Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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Existence for implicit differential equations in Banach spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **3** (1992), n.3, p. 203–215.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1992_9_3_3_203_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1992.

Rend. Mat. Acc. Lincei
s. 9, v. 3:203-215 (1992)

Analisi matematica. — *Existence for implicit differential equations in Banach spaces.* Nota di VIOREL BARBU e ANGELO FAVINI, presentata (*) dal Socio E. Magenes.

ABSTRACT. — We prove two existence results on abstract differential equations of the type $d(Bu)/dt + A(u) = f$ and we give some applications of them to partial differential equations.

KEY WORDS: Abstract differential equations; Degenerate differential equations; Monotone operators.

RIASSUNTO. — *Esistenza per equazioni differenziali implicite in spazi di Banach.* Si dimostrano due risultati di esistenza per equazioni differenziali astratte del tipo $d(Bu)/dt + A(u) = f$ e si danno alcune applicazioni di essi ad equazioni alle derivate parziali.

INTRODUCTION

This work is concerned with differential equations of the form

$$\begin{cases} \dfrac{d}{dt}(Bu) + Au = f, & \text{in } (0, T), \\ u(0) = u_0, \end{cases} \tag{0.1}$$

in a Banach space X, where A and B are unbounded (possibly nonlinear) operators in X.

There is a large variety of results and methods involved in the study of problems of this type for which in general the standard theory of linear and nonlinear continuous semigroups of operators is not directly applicable (see *e.g.* [2]).

Here we will confine ourselves to study eq.(0.1) in the following two situations:

1) A and B are nonlinear monotone operators;

2) B is linear and A is a semilinear operator in a Hilbert space.

In the first case we use the general methods of nonlinear monotone operators theory, while in the second one we make use of the methods and regularity results connected with degenerate differential linear abstract equations in Banach spaces [3, 10, 12, 13].

1. NONLINEAR MONOTONE IMPLICIT EQUATIONS

We shall study here the nonlinear equation (0.1) essentially under the following hypotheses:

(*) Nella seduta del 14 marzo 1992.

(H$_1$) *H is a real Hilbert space and W is a reflexive Banach space dense in H and such that $W \subset H \subset W'$ algebraically and topologically (W' is the dual of W). Moreover, the injection of W into H is compact.*

We shall denote by $|\cdot|$ and $\|\cdot\|_W$ the norm of H and W, respectively and we shall use the same symbol $(\cdot,\cdot)$ to denote the scalar product of H and the pairing between W and W'.

(H$_2$) *A: $W \to W'$ is maximal monotone and everywhere defined. Moreover, A is bounded on bounded subsets of W and $A = \partial\psi$, where ψ: $W \to \overline{\boldsymbol{R}} =]-\infty, +\infty]$ is a lower semicontinuous convex function on W such that*

$$\psi(u) \geq \omega \|u\|_W^p + C, \qquad \forall u \in W, \tag{1.1}$$

for some $p \in [1, +\infty[$, $\omega > 0$ and $C \in \boldsymbol{R}$.

(H$_3$) *B: $H \to H$ is maximal monotone in $H \times H$ and $B = \partial\varphi$, where φ: $H \to \overline{R}$, is a lower semicontinuous convex function on H.*

We have denoted, as usual in the literature, by $\partial\psi$: $W \to W'$ and $\partial\varphi$: $H \to H$ the subdifferentials of ψ and φ, rispectively (see [2]).

Note that hypothesis (H$_3$) allows multivalued and unbounded (not everywhere defined) operators B from H into itself. On the other hand, hypothesis (H$_2$) implies that the operator A_H: $H \to H$ defined by $A_H u = Au \cap H$, $\forall u \in D(A_H)$, $D(A_H) = \{u \in H; Au \cap H \neq \emptyset\}$, is maximal monotone in H (see *e.g.* [2]).

If A_λ denotes the Yosida approximation of A_H, *i.e.*, $A_\lambda = \lambda^{-1}(I - (I + \lambda A_H)^{-1})$, $\lambda > 0$, it is well known [2] that $A_\lambda = \partial\psi_\lambda$, where

$$\psi_\lambda(u) = \psi((I + \lambda A_H)^{-1}u) + 2^{-1}\lambda\,|A_\lambda u|^2, \qquad \forall u \in H, \ \lambda > 0. \tag{1.2}$$

See [2, p. 57].

(H$_4$) *There is a real constant C such that*

$$(A_\lambda u, v) \geq C(|v|^2 + |(I + \lambda A_H)^{-1}u|^2 + 1) \tag{1.3}$$

for all $u \in D(B)$, $v \in Bu$ and $\lambda > 0$.

We also introduce the following hypothesis which will be used to obtain supplementary regularity results.

(H$_5$) *B^{-1} is single-valued and*

$$|B^{-1}x - B^{-1}y|^2 \leq \alpha(B^{-1}x - B^{-1}y, x - y), \qquad \forall x, y \in H, \ \alpha > 0. \tag{1.4}$$

Now we are ready to formulate the first existence result for problem (0.1).

THEOREM 1. *Under assumptions* (H$_1$)-(H$_4$), *let $f \in L^\infty(0, T; H) \cap W^{1,q}(0, T; W')$, $1/p + 1/q = 1$, and $u_0 \in W$, $\xi_0 \in Bu_0$.*

Then there exist $u \in L^\infty(0, T; W)$, $y \in C_w([0, T]; H) \cap W^{1,\infty}(0, T; W')$ *such that*

$$(1.5) \qquad \begin{cases} \dfrac{dy}{dt}(t) = y'(t) = -w(t) + f(t), & \text{a.e. } t \in (0, T), \\ y(0) = \xi_0, \end{cases}$$

$$(1.6) \qquad y(t) \in Bu(t), \quad w(t) \in Au(t), \quad \text{a.e. } t \in (0, T).$$

If in addition hypothesis (H_5) *holds, then* $u \in W^{1,2}(0, T; H)$.

Here we have denoted by $W^{1,q}([0, T]; W')$, $1 \leq q \leq \infty$, the space $\{v \in L^q(0, T; W'); dv/dt \in L^q(0, T; W')\}$ where dv/dt is taken in the sense of distributions; $C_w([0, T]; H)$ is the space of weakly continuous functions from $[0, T]$ to H and $W^{1,2}([0, T]; H) = \{u \in L^2(0, T; H);\ du/dt \in L^2(0, T; H)\}$

Theorem 1 has been established under a slight different form in [3] (see also [4]) and it is related to a well-known result of Grange and Mignot [14]. Other results of this type have been obtained by Di Benedetto and Showalter [11], Bernis [6], Colli-Visentin [9].

EXAMPLES. 1) A model example to which it applies is the boundary value problem

$$(1.7) \qquad \begin{cases} \dfrac{\partial}{\partial t}\beta(u) - \displaystyle\sum_{i=1}^{N} \dfrac{\partial}{\partial x_i}\left(a_i\left(\dfrac{\partial u}{\partial x_i}\right)\right) = f(x, t), & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_0(x), & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega \times (0, T), \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$, β is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ and a_i are continuous increasing functions such that

$$(1.8) \qquad \begin{cases} 0 \leq a_i(r) \leq C_1 |r|^{p-1} + C_2, & \forall r \in \mathbf{R}, \\ a_i(r) r \geq \omega |r|^p + C_3, & \forall r \in \mathbf{R},\ i = 1, \ldots, N,\ \omega > 0. \end{cases}$$

In this case $H = L^2(\Omega)$, $W = W_0^{1,p}(\Omega)$,

$$(Au, v) = \sum_{i=1}^{N} \int_\Omega a_i\left(\frac{\partial u}{\partial x_i}\right) \frac{\partial v}{\partial x_i}\, dx, \qquad \forall u, v \in W,$$

and $Bu = \{v \in L^2(\Omega);\ v(x) \in \beta(u(x)), \text{ a.e. } x \in \Omega\}$.

Let us remark that Theorem 1 applies also to the problem

$$\begin{cases} \dfrac{\partial}{\partial t}\beta_1(u) + \gamma(u - v) \ni f(x, t), & \text{in } \Omega \times (0, T), \\ \dfrac{\partial}{\partial t}\beta_2(v) - \gamma(u - v) - \displaystyle\sum_{i=1}^{N} \dfrac{\partial}{\partial x_i}\left(a_i\left(\dfrac{\partial v}{\partial x_i}\right)\right) \ni g(x, t), & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \\ u = 0, & \text{in } \partial\Omega \times (0, T), \end{cases}$$

where β_i, $i = 1, 2$, are maximal monotone graphs in $\boldsymbol{R} \times \boldsymbol{R}$, $\gamma: \boldsymbol{R} \to \boldsymbol{R}$ is a monotone continuous function such that $\gamma(0) = 0$ and $|\gamma(u)| \le C|u|^{Np/2(N-p)}$, $\forall u \in \boldsymbol{R}$, while a_i satisfies the above assumptions.

Such a problem has been studied recently by Showalter and Walkington [15].

2) Consider the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\varphi(x)\,\dfrac{u}{\sqrt{|u|}}\right) - \operatorname{div} F(\nabla u) = 0 & \text{in } \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ F(\nabla u)\cdot \nu = 0 & \text{in } \partial\Omega \times (0, T), \end{cases}$$

where $F(v) = K((1 + \sigma|v|)^{1/2} - 1)(\sigma|v|)^{-1}v$, $\forall v \in \boldsymbol{R}^m$.

Here φ, k, σ are $L^\infty(\Omega)$ functions such that $0 < \varphi_- \le \varphi(x) \le \varphi_+$ a.e. in Ω, $0 < K_- \le K(x) \le K_+$ a.e. in Ω, $0 < \sigma_- \le \sigma(x) \le \sigma_+$ a.e. in Ω.

This problem models the transient gas flow through a porous medium (Amirat [1]).

This equation can be written in the form (0.1) where $H = L^2(\Omega)$, $W = W^{1,3/2}(\Omega)$, $A: W \to W'$ is defined by

$$(Au, v) = \int_\Omega F(\nabla u)\cdot \nabla v\,dx, \qquad \forall v \in W$$

and B: $D(B) = L^2(\Omega) \to L^2(\Omega)$ is given by $(Bu)(x) = \varphi(x)\,u(x)/\sqrt{|u|}$, a.e. $x \in \Omega$ for $u \in D(B) = L^2(\Omega)$. It is readly seen that assumptions (H_1)-(H_4) are satisfied in the present case.

Then by Theorem 1 we may conclude that if $u_0 \in W^{1,3/2}(\Omega) \cap L^2(\Omega)$, then the above problem has a solution $u \in L^\infty(0, T; W^{1,3/2}(\Omega))$, with $\varphi u/\sqrt{|u|} \in C_w([0, T]; L^2(\Omega)) \cap W^{1,\infty}(0, T; W^{-1,3}(\Omega))$.

PROOF OF THEOREM 1. Consider the approximating equation

$$\begin{cases} \dfrac{d}{dt}(\lambda u_\lambda + Bu_\lambda) + A_\lambda u_\lambda \ni f, & \text{a.e. } t \in (0, T), \\ u_\lambda(0) = u_0, \quad \xi_0 \in Bu_0, \end{cases} \tag{1.9}$$

which clearly has a unique solution $u_\lambda \in W^{1,\infty}([0, T]; H)$.

Multiplying (scalarly in H) eq. (1.9) by u_λ and integrating on $[0, t]$, we get

$$\frac{\lambda}{2}|u_\lambda(t)|^2 + \varphi^*(Bu_\lambda(t)) + \int_0^t \psi_\lambda(u_\lambda(s))\,ds =$$

$$= \frac{\lambda}{2}|u_0|^2 + \varphi^*(Bu_0) + \int_0^t (f, u_\lambda)\,ds + \psi_\lambda(0),$$

where $\varphi^*(p) = \sup\{(p, v) - \varphi(v), v \in H\}$ is the conjugate of φ.

Taking into account (1.2) and hypothesis (H_2) we deduce the estimate

$$\int_0^T \|(I+\lambda A_H)^{-1}u_\lambda\|^p dt + \lambda \int_0^T |A_\lambda u_\lambda|^2 dt \le C\left(1+\int_0^T |f|^2 dt\right). \tag{1.10}$$

Next we multiply eq. (1.9) by du_λ/dt and integrate on $(0, t)$. Noting that, by the monotonicity of B and the relation between A_λ and ψ_λ, $(dBu_\lambda/dt, du_\lambda/dt) \ge 0$; $(du_\lambda/dt, A_\lambda u_\lambda) = d\psi_\lambda(u_\lambda)/dt$, a.e. $t \in (0, T)$, we infer

$$\lambda \int_0^T \left|\frac{du_\lambda}{ds}\right|^2 ds + \psi_\lambda(u_\lambda(t)) \le \int_0^T \left(f, \frac{du_\lambda}{ds}\right) ds. \tag{1.11}$$

Now the estimate (1.10) ensures that there is $\{f_\lambda\} \subset W^{1,2}([0,T]; H)$ such that for $\lambda \to 0_+$,

$$\begin{cases} \lambda \displaystyle\int_0^T \left(\frac{df_\lambda}{dt}, A_\lambda u_\lambda\right) dt \to 0; \qquad \{f_\lambda\} \text{ bounded in } L^\infty(0, T; H); \\ f_\lambda \to f \text{ strongly in } L^2(0, T; H) \cap W^{1,q}([0,T]; W'). \end{cases} \tag{1.12}$$

Then substituting in (1.9) and (1.11) f by f_λ, we get

$$\lambda \int_0^t \left|\frac{du_\lambda}{ds}\right|^2 ds + \psi_\lambda((1+\lambda A_H)^{-1}u_\lambda(t)) + \lambda\,|A_\lambda u_\lambda(t)|^2 \le$$

$$\le (f_\lambda(t), u_\lambda(t)) - (f_\lambda(0), u_0) - \int_0^t \left(\frac{df_\lambda}{ds}, u_\lambda\right) ds.$$

Using estimates (1.10), hypothesis (H_2) and relations (1.12), we obtain after some calculation

$$\lambda \int_0^t \left|\frac{du_\lambda}{ds}\right|^2 ds + \|(I+\lambda A_H)^{-1}u_\lambda(t)\|_W + |u_\lambda(t)| +$$

$$+\lambda^{-1}|(I+\lambda A_H)^{-1}u_\lambda(t) - u_\lambda(t)| \le C, \qquad \forall \lambda > 0. \tag{1.13}$$

Denote by $\theta_\lambda(t)$ the function $\int_0^t A_\lambda u_\lambda(s)\,ds$, so that eq. (1.9) is written as

$$\lambda u_\lambda(t) + Bu_\lambda(t) + \theta_\lambda(t) = \lambda u_0 + \xi_0 + \int_0^t f_\lambda(s)\,ds, \qquad t \in (0, T). \tag{1.14}$$

Multiplying it (scalarly in H) by $A_\lambda u_\lambda$, using (H.4) and integrating by parts, we get

$$|\theta_\lambda(t)| \le C, \qquad \forall \lambda > 0. \tag{1.15}$$

In virtue of the estimate (1.11), $\{(I+\lambda A_H)^{-1}u_\lambda\}$ is bounded in $L^\infty(0, T; W)$ and thus, by hypothesis (H_1), $\{A_\lambda u_\lambda\}$ is bounded in $L^\infty(0, T; W')$.

Since, by (1.13), $\{\theta_\lambda\}$ is bounded in $L^\infty(0, T; H)$ and the injection of H into W' is compact, we conclude that $\{\theta_\lambda\}$ is compact in $C([0, T]; W')$.

Hence, extracting a subsequence if necessary, we may deduce that there are functions u, y, w such that $u \in L^\infty(0, T; W)$, $y \in W^{1,\infty}([0, T]; W') \cap L^\infty(0, T; H)$, $w \in L^\infty(0, T; W)$ and for $\lambda \to 0$

$$(1.16) \qquad \begin{cases} (I + \lambda A_H)^{-1} u_\lambda \to u \text{ weak-star in } L^\infty(0, T; W), \\ u_\lambda \to u \text{ weak-star in } L^\infty(0, T; H), \\ A_\lambda u_\lambda \to w \text{ weak-star in } L^\infty(0, T; W'), \\ \theta_\lambda \to \theta \text{ strongly in } C([0, T]; W') \text{ and weak-star in } L^\infty(0, T; H), \\ y_\lambda \in Bu_\lambda \to y \text{ weak-star in } L^\infty(0, T; H), \text{ and strongly in } C([0, T; W'), \\ \lambda A_\lambda u_\lambda \to 0 \text{ strongly in } L^\infty(0, T; H), \\ \lambda u_\lambda \to 0 \quad \text{strongly in } \mathrm{L}^\infty(0, \mathrm{T}; \mathrm{H}). \end{cases}$$

Letting λ tend to zero in (1.14), we see that

$$y(t) + \int_0^t w(s)\, ds = \xi_0 + \int_0^t f(s)\, ds; \quad \theta(t) = \int_0^t w(s)\, ds \qquad \forall t \in (0, T).$$

Hence $y \in W^{1,\infty}([0, T]; W') \cap L^\infty(0, T; H)$ and $dy/dt + w = f$ a.e. $t \in (0, T)$. In particular, we deduce that y is weakly continuous from $[0, T]$ into H.

It remains to prove that

$$(1.17) \qquad y(t) \in Bu(t), \qquad \text{a.e. } t \in (0, T),$$

$$(1.18) \qquad w(t) \in Au(t), \qquad \text{a.e. } t \in (0, T).$$

Let us denote by $\mathcal{B}: L^2(0, T; H) \to L^2(0, T; H)$ the realization of the operator B in $\mathcal{H} = L^2(0, T; H)$, *i.e.*, $\mathcal{B} = \{[v, z] \in \mathcal{H} \times \mathcal{H},\ z(t) \in Bv(t), \text{ a.e. } t \in [0, T]\}$. To prove (1.17), it suffices to show that $[u, y] \in \mathcal{B}$.

Let $[v, z]$ be arbitrary but fixed in $\mathcal{B}$. By the monotonicity of $\mathcal{B}$ we have

$$\int_0^T (y_\lambda(t) - z(t), u_\lambda(t) - v(t))\, dt \geq 0, \qquad \forall \lambda > 0.$$

Equivalently,

$$\int_0^T (y_\lambda(t) - z(t), (I + \lambda A_H)^{-1} u_\lambda(t) - v(t))\, dt + \lambda \int_0^T (y_\lambda(t) - z(t),\ A_\lambda u_\lambda(t))\, dt \geq 0.$$

Letting λ tend to zero, we obtain by (1.15) and (1.16) that

$$\int_0^T (y(t) - z(t), u(t) - v(t))\, dt \geq 0 \qquad \forall [v, z] \in \mathcal{B}.$$

Since $\mathcal{B}$ is maximal monotone, (1.17) follows.

To prove (1.18), we observe first that

$$\limsup_{\lambda\to 0} \int_0^T (A_\lambda u_\lambda(t), u_\lambda(t))\,dt = \limsup_{\lambda\to 0} \int_0^T \left(f_\lambda(t) - \frac{d}{dt}(\lambda u_\lambda(t) + Bu_\lambda(t), u_\lambda(t))\,dt \right) =$$

$$= \limsup_{\lambda\to 0} \left(\varphi^*(\xi_0) - \varphi^*(y_\lambda(T)) + \int_0^T (f_\lambda(t), u_\lambda(t))\,dt \right).$$

In view of (1.16), $y_\lambda(t) \to y(t)$ weakly in H, and φ^* is weakly lower semicontinuous; hence, we infer that

$$\limsup_{\lambda\to 0} \int_0^T (A_\lambda u_\lambda(t), u_\lambda(t))\,dt \le \varphi^*(\xi_0) - \varphi^*(y(T)) +$$

$$+ \int_0^T (f(t), u(t))\,dt = \int_0^T \left(f(t) - \frac{d}{dt} Bu(t), u(t) \right) dt = \int_0^T (w(t), u(t))\,dt.$$

This implies as above that $w(t) \in Au(t)$, thereby completing the proof. If in addition Hypothesis (H_5) holds, multiplying eq. (1.9) by du_λ/dt and noting that then $(d\,Bu_\lambda/dt, du_\lambda/dt) \ge \alpha^{-1} |du_\lambda/dt|^2$, a.e. $t \in (0, T)$, we deduce that $\{du_\lambda/dt\}$ is bounded in $L^2(0, T; H)$ and thus $du/dt \in L^2(0, T; H)$ as claimed. This finishes the proof of Theorem 1.

REMARK 1. The existence result of Grange and Mignot [14] mentioned above assumes instead of Hypothesis (H_4) that B: $W_0 \to W_0'$ is maximal monotone, subpotential and bounded on bounded subsets, W_0 being a reflexive Banach space such $W \subset \subset W_0$ with compact embedding.

If one assumes in addition that the restriction of B to H is maximal monotone and $f \in L^\infty(0, T; H) \cap W^{1,q}([0, T]; W')$, we may use the approximating process (1.9) to obtain existence in problem (0.1) arguing as in the proof of Theorem 1.

The next Theorem is a variant of Theorem 1 under weaker assumptions on f and u_0

THEOREM 2. *Under assumptions* (H_1)-(H_4), *let us assume further that* $0 \in \operatorname{int} D(B)$ *and*

$$\sup\{\|w\|_{W'};\ w \in Au\} \le C_1 \|u\|_W^{p-1} + C_2, \qquad \forall u \in W. \tag{1.19}$$

Then for $f \in L^q(0, T; W')$, $u_0 \in D(B)$ *and* $\xi_0 \in Bu_0$, *there are* $u \in L^p(0, T; W)$ *and* $y \in L^\infty(0, T; H) \cap W^{1,q}([0, T]; W')$ *such that* ($y'(t)$ *denoting the derivative of* $y(t)$ *with respect to* t),

$$\begin{cases} y'(t) + w(t) = f(t), & \text{a.e. } t \in (0, T), \\ y(0) = \xi_0, \\ y(t) \in Bu(t), \quad w(t) \in Au(t), & \text{a.e. } t \in (0, T). \end{cases} \tag{1.20}$$

PROOF. Let $f_n \in L^\infty(0, T; H) \cap W^{1,q}([0, T]; W')$ and $u_0^n \in W$, $\xi_0^n \in H$ be such that $f_n \to f$ strongly in $L^q([0, T]; W')$, $u_0^n \to u_0$ strongly in H, $\xi_0^n \in Bu_0^n \to \xi_0$ strongly in H.

We may choose for instance $u_0^n = (I + n^{-1}B)^{-1} u_0$.

Now let (y_n, u_n) be the corresponding solution to (1.5), (1.6), *i.e.*,

$$\text{(1.21)}\qquad \begin{cases} y_n'(t) + w_n(t) = f_n(t), & \text{a.e. in } (0, T), \\ y_n(0) = \xi_0^n,\ y_n(t) \in Bu_n(t),\ w_n(t) \in Au_n(t), & \text{a.e. } t \in (0, T). \end{cases}$$

Multiplying the latter by u_n and integrating on $(0, t)$, we get

$$\text{(1.22)}\qquad \varphi^*(y_n(t)) + \omega \int_0^t \|u_n(s)\|_W^p\, ds \le \varphi^*(\xi_0^n) + \int_0^t (f_n(s), u_n(s))\, ds + C_0 \quad \forall t \in (0, T].$$

On the other hand, since $0 \in \operatorname{int} D(B) = \operatorname{int} D(\varphi)$, we have $\varphi^*(y_n(t)) \ge \rho |y_n(t)| + C$, $\forall t \in [0, T]$.

This yields

$$|y_n(t)| + \int_0^t \|u_n(s)\|_W^p\, ds \le C, \qquad \forall t \in [0, T].$$

(We have denoted by C several positive constants independent of n).

Hence on a subsequence we have, in virtue of assumption (1.19), $y_n \to y$ weakly star in $L^\infty(0, T; H)$ and strongly in $C([0, T]; W')$, $u_n \to u$ weakly in $L^p(0, T; W)$, $w_n \to w$ weakly in $L^q(0, T; W')$.

Then arguing as in the previous proof we infer that $w(t) \in Au(t)$, $y(t) \in Bu(t)$, a.e. $t \in (0, T)$, as claimed. (We note that $\varphi^*(\xi_0^n) \to \varphi^*(\xi_0)$).

Let us return back to problem (1.7), *i.e.*, $W = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$, $Bu = \{w \in H;\ w(x) \in \beta(u(x)) \text{ a.e. } x \in \Omega\}$, $A\colon W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ defined by

$$(Au, v) = \sum_{i=1}^N \int_\Omega a_i\left(\frac{\partial u}{\partial x_i}\right) \frac{\partial v}{\partial x_i}\, dx$$

where β is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ and a_i satisfy (1.8).

We shall further assume that $D(\beta) = R$. Though Theorem 2 is not directly applicable (because int $D(B)$ could be empty), its conclusions remain true in this case.

Namely one has (related results have been recently obtained in [7])

PROPOSITION 1. *Let the above assumptions hold where $p > N$. Then for $f \in L^q(0, T; W^{-1,q}(\Omega))$ and $u_0 \in L^2(\Omega)$ such that $\exists\, \xi_0 \in L^2(\Omega)$, $\xi_0(x) \in \beta(u_0(x))$ a.e. $x \in \Omega$, there are $\mu \in W^{1,q}([0, T]; W^{-1,q}(\Omega)) \cap L^\infty(0, T; L^1(\Omega))$ and $u \in L^p(0, T; W_0^{1,p}(\Omega))$ satisfying*

$$\text{(1.23)}\qquad \begin{cases} \dfrac{\partial}{\partial t}\mu(x, t) - \displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i\left(\frac{\partial}{\partial x_i} u(x, t)\right) = f(x, t), & \text{in } \Omega \times (0, T), \\ \mu(x, 0) = \xi_0(x), & x \in \Omega, \\ \mu(x, t) \in \beta(u(x, t)), & \text{a.e. } (x, t) \in \Omega \times (0, T). \end{cases}$$

PROOF. We will use the notations and the scheme of proof of Theorem 2. Note that in this case assumptions (H_1)-(H_4) and (1.19) hold and

$$\varphi^*(y) = \int_\Omega j^*(y(x))\,dx, \qquad \forall y \in L^2(\Omega),$$

where $\partial j = \beta$ and $j^* : \boldsymbol{R} \to \overline{\boldsymbol{R}}$ is the conjugate of $j: \boldsymbol{R} \to \overline{\boldsymbol{R}}$. Since $D(\beta) = R$ we have (see *e.g.* [2, p. 56])

$$\lim_{|r| \to \infty} j^*(r)/|r| = +\infty. \tag{1.24}$$

Then by (1.22) it follows in particular that $\|y_n(t)\|_{L^1(\Omega)} \le C$, $\forall t \in (0, T)$.

Now if $p > N$, then $L^1(\Omega) \subset W' = W^{-1,q}(\Omega)$ and so by the Ascoli-Arzelà Theorem, $y_n \to y$ strongly in $C([0, T]; W')$, because as seen in the proof of Theorem 2 $\{y_n\}$ is bounded in $W^{1,q}(0, T; W')$.

Since $u_n \to u$ weakly in $L^p(0, T; W)$ we infer as above that $y(t) \in \tilde{B}u(t)$, a.e. $t \in (0, T)$, where $\tilde{B} \subset W \times W'$ is an extension of B given by $\tilde{B} = \partial\tilde{\varphi}$ where $\tilde{\varphi}$: $W \to \boldsymbol{R}$ is defined by $\tilde{\varphi}(u) = \varphi(u)$ $\forall u \in W$. We know that (see *e.g.* [8])

$$\tilde{B}u = \{\mu \in W' \cap L^1(\Omega);\ \mu(x) \in \beta(u(x)) \quad \text{a.e. } x \in \Omega\}. \tag{1.25}$$

Moreover,we have by eq. (1.21)

$$\int_0^T (w_n(t), u_n(t))\,dt + \varphi^*(y_n(T)) - \varphi^*(y_n(0)) = 0.$$

Note also that by (1.22) and (1.24) it follows via Dunford-Pettis theorem that $\{y_n(T)\}$ is weakly compact in $L^1(\Omega)$.Since

$$y \to \int_\Omega j^*(y)\,dx$$

is weakly lower semicontinuous in $L^1(\Omega)$, we infer therefore that

$$\liminf_{n \to \infty} \varphi^*(y_n(T)) \ge \varphi^*(y(T)).$$

This implies as in the proof of Theorem 1 that $w(t) \in Ay(t)$ a.e. $t \in (0, T)$, as claimed.

2. Strongly degenerate implicit equations

In this Section we confine ourselves to a linear operator B in eq. (0.1), but we permit to B to be strongly degenerate and we avoid the angle condition (1.3).

Let V, H be two complex Hilbert spaces such that $V \subset H \subset V'$ algebraically and topologically. Denote by $\|\cdot\|$ and $\|\cdot\|_*$ the norms in V and V', respectively $(\cdot,\cdot)$ denotes the inner product in H and $\langle\cdot,\cdot\rangle$ is the pairing between V and V', so that $\langle u, v\rangle = (u, v)$ for all $u, v \in V$. We assume

[i] *A is a bounded linear operator from V to V' such that* $\operatorname{Re}\langle Au, u\rangle \ge a_0\|u\|^2$, $\forall u \in V$, *$a_0 > 0$, and B is a self-adjoint non negative bounded operator from H into H.*

Let us remark that the adjoint operator A^* has analogous properties.

If $(\lambda B + A)u = f$, where λ is a complex number with real part $\operatorname{Re}\lambda \geq 0$, $u \in V$, we deduce that $\operatorname{Re}\lambda(Bu, u) + \operatorname{Re}\langle Au, u\rangle = \operatorname{Re}\langle f, u\rangle$ and also $a_0\|u\| \leq \|f\|_*$.

On the other hand, if $y \in V$ satisfies $0 = \langle(\lambda B + A)u, y\rangle$, for any $u \in V$, then $0 = \langle u, (\bar{\lambda}B + A^*)y\rangle = (u, \bar{\lambda}By) + \langle Au, y\rangle$ and hence $(\bar{\lambda}B + A^*)y = 0$.

Arguing as before, $a_0\|y\| \leq \|(\bar{\lambda}B + A^*)y\|_*$, and thus $y = 0$.

We conclude that $\lambda B + A$ has a bounded inverse from V' into V and, since A is bounded, $\|A(\lambda B + A)^{-1}y\|_* \leq C\|y\|_*$, $\operatorname{Re}\lambda \geq 0$, $y \in V^*$.

But one easily recognizes that such an estimate may be extended to all λ's in a sector containing $\operatorname{Re}\lambda \geq 0$.

We are then in a position to apply the method described in [12], arguing that if $S = BA^{-1}$, then the representation

$$V' = N(S) \oplus \overline{R(S)} \tag{2.1}$$

holds, where $N(S)$ denotes the null-space of S and $\overline{R(S)}$ is the closure in V' of the range $R(S)$ of S.

Furthermore, if $\tilde{S}$ is the restriction of S to $\overline{R(S)} = Z$, $-\tilde{S}^{-1}$ generates a bounded analytic semigroup in Z.

Denote by P the projection operator onto $N(S)$ associated to (2.1), and consider the linear problem

$$\begin{cases} \dfrac{d}{dt}(Bu) + Au = f, & 0 < t < T, \\ (Bu)(0) = \xi = Bu_0, & u_0 \in V; \end{cases} \tag{2.2}$$

where $f \in L^2(0, T; V')$. We have

LEMMA 1. *Assume Hypothesis* [i]. *Then for any $u_0 \in V$ and $f \in L^2(0, T; V')$, problem (2. 2) has a unique solution u such that $Au(\cdot)$, $d(Bu(\cdot))/dt \in L^2(0, T; V')$.*

PROOF. Put $Lu = v$. Then the representation (2.1) permits to decouple (2.2) into $Pv(t) = Pf(t)$ and

$$\begin{cases} \dfrac{d}{dt}(\tilde{S}(I - P)v) + (I - P)v = (I - P)f, & 0 < t < T, \\ (\tilde{S}(I - P)v)(0) = \tilde{S}(I - P)v_0, \end{cases} \tag{2.3}$$

where $v_0 = Au_0$.

We are then reduced to find a solution, in a sense which shall be precised at once, to

$$\begin{cases} z' + \tilde{S}^{-1}z = (I - P)f, & 0 < t < T, \\ z(0) = \tilde{S}^{-1}(I - P)v_0. \end{cases} \tag{2.4}$$

In view of [9, Theorem 4.19, p. 338], since $\tilde{S}^{-1}(I - P)v_0 \in D(\tilde{S})$, if $f \in L^2(0, T; V')$ (so that $(I - P)f \in L^2(0, T; Z)$), problem (2.4) has a unique strict solution z such that z' and $\tilde{S}^{-1}z \in L^2(0, T; Z)$.

This is exactly the sense to give to the solution of (2.4).If $V(t)$, $t \geq 0$, denotes the semigroup in Z generated by $-\tilde{S}^{-1}$, then

$$v(t) = Pf(t) + V(t)(I-P)v_0 + \tilde{S}^{-1}\int_0^t V(t-s)(I-P)f(s)\,ds,$$

and $u(t) = A^{-1}v(t)$ solves problem (2.2).

Let $u_0 \in V$ be fixed and consider the map K: $L^2(0, T; V') \to L^2(0, T; V)$, $Kf = u$, u the solution to problem (2.2); such a map has the Lipschitz property, since, if $u_i = Kf_i$, $f_i \in L^2(0, T; V')$, $i = 1, 2$,

$$\left(\int_0^T \|Lu_1(t) - Lu_2(t)\|_*^2\,dt\right)^{1/2} \leq \left(\int_0^T \|P(f_1(t) - f_2(t))\|_*^2\,dt\right)^{1/2} +$$

$$+C\left(\int_0^T \|(I-P)(f_1(t) - f_2(t))\|_*^2\,dt\right)^{1/2} \leq C_1\left(\int_0^T \|f_1(t) - f_2(t)\|_*^2\,dt\right)^{1/2}.$$

We introduce a, possibly nonlinear, operator F from V to V', such that

[ii] *There exists* $\omega > 0$ *for which* $\|F(u_1) - F(u_2)\|_* \leq \omega\|u_1 - u_2\|, u_1, u_2 \in V.$

We then prove

THEOREM 3. *Assume Hypothesis* [i-ii]. *If the constant* ω *in* [ii] *is sufficiently small, then for any* $u_0 \in V$ *and* $f \in L^2(0, T; V')$ *the problem*

$$\tag{2.5} \begin{cases} \dfrac{d}{dt}(Bu) + Au + F(u) = f, & 0 < t < T, \\ (Bu)(0) = Bu_0, \end{cases}$$

has a strict solution u *such that* $Au(\cdot)$, $d(Bu(\cdot))/dt \in L^2(0, T; V')$.

PROOF. We seek a solution to problem (2.5) under the form $u = K(h)$, $h \in L^2(0, T; V')$.

This u satisfies (2.5) iff h is a fixed point for $-F \circ K + f$.

Now, $F \circ K$ satisfies a Lipschitz condition as an operator from $L^2(0, T; V')$ into itself with a Lipschitz constant $\leq C_1\omega$ and hence the result follows immediately.

REMARK 2. Obviously, one needs no assumption on the smallness of ω in [ii] if F replaced by εF, $\varepsilon \in \boldsymbol{R}$, provided that $|\varepsilon|$ is in its turn suitably small.

REMARK 3. In a case strongly degenerate as the one under consideration, the smallness of ω is essential.

For example, if $V = H = V' = \boldsymbol{R}^2$, $B(u, v) = (0, v)$, $A(u, v) = (u, v)$, $F(u, v) = (-u - v, 1)$, $f(t) = (0, 0)$, $0 \leq t \leq T$, then problem (2.5) has no solution for any initial condition even if in this case the angle condition $\langle Au, Bu\rangle \geq 0$ for all $u \in V$ holds, (see Assumption (1.3)).

EXAMPLE. Let $m(\cdot)$ be a continuous non negative function on [0, 1]. Define the operators A, B by means of $Au = u''$, (weak derivative), $u \in H_0^1(0,1)$, $Bu = mu$, $u \in L^2(0,1)$.

It has been proved in [12] that $(\lambda B + A)^{-1}$ exists for all λ, $\mathrm{Re}\,\lambda \geq 0$, and $\|A(\lambda B + A)^{-1}u\|_{H^{-1}(0,1)} \leq C\|u\|_{H^{-1}(0,1)}$: in fact, Hypothesis [i] is verified.

This result can be extended to more general operators A defined in variational way by the sesquilinear form

$$\sum_{|p|,\,|q|=m} \int_\Omega a_{p,q}(x)\left(\frac{\partial}{\partial x}\right)^q u \left(\frac{\partial}{\partial x}\right)^p v\,dx, \qquad u, v \in H_0^m(\Omega), \quad m \geq 1,$$

Ω a bounded domain in $\boldsymbol{R}^n$ with a smooth boundary, under the usual coerciveness assumptions on the coefficients $a_{p,q}(x)$. Let us introduce

$$F(u, \phi) = \sum_{|\alpha| \leq r} (-1)^{|\alpha|} \int_\Omega A_\alpha(x, u(x), \ldots, D^r u(x))\, D^\alpha \phi(x),$$

$u, \phi \in C_0^\infty(\Omega)$, where the real-valued functions $A_\alpha(x, u, u_1, \ldots, u_r)$ are continuous from $\bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^n \times \ldots \times \boldsymbol{R}^{n^r}$ into $\boldsymbol{R}$ and $|A_\alpha(x, u, u_1, \ldots, u_r) - A_\alpha(x, v, v_1, \ldots, v_r)| \leq \leq \eta(|u - v| + \|u_1 - v_1\| + \ldots + \|u_r - v_r\|)$, $|\alpha| \leq r$.

Here $\|\cdot\|$ denotes the usual norm in $\boldsymbol{R}^N$ for suitable N and $\eta > 0$.

Suppose $0 \leq r < m - n/2$. Then in view of Sobolev embedding Theorem we deduce that there are two positive constants C_i, $i = 1, 2$, such that for all $\phi, u, v \in \in C_0^\infty(\Omega)$,

$$|F(u, \phi) - F(v, \phi)| \leq C_1 \eta \sum_{j=0}^{r} \|D^j u - D^j v\|_{C(\bar{\Omega})} \|\phi\|_{H_0^m(\Omega)} \leq C_2 \eta \|u - v\|_{H^m(\Omega)} \|\phi\|_{H_0^m(\Omega)}.$$

This implies that the operator F defined by duality [5, p. 83] by means of $F(u, \phi)$ satisfies [ii] and we may apply either Theorem 3 or Remark 2.

Partially supported by the G.N.A.F.A. of C.N.R. The first author was visiting professor at the University of Bologna in preparing this work.

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