretical profile of inviscid liquid subject only to gravity, fits satisfactorily the experimental points corresponding to the larger values of critical depth tested by D'Alpaos.

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