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Some properties of collision and non-collision orbits for a class of singular dynamical systems

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Equazioni differenziali ordinarie. — *Some properties of collision and non-collision orbits for a class of singular dynamical systems.* Nota di VITTORIO COTI ZELATI e ENRICO SERRA, presentata (*) dal Corrisp. A. Ambrosetti.

ABSTRACT. — We present some regularity properties of periodic solutions to a class of singular potential problems and we discuss the existence of a regular solution.

KEY WORDS: Periodic solutions; Kepler problem; Variational methods.

RIASSUNTO. — *Alcune proprietà delle soluzioni di collisione e di non collisione per una classe di sistemi dinamici singolari*. Si presentano alcune proprietà di regolarità delle soluzioni periodiche di una classe di sistemi dinamici con potenziale singolare e si prova l'esistenza di una soluzione regolare.

1. INTRODUCTION

In this *Note* we are concerned with the existence of periodic solutions for second order Hamiltonian systems of the form

$$-q''(t) = \nabla V(t, q(t)) \tag{1}$$

where $q(t) \in \boldsymbol{R}^N$ and $V \in C^1(\boldsymbol{R} \times \boldsymbol{R}^n \setminus \{0\}; \boldsymbol{R})$ is T-periodic in t and $V(t,x) \to -\infty$ as $|x| \to 0$.

In order to find classical solutions of (1) (*i.e.* solutions $q(t) \in C^2([0,T]; \boldsymbol{R}^N \setminus \{0\})$), an important role is played by the behavior of $V(t,x)$ near the singularity $x = 0$.

More precisely, defining as usual the functional $f: \Lambda \to \boldsymbol{R}$, where $\Lambda = \{u \in H^1(S^1; \boldsymbol{R}^N) \mid u(t) \neq 0 \ \forall t\}$ by

$$f(u) = \frac{1}{2} \int_0^T |u'(t)|^2 dt - \int_0^T V(t, u(t))\, dt \tag{2}$$

we recall that one has $f(u) \to +\infty$ as $u \to \partial\Lambda$ weakly in H^1 if V is a «Strong Force», *i.e.* if

$$-V(t,x) \geq c/|x|^2 \tag{3}$$

for some $c > 0$ when $|x|$ is close to 0, while, in case (3) is violated, one can have situations in which $f'(u_n) \to 0$, $f(u_n) \to c < +\infty$, $u_n \to \partial\Lambda$. As a consequence, one can use standard variational arguments to study (1) if (3) holds, while they generally fail if (3) is not satisfied.

To prove existence for (1) when (3) does not hold one possible approach, used for the first time by A. Bahri and P. H. Rabinowitz in [3], is the following:

a) Perturb V by a strong force, for example setting $V_\varepsilon(t,x) = V(t,x) - \varepsilon/|x|^2$;

(*) Nella seduta del 14 marzo 1992.

b) Prove existence of a solution q_ε for

$$-q''(t) = \nabla V_\varepsilon(t, q(t)). \tag{1$_\varepsilon$}$$

This is possible using variational techniques since V_ε now satisfies the strong force condition (3).

c) Try to pass to the limit as $\varepsilon \to 0$ to find a solution $\bar{q}$ of (1).

This approach indeed works, but one cannot prove in general that such a solution $\bar{q}$ is a classical solution of (1). This is the reason which led Bahri and Rabinowitz to the introduction in [3] (Definition 3.1) of the concept of *generalized solution* q of (1).

In the paper [3] existence of at least one generalized solution is proved. In particular, in such a paper it is asked if such a generalized solution has additional regularity.

The main object of this *Note* (we refer to [5] for a more detailed discussion and for the complete proofs) is to show that generalized solutions indeed have additional regularity under mild assumptions on the behaviour of V near the singularity. In particular we show that every generalized solution has only finitely many collisions; and we also show that the number of collisions can be bounded in terms of the value of the action functional or the Morse index of a sequence of approximated solutions.

Lastly we prove existence of a noncollision (*i.e.*, of a classical) solution for a class of singular systems when V behaves near the singularity as $-|x|^{-\alpha}$, $1 < \alpha < 2$. Such a problem has been studied by many authors; we recall here [1, 6, 11]. In all these papers global assumptions are made on the potential V which imply that it is not too far from a radial one; here we only make assumptions on the behaviour of V for x close to 0. Such a result is similar to that of [4], valid only for planar systems, and to that of [7], valid only for even potentials.

As far as the case $\alpha = 1$ is concerned, we cannot prove the existence of a non-collision solution, but we prove that the solution found in [3] if it does collide, it is reflected back by the singularity.

After completing the paper, we received the paper [10], where some results related to ours are obtained.

2. Regularity properties of generalized solutions

In this Section we establish some properties of generalized solutions by making some assumptions on the potential V near its singularity and by considering solutions obtained as limit of classical solutions of perturbed problems. When q is a generalized solution of (1) we denote by $\mathcal{C}(q)$ its *collision set*, namely the set $\{t \in [0, T]/q(t) = 0\}$.

Let $\mathcal{G} = \{G \in \mathcal{C}^\infty(\mathbf{R}^N \setminus \{0\}; \mathbf{R})/G(x) \le -a/|x|^2 + b \text{ for some } a, b > 0\}$; then we give the following definition

DEFINTION 2.1. We say that a generalized solution q of (1) is a variational solution of (1) if $\exists G \in \mathcal{G}$ such that $\forall \varepsilon > 0$ $\exists q_\varepsilon \in \Lambda$ satisfying

(i) q_ε is a classical solution of

$$(4)_\varepsilon \qquad -\ddot{q}_\varepsilon = \nabla V(t, q_\varepsilon) + \varepsilon \nabla G(q_\varepsilon);$$

(ii) $f(q_\varepsilon) = \dfrac{1}{2}\displaystyle\int_0^T |\dot{q}_\varepsilon|^2 - \int_0^T V(t, q_\varepsilon) - \varepsilon \int_0^T G(q_\varepsilon) \leq C,$

where C does not depend on ε;

(iii) $q_\varepsilon \to q$ in $H^1(S^1; \mathbf{R}^N)$.

REMARK 2.2. (i) The generalized solution whose existence is proved in [3] is actually variational.

(ii) Since $q_\varepsilon \to q$ in H^1, q_ε is a classical solution in $[0, T]$ and q is a classical solution in $[0, T] \setminus \mathcal{C}(q)$, we easily deduce that $q_\varepsilon \to q$ in $C^2(B)$ $\forall B$ compact subset of $[0, T] \setminus \mathcal{C}(q)$.

We now make some assumptions on V near the singularity and show that these imply additional regularity for a variational solution.

Consider $V \in C^1(\mathbf{R} \times \mathbf{R}^N \setminus \{0\}; \mathbf{R})$ and assume that $\exists 0 < \alpha < 2$ such that

(V1) V is T-periodic in t;

(V2) $V(t, x) = -|x|^{-\alpha} + U(t, x)$;

(V3) $|\nabla U(t, x)|\, |x|^{\alpha+1} \to 0$ as $|x| \to 0$, uniformly in t;

(V4) $\exists \alpha' < \alpha$ such that $|\partial U(t, x)/\partial t|\, |x|^{\alpha'} \to 0$ as $|x| \to 0$, uniformly in t.

We start by

LEMMA 2.3. Suppose $V \in C^1(\mathbf{R} \times \mathbf{R}^N \setminus \{0\}; \mathbf{R})$ satisfies (V1-V4). Let q be a variational solution of (1) with collision set $\mathcal{C}(q)$. Then $\exists E \in \mathbf{R}$ such that

$$(5) \qquad \frac{1}{2} |\dot{q}(t)|^2 + V(t, q(t)) = E + \int_0^t \frac{\partial U}{\partial t}(s, q(s))\, ds \qquad \forall t \in [0, T] \setminus \mathcal{C}(q).$$

We can now state a first regularity result.

THEOREM 2.4. Suppose V satisfies (V1-V4). Let q be a variational solution of (1). Then q has only finitely many collisions. Moreover $\exists \delta > 0$ such that $d^2 |q(t)|^2/dt^2 > 0$ $\forall t$ such that $|q(t)| \leq \delta$.

REMARK 2.5. If the potential V does not depend explicitly on time the previous results hold true under less restrictive assumptions, and their proofs are very simple. Moreover, if $V(x) + (\nabla V(x), x)/2 \leq -a/|x|^\alpha$ $\forall x \in \mathbf{R}^N \setminus \{0\}$ for some $a > 0$ and $\alpha \in]0, 2[$, then the number of collisions $\mathcal{N}(q)$ of a generalized solution q can be estimated in terms of the value of the action functional as $\mathcal{N}(q) \leq \alpha^{-2/\alpha} \alpha (2+\alpha)^{-(2+\alpha)/2} f(q)^{2+\alpha} T^{(\alpha-2)/\alpha}$.

We now turn to the relationship between the Morse index and the number of collisions of a generalized solution.

Actually, one cannot give a meaning to the Morse index $\underline{m}$ of a generalized solution (the functional is not C^2 in such a point), so we will take q to be a variational solution and bound the number of its collisions by the Morse index of the sequence of classical solutions of $(1)_\varepsilon$ which converge to q. We recall that the Morse index $\underline{m}(x)$ of a critical point x of a functional $J \in \mathcal{C}^2(H; \boldsymbol{R})$ is the dimension of the maximum subspace of H where $d^2J(x)$ is negative definite.

THEOREM 2.6. Suppose $V \in C^2(\boldsymbol{R} \times \boldsymbol{R}^N \setminus \{0\}; \boldsymbol{R})$, $N \geq 3$, satisfies (V1-V4). Assume there exist $\sigma > 0$ and $C > 0$ such that

(V5) $|\nabla^2 U(t, y)|\, |y|^{\alpha + 2 - \sigma} \leq C$ as $|y| \to 0$ uniformly in t,

(V6) $|\nabla U(t, y)|\, |y|^{-1} \to 0$ as $|y| \to \infty$ uniformly in t.

Let q be a variational solution of (1). Then

$$\mathcal{N}(q)(N-2) \leq \liminf_{\varepsilon \to 0} \underline{m}(q_\varepsilon)$$

where q_ε are classical solutions of $(1)_\varepsilon$ such that $q_\varepsilon \to q$.

The proof relies on the results obtained by E. Serra and S. Terracini in [7] to which we refer for the details.

REMARK 2.6. It is not difficult, using results contained in [2, 8, 9] and [12] to show that, in the setting of [3] (and also in the setting of Sect. 3), $\underline{m}(q_\varepsilon) \leq N - 2$. This implies that the generalized solution found in [3] has at most one collision.

3. EXISTENCE RESULTS

In this Section we will sketch how to prove existence of a noncollision solution in the case V has the form $V(x) = -|x|^{-\alpha} + U(t, x)$ with $1 < \alpha < 2$.

Actually, we will show how the generalized solution found in [3] is a noncollision one in our situation.

Let us assume $V \in C^1(\boldsymbol{R} \times \boldsymbol{R}^N \setminus \{0\}; \boldsymbol{R})$ satisfies (V1), (V2) and

(V7) $U(t, x) < 0 \;\; \forall (t, x) \in \boldsymbol{R} \times \boldsymbol{R}^N \setminus \{0\}$,

(V8) $\lim_{|x| \to \infty} U(t, x) = \lim_{|x| \to \infty} |\nabla U(t, x)| = 0$,

(V9) $\exists r > 0$, $\exists \phi \in C^1(]0, r]; \boldsymbol{R})$ such that $U(t, x) = \phi(|x|)$, $\forall 0 < |x| \leqslant r$, $\forall t$,

(V10) $\lim_{s \to 0} \phi'(s) s^{\alpha + 1} = 0$.

Our main result is the following

THEOREM 3.1. Let $V \in C^1(\boldsymbol{R} \times \boldsymbol{R}^N \setminus \{0\}; \boldsymbol{R}^N)$ satisfy (V1), (V2) and (V7)-(V10), with $\alpha > 1$ and $N \geqslant 3$. Then there exists at least one noncollision solution of (1).

SKETCH OF THE PROOF. The proof is divided in various steps.

STEP 1. Existence of a variational solution q (limit of a sequence q_ε of classical solution of approximated problems): such a proof follows the one in [3], to which we refer for details. The solutions q_ε are found by minimaximizing the functional

$$f_\varepsilon(q) = f(q) - \varepsilon \int_0^T G(q)\,dt \qquad (G \in \mathcal{G})$$

over a suitable class Γ of subsets of Λ. In this way one finds for each $\varepsilon > 0$ critical levels c_ε for f_ε and corresponding solutions q_ε of $(1)_\varepsilon$ which can be shown to converge to a (variational) solution q of (1).

STEP 2. Properties of q and $q_\varepsilon (1 < \alpha < 2)$.

We show that if q has a collision then q_ε has a self-intersection for ε small. The proof is similar to that of [4].

We now use the fact that q_ε has a self-intersection to find a contradiction.

STEP 3. The solutions q_ε cannot have self-intersections for ε small $(0 < \alpha < 2)$.

More precisely, one can show that, whenever q_ε has a self intersection, then $\exists A \in \Gamma$ such that

$$\max_{x \in A} f_\varepsilon(x) = c_\varepsilon, \qquad \nabla f_\varepsilon(x) \neq 0 \quad \forall x \in A$$

contradicting the fact that c_ε is a minimax value. This proves step 3 and the Theorem.

In the case $\alpha = 1$ Theorem 3.1 cannot hold. However the method used to prove Theorem 3.1 shows that the solution found as a limit of solutions of approximated problems has still some additional properties. Precisely we have

THEOREM 3.2. *Suppose that the assumptions of Theorem 3.1 hold with $\alpha = 1$. Then there exists a generalized solution q of (1) such that for every $\bar{t} \in \mathcal{C}(q)$ one has $q(\bar{t} + t) = q(\bar{t} - t)$. Moreover, such a solution has at most one collision.*

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