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A simple proof of Baer;s and Sato's theorems on lattice-isomorphisms between groups

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Teoria dei gruppi. — *A simple proof of Baer's and Sato's theorems on lattice-isomorphisms between groups.* Nota di MARIO MAINARDIS, presentata (*) dal Corrisp. G. Zacher.

ABSTRACT. — A simple proof is given of a well-known result of the existance of lattice-isomorphisms between locally nilpotent quaternionfree modular groups and abelian groups.

KEY WORDS: Group; Lattice; Lattice-isomorphism.

RIASSUNTO. — *Una semplice dimostrazione dei teoremi di Baer e Sato sugli isomorfismi reticolari tra gruppi.* Si dà una semplice dimostrazione di un noto risultato sull'esistenza di isomorfismi reticolari tra gruppi localmente nilpotenti modulari liberi da quaternioni e gruppi abeliani.

INTRODUCTION

We say that a group G is an *M^*-group* if all subgroups of G are quasinormal and G is quaternionfree. The aim of this *Note* is to give a simple and unified proof of the following result:

THEOREM 1. *If G is an M^*-group, then there exists an abelian group A such that the lattice of subgroups of A is isomorphic to the one of G.*

This Theorem was proved by Baer [1, Theorem 7.2] in case G is a torsion group (actually Baer proved the stronger result that the right coset lattice of A is isomorphic to the one of G) and later completed by Sato [6, Theorem 1] in case G contains elements of infinite order.

We shall first show how both cases can be reduced to split extensions of divisible abelian torsion groups by groups of rank 1 and then apply Sato's arguments, which become easier in this particular situation. We remark that in case of torsion groups our result is weaker than Baer's since, as it can easily be checked, the subgroup lattice-isomorphisms we obtain are not induced by coset preserving maps.

All notations are the same as in [4].

PRELIMINARY RESULTS

THEOREM 2 (Iwasawa [2, 3]). *A group G is an M^*-group if and only if G is abelian or an extension of an abelian torsion group T by an abelian group of rank* 1 *such that all subgroups of T are normal and those of prime order and of order* 4 *are central in G.*

REMARK. We recall that M^*-groups are locally nilpotent. Moreover if G is as in Theorem 2 and nonabelian, then for every $z \in G - T$ and $n \in \mathbb{N}$ there exists $\phi'(z, n)$ in

(*) Nella seduta dell'8 febbraio 1992.

Pot(T) such that for every $a, b \in T$ the following identity holds:

(1) $$(abz)^n = (a^n)^{\phi'(z,n)} (bz)^n \quad \text{(s. [4], vi)}.$$

Let m be a negative integer. With the same notations as above, define $\phi'(z,m)$ by

$$a^{\phi'(z,m)} := z^m a^{\phi'(z,-m)} z^{-m}.$$

By Theorem 2 $\phi'(z,m)$ is a power automorphism and, since z and bz induce the same automorphism on $\langle a \rangle$, we have by (1)

$$(abz)^m = ((abz)^{-m})^{-1} = ((a^{-m})^{\phi'(z,-m)} (bz)^{-m})^{-1} = (bz)^m (a^m)^{\phi'(z,-m)} =$$
$$= (bz)^m (a^m)^{\phi'(z,-m)} (bz)^{-m} (bz)^m = (a^m)^{\phi'(z,m)} (bz)^m,$$

so that the identity (1) holds also for negative integers.

If G is also a torsion group, then there exists a supplement Z of rank 1 to T in G.

Lemma 3. *Let G be an extension of a group T and let Z be a supplement of T in G. Then, for any group X such that there is an epimorphism μ from X to Z, G is isomorphic to a factor group of a split extension H of T by X. Moreover if $N \leq T$, then N is normal (central) in G if and only if N is normal (central) in H.*

Proof. Define an action of X on T by $t^x := (x^\mu)^{-1} t(x^\mu)$ for every t in T and x in X. Let H be the semidirect product of T by X with respect to this action. Consider the mapping from H to G which sends the pair $(t, x) \in T \times X$ to $tx^\mu \in G$. Straightforward computation shows that this mapping is an epimorphism. The rest follows immediately.

Corollary 4. *Let G be an M^*-group. Then there is an M^*-group G^0 with $T(G^0)$ divisible abelian such that G is isomorphic to a factor group of a subgroup of G^0.*

Proof. For abelian groups this result is well known. If G is non-abelian, then, by [4, Satz 2], we just have to prove that a torsion M^*-group G is isomorphic to a factor group of an M^*-group H with elements of infinite order. Hence let T and Z be as in the previous remark. Clearly there is a torsion-free group X of rank 1 such that Z is isomorphic to factor group of X. Define H as in the proof of Lemma 3. Then by Theorem 2 and Lemma 3 H has the required properties.

Proof of Theorem 1

Clearly a lattice-isomorphism induces lattice-isomorphisms between corresponding sublattices. Hence, by Corollary 4 and [4, Satz 3], we may assume that the M^*-group G is a split extension of a divisible abelian torsion group T by a torsion-free group Z of rank 1. Suppose first that Z is cyclic and let z be a generator. Using (1) and the fact that T is divisible it is easy to see that every element g of infinite order of G has the following representation

(2) $$g = (tz)^n \quad \text{with } t \in T \quad \text{and } n \in \mathbb{Z} - \{0\}.$$

Let A be the external direct product of T by Z and $\beta \in \mathrm{Pot}(T)$. Define for every (a, y) in A an element $(a,y)^{\pi_\beta}$ in G in the following way:

if $y = 1$, $(a,y)^{\pi_\beta} := a$;

if $y \neq 1$, then, since T is divisible, there is $n \in \mathbb{Z} - \{0\}$ and $t \in T$ such that $(a,y) = (t,z)^n$; in this case put $(a,y)^{\pi_\beta} := (t^\beta z)^n$.

To simplify notations we shall write π instead of π_β. Suppose $(a,z)^n = (b,z)^m$ in A with $a, b \in T$ and $n, m \in \mathbb{Z}$. Then $n = m$ and $a^n = b^n$. Hence $((a,z)^n)^\pi = (a^\beta z)^n = ((a^\beta)^n)^{\phi'(z,n)} z^n = ((a^n)^\beta)^{\phi'(z,n)} z^n = ((b^n)^\beta)^{\phi'(z,n)} z^n = (b^\beta z)^n = ((b,z)^n)^\pi$. Thus π is a well-defined map from A to G and, by (2), surjective. Moreover, by definition, π induces an isomorphism between every cyclic subgroup of A and its image in G. It follows that π is bijective and induces an inclusion preserving bijection between the partially ordered set of cyclic subgroups of A and the one of G. Let now W, X, Y be cyclic subgroups of A. We show that

$$(*) \qquad W \leqslant XY \quad \text{if and only if} \quad W^\pi \leqslant X^\pi Y^\pi .$$

This is obvious if $X, Y \leqslant T(A)$. Suppose first that $X \leqslant T(A)$ and $|Y| = \infty$. Then $X = \langle (a,1) \rangle$ and $Y = \langle (b,z)^n \rangle$ with $a, b \in T$ and $n \in \mathbb{N} - \{0\}$. Let $W \leqslant XY$. Then there are $h \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $W = \langle (a^h b^{nk}, z^{nk}) \rangle$. Since T is divisible, there is $c \in T$ such that $c^{nk} = a^h$. Hence $W^\pi = \langle (cb,z)^{nk} \rangle^\pi = \langle ((cb)^\beta z)^{nk} \rangle = \langle (c^{nk})^{\beta\phi'(z,nk)} (b^\beta z)^{nk} \rangle \leqslant X^\pi Y^\pi$. Viceversa, suppose $W^\pi \leqslant X^\pi Y^\pi$. Then $W^\pi = \langle a^{h'} (b^\beta z)^{nk'} \rangle$ with $h' \in \mathbb{Z}$ and $k' \in \mathbb{N}$. Since T is divisible and $\phi'(z, nk') \in \mathrm{Pot}(T)$, there is $c' \in T$ such that $a^{h'} = (c^{nk'})^{\beta\phi'(z,nk')}$. It follows that $W^\pi = \langle (c^{nk'})^{\beta\phi'(z,nk')} (b^{nk'})^{\beta\phi'(z,nk')} z^{nk'} \rangle = \langle ((cb)^\beta z)^{nk'} \rangle = \langle (cb,z)^{nk'} \rangle^\pi = \langle (c,1)^{nk'} (b,z)^{nk'} \rangle^\pi \leqslant (XY)^\pi$, that is $W \leqslant XY$. Hence

$$(3) \qquad (*) \text{ is true if } X \in T(A).$$

Suppose now $|X| = |Y| = \infty$. By the structure theorem of finitely generated abelian groups there are cyclic subgroups X_1, Y_1 in XY such that $XY = X_1 Y_1$ and $|X_1|$ is finite. From (3) it follows that $X^\pi Y^\pi \leqslant X_1^\pi Y_1^\pi$. On the other hand $T(X^\pi Y^\pi)$ is contained in $T(X_1^\pi Y_1^\pi) = X_1^\pi$ which is cyclic and $(X^\pi Y^\pi)/T(X^\pi Y^\pi)$ is also cyclic since $X^\pi Y^\pi$ is an M^*-group. Hence there are cyclic subgroups X_2 and Y_2 in A such that $X_2^\pi = T(X^\pi Y^\pi)$ and

$$(4) \qquad X_2^\pi Y_2^\pi = X^\pi Y^\pi \leqslant X_1^\pi Y_1^\pi .$$

By (3), $X_2 \leqslant X_1 Y_1$ and $Y_2 \leqslant X_1 Y_1$. By (4) X^π and Y^π are contained in $X_2^\pi Y_2^\pi$ and, since X_2 is finite, by (3) we have

$$(5) \qquad X_1 Y_1 = XY \leqslant X_2 Y_2 .$$

Thus $X_1 Y_1 = XY = X_2 Y_2$ and $X_2^\pi Y_2^\pi = X^\pi Y^\pi = X_1^\pi Y_1^\pi$. It follows that $(*)$ is true for all $X, Y \in L_1(A)$. By [5, Corollary], π induces a lattice-isomorphism between $L(A)$ and $L(G)$.

Let finally Z be locally cyclic so that Z is generated by a set $\{z_i | i \in \mathbb{N}\}$ with $(z_{i+1})^{n_{i+1}} = z_i$ with $n_i \in \mathbb{N}$ for every $i \in \mathbb{N}$. Let A be the external direct product of T by Z and define $A_i := T \times \langle z_i \rangle$, as well as $G_i := T\langle z_i \rangle$ for every $i \in \mathbb{N}$. Define by induction $\beta_1 := 1$ and $\beta_i := \beta_{i-1} \phi'(z_i, n_i)^{-1}$. As above we obtain a family of bijections

$\pi_i := \pi_{\beta_i}$ from A_i to G_i such that, for every $i \in \mathbb{N}$, π_i induces a lattice-isomorphism from $L(A_i)$ to $L(G_i)$. Observe that if $(a, z_i)^n \in A_i$ with $a \in T$ and $n \in \mathbb{N}$, then there exists $b \in T$ such that $b^{n_i} = a$ and

$$((a, z_{i-1})^n)^{\pi_i} = ((a, z_i^{n_i})^n)^{\pi_i} = ((b^{n_i}, z_i^{n_i})^n)^{\pi_i} = ((b, z_i)^{n_i n})^{\pi_i} = (b^{\beta_i} z_i)^{n_i n} =$$

$$= (b^{\beta_{i-1}\phi'(z_i, n_i)^{-1}} z_i)^{n_i n} = (b^{\beta_{i-1} n_i} z_i^{n_i})^n = (a^{\beta_{i-1}} z_{i-1})^n = ((a, z_{i-1})^n)^{\pi_{i-1}}.$$

It follows that the restriction of π_i on A_{i-1} is π_{i-1}. Hence this family is inductive and its inverse limit defines a lattice-isomorphism between $L(A)$ and $L(G)$.

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