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On a construction of regular Hadamard matrices

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Teorie combinatorie. — *On a construction of regular Hadamard matrices.* Nota di DAVID BENJAMIN MEISNER, presentata (*) dal Socio G. Zappa.

ABSTRACT. — We give a construction for regular Hadamard matrices of order a^2v where $a \neq 1$ is the order of a Hadamard matrix and v is the order of a regular Hadamard matrix. The construction can be used to construct regular Hadamard matrices with special properties and includes several constructions which have been given previously. In the final section we consider the case $a = 2$ in more detail.

KEY WORDS: Hadamard matrices; Regular Hadamard matrices; Menon designs.

RIASSUNTO. — *Su una costruzione per le matrici regolari di Hadamard.* Si dà una costruzione per le matrici di Hadamard di ordine a^2v dove a è l'ordine di una matrice di Hadamard e v è l'ordine di una matrice regolare di Hadamard. Questa costruzione può essere usata per costruire matrici regolari di Hadamard con particolari proprietà e comprende diverse costruzioni date in precedenza. Nell'ultima sezione si considera più dettagliatamente il caso $a = 2$.

1. INTRODUCTION

A ± 1-matrix is a matrix whose entries are either $+1$ or -1. We denote the $a \times a$ identity matrix by I_a and the transpose of a matrix H by H^T.

DEFINITION. A *Hadamard Matrix* H of order a is an $a \times a$ ± 1-matrix with the property that $HH^T = a I_a$.

For an integer a we denote by $\langle a \rangle$ the index set $\{1, \ldots, a\}$ A square $a \times a$ ± 1-matrix

$$H = (h_{ij})_{j \in \langle a \rangle}^{i \in \langle a \rangle}$$

with rows labelled $H_i \, (i \in \langle a \rangle)$ is a Hadamard matrix if and only if

$$H_i H_{i'}^T = \sum_{j=1}^{a} h_{ij} h_{i'j} = a\delta_{ii'}$$

where

$$\delta_{ii'} \begin{cases} = 1 & \text{if } i = i', \\ = 0 & \text{if } i \neq i'. \end{cases}$$

Note that $(H^T)^{-1} = H/a$ so a matrix H is a Hadamard matrix if and only if its transpose H^T is a Hadamard matrix. Hence the entries h_{ij} of H also satisfy the equations

$$\sum_{i=1}^{a} h_{ij} h_{ij'} = a\delta_{jj'} .$$

From these equations it is clear that permutation of the rows and columns of a Hadamard matrix H, and multiplication by -1 of the rows and columns of H leave the Hadamard property unchanged. Two Hadamard matrices are said to be *equivalent*

(*) Nella seduta del 12 giugno 1992.

if one can be obtained from the other by permutation or multiplication by -1 of the rows and columns.

In this paper we are especially concerned with regular Hadamard matrices.

DEFINITION. A Hadamard matrix

$$H = (h_{ij})_{j \in \langle a \rangle}^{i \in \langle a \rangle}$$

is said to be *regular* if the sum of the entries in each row,

$$\sum_{j=1}^{a} h_{ij}, \qquad (i \in \langle a \rangle),$$

and the sum of the entries in each column

$$\sum_{i=1}^{a} h_{ij}, \qquad (j \in \langle a \rangle),$$

are all equal to one another.

Regular Hadamard matrices correspond to a particular type of symmetric 2-design.

DEFINITION. A *symmetric 2-design* on the point set P of size v is a collection of v distinct subsets of P each of size k, which we call blocks, such that every pair of distinct points is contained in exactly λ blocks. A *Menon design* is a symmetric 2-design whose parameters satisfy the relation $v = 4(k - \lambda)$.

By identifying points $\{p_i \mid i \in \langle v \rangle\}$ and blocks $\{B_j \mid j \in \langle v \rangle\}$ with the rows and columns of a ± 1-matrix

$$H = (h_{ij})_{j \in \langle a \rangle}^{i \in \langle a \rangle}$$

and the containment of points in blocks with the entries of H by $p_i \in B_j$ if and only if $h_{ij} = +1$ we get a correspondence between regular Hadamard matrices of orders greater than 1 and Menon designs [5].

From counting arguments (see [4]) it follows that the parameters (v, k, λ) of a Menon design are related by a single integer parameter t with $v = (4t^2)$, $k = (2t^2 \pm t)$, $\lambda = (t^2 \pm t)$. Hence regular Hadamard matrices exist only for orders of the form 1 or $4t^2$ with constant row/column sum $\pm 2t$ in the latter case.

Regular Hadamard matrices are already known to exist for the orders constructed in this paper. We show how some known constructions can be considered to be special cases of our construction. For the case $a = 2$ we obtain a result about the number of inequivalent regular Hadamard matrices of a given order. It is noted that inequivalent regular Hadamard matrices correspond to non-isomorphic Menon designs.

2. A CONSTRUCTION FOR HADAMARD MATRICES

We give a general construction for Hadamard matrices from other Hadamard matrices which uses the Kronecker product.

DEFINITION. Let

$$M = (m_{ij})_{j \in \langle s \rangle}^{i \in \langle r \rangle}$$

be an $r \times s$ matrix and

$$N = (n_{i'j'})_{j' \in \langle p \rangle}^{i' \in \langle q \rangle}$$

be a $q \times p$ matrix. Then the *Kronecker product* of M and N is given by

$$M \otimes N = (k_{(i,i')(j,j')})_{(j,j') \in \langle s \rangle \times \langle p \rangle}^{(i,i') \in \langle r \rangle \times \langle q \rangle} \qquad \text{where} \qquad K_{(i,i')(j,j')} = m_{ij} n_{i'j'} \, .$$

The Kronecker product of ± 1-matrices is a ± 1-matrix and the Kronecker product of two identity matrices I_a and $I_{a'}$ is the identity matrix $I_{aa'}$. The following properties of the Kronecker product, which we use in the proofs of Theorems 1 and 2, are given by Wallis Street Wallis [7].

RESULT 1. *Let x be any scalar and M, N, M_1, M_2, N_1, N_2, R matrices with sizes which make sense in the context of the operations in the following:*

(*i*) $x(M \otimes N) = M \otimes xN = xM \otimes N$,

(*ii*) $(M_1 + M_2) \otimes N = M_1 \otimes N + M_2 \otimes N$,

(*iii*) $M \otimes (N_1 + N_2) = M \otimes N_1 + M \otimes N_2$,

(*iv*) $(M_1 \otimes N_1)(M_2 \otimes N_2) = M_1 M_2 \otimes N_1 N_2$,

(*v*) $(M \otimes N)^T = M^T \otimes N^T$,

(*vi*) $M \otimes (N \otimes R) = (M \otimes N) \otimes R$.

From (*iv*) it is clear that the Kronecker product of Hadamard matrices is a Hadamard matrix.

Let $a \neq 1$ and v be orders for which Hadamard matrices exist. In our construction we use a^2 Hadamard matrices of order v which we label

$$M^{il} = (m_{bc}^{il})_{c \in \langle v \rangle}^{b \in \langle v \rangle}, \qquad (i, l \in \langle a \rangle),$$

and we use $2a$ Hadamard matrices of order a which we label

$$G^l = (g_{ir}^l)_{r \in \langle a \rangle}^{i \in \langle a \rangle}, \quad (l \in \langle a \rangle), \qquad \text{and} \qquad F^i = (f_{el}^i)_{l \in \langle a \rangle}^{e \in \langle a \rangle}, \quad (i \in \langle a \rangle).$$

The rows of G^l are labelled G_i^l, $(i \in \langle a \rangle)$, and the columns of F^i are labelled F_l^i, $(l \in \langle a \rangle)$. We refer to the Hadamard matrices M^{il}, G^l and F^i collectively as the *ingredients* of the construction.

Using the ingredients we form the following partition matrix Z.

$$Z = \begin{bmatrix} Z^{11} & | & Z^{12} & | & \cdots & | & Z^{1a} \\ - & & - & & & & - \\ Z^{21} & | & Z^{22} & | & \cdots & | & Z^{2a} \\ - & & - & & & & - \\ \vdots & | & \vdots & | & \ddots & | & \vdots \\ - & & - & & & & - \\ Z^{a1} & | & Z^{a2} & | & \cdots & | & Z^{aa} \end{bmatrix}$$

where

$$Z^{il} = F_l^i \otimes G_i^l \otimes M^{il} = (f_{el}^i \, g_{ir}^l \, m_{bc}^{il})_{(r,c) \in \langle a \rangle \times \langle v \rangle}^{(e,b) \in \langle a \rangle \times \langle v \rangle} \qquad (i, l \in \langle a \rangle)$$

so that

$$Z = (f_{el}^i \, g_{ir}^l \, m_{bc}^{il})_{(l,r,c) \in \langle a \rangle \times \langle a \rangle \times \langle v \rangle}^{(i,e,b) \in \langle a \rangle \times \langle a \rangle \times \langle v \rangle} .$$

THEOREM 1. *With the notation above Z is a Hadamard matrix.*

PROOF.

$$ZZ^T = \begin{bmatrix} A^{11} & | & A^{12} & | & \cdots & | & A^{1a} \\ - & & - & & & & - \\ A^{21} & | & A^{22} & | & \cdots & | & A^{2a} \\ - & & - & & & & - \\ \vdots & | & \vdots & | & \ddots & | & \vdots \\ - & & - & & & & - \\ A^{a1} & | & A^{a2} & | & \cdots & | & A^{aa} \end{bmatrix}$$

where for $i, j \in \langle a \rangle$

$$A^{ij} = (Z^{i1} \,|\, Z^{i2} \,|\, \cdots \,|\, Z^{ia})(Z^{j1} \,|\, Z^{j2} \,|\, \cdots \,|\, Z^{ja})^T = \sum_{l=1}^{a} (Z^{il})(Z^{jl})^T =$$

$$= \sum_{l=1}^{a} (F_l^i \otimes G_i^l \otimes M^{il})(F_l^j \otimes G_j^l \otimes M^{jl})^T = \sum_{l=1}^{a} (F_l^i)(F_l^j)^T \otimes (G_i^l)(G_j^l)^T \otimes (M^{il})(M^{jl})^T =$$

$$= \sum_{l=1}^{a} (F_l^i)(F_l^j)^T \otimes \delta_{ij} a \otimes (M^{il})(M^{jl})^T = \delta_{ij} \sum_{l=1}^{a} (F_l^i)(F_l^i)^T \otimes a (M^{il})(M^{il})^T$$

since $\delta_{ij} N = \delta_{ij} N'$ where N, N' are matrices of the same size for $i \neq j$. So

$$A^{ij} = \delta_{ij} \sum_{l=1}^{a} (F_l^i)(F_l^i)^T \otimes av I_v = \delta_{ij} \left(\sum_{l=1}^{a} (f_{rl}^i \, f_{r'l}^i)_{r' \in \langle a \rangle}^{r \in \langle a \rangle} \right) \otimes av \, I_v =$$

$$= \delta_{ij} (\delta_{rr'} \, a)_{r' \in \langle a \rangle}^{r \in \langle a \rangle} \otimes av \, I_v = \delta_{ij} (a \, I_a \otimes av \, I_v) = \delta_{ij} a^2 v \, I_{av} .$$

Hence $ZZ^T = a^2 v \, I_{a^2 v}$ as required.

If all the Hadamard matrices M^{il}, $(i, l \in \langle a \rangle)$, are equal to the same matrix M then the constructed matrix Z has the form $Z = X \otimes M$ where X is the Hadamard matrix of order a^2 constructed using our construction from ingredients G^l, F^i with $v = 1$ and all ingredients of order v equal to the regular Hadamard matrix (1).

3. Regular Hadamard matrices

By placing further restrictions on the ingredients of the construction in Sect. 2 we show how it may be used to form regular Hadamard matrices. We now assume that v is the order of a regular Hadamard matrix and that the ingredients M^{ii}, $(i \in \langle a \rangle)$, are regular, each with the same row/column sum σ. We suppose further that the row G_l^l of G^l has all entries $+1$ and that the column F_i^i of F^i has all entries $+1$ so that $g_{lr}^l = f_{ei}^i = 1$, $(i, l, r, e \in \langle a \rangle)$. These last conditions are always possible to achieve when a Hadamard matrix of order a exists by multiplying appropriate rows and columns by -1.

Theorem 2. *With ingredients as above the constructed Hadamard matrix Z of the previous section is a regular Hadamard matrix.*

Proof. By Theorem 1 we need only show that Z is regular. We note that

$$\sum_{r=1}^{a} g_{ir}^l = \sum_{r=1}^{a} g_{ir}^l g_{lr}^l = \delta_{il} a \quad \text{and that} \quad \sum_{e=1}^{a} f_{el}^i = \sum_{e=1}^{a} f_{el}^i f_{ei}^i = \delta_{il} a .$$

Hence the row sum of row (i, e, b) is

$$\sum_{l=1}^{a} \sum_{r=1}^{a} \sum_{c=1}^{v} f_{el}^i g_{ir}^l m_{bc}^{il} = \sum_{l=1}^{a} \sum_{c=1}^{v} f_{el}^i m_{bc}^{il} \left(\sum_{r=1}^{a} g_{ir}^l \right) = a \sum_{c=1}^{v} f_{el}^i m_{bc}^{ii} = a \sum_{c=1}^{v} m_{bc}^{ii} = a\sigma$$

and the column sum of column (l, r, c) is

$$\sum_{i=1}^{a} \sum_{e=1}^{a} \sum_{b=1}^{v} f_{el}^i g_{ir}^l m_{bc}^{il} = \sum_{i=1}^{a} \sum_{b=1}^{v} g_{ir}^l m_{bc}^{il} \left(\sum_{e=1}^{a} f_{el}^i \right) = a \sum_{b=1}^{v} g_{lr}^l m_{bc}^{ll} = a \sum_{c=1}^{v} m_{bc}^{ll} = a\sigma .$$

In the following two Sections we show how the construction for regular Hadamard matrices given above can be used to construct regular Hadamard matrices with further properties. We also indicate how two known constructions can be described in terms of our construction.

4. Symmetric regular Hadamard matrices

Definition. A Hadamard matrix H is said to be *symmetric* if $H = H^T$.

The constructed regular Hadamard matrix Z of Sect. 3 is symmetric if for all pairs (i, e, b), $(l, r, c) \in \langle a \rangle \otimes \langle a \rangle \otimes \langle v \rangle$ the entries of the ingredients satisfy $f_{el}^i g_{ir}^l m_{bc}^{il} = f_{ri}^l g_{le}^i m_{cb}^{li}$. This condition is satisfied when the ingredients are chosen so that $F^i = (G^i)^T$ and $M^{il} = (M^{li})^T$, in which case the regular Hadamard matrices M^{ii} are themselves symmetric.

A Hadamard matrix of order 1 is symmetric so we have a construction for symmetric regular Hadamard matrices of order a^2 whenever a Hadamard matrix of order a exists. Such a construction, due to W.D. Wallis (see [1]), is well known and corresponds to a case of the construction above with $v = 1$ and all the matrices $F^i = (G^i)^T$, $(i \in \langle a \rangle)$, equivalent to one another.

5. Diagonal-skew regular Hadamard matrices

Definition. A Hadamard matrix

$$H = (h_{ij})_{j \in \langle a \rangle}^{i \in \langle a \rangle}$$

is said to be *diagonal-skew* if for $i \in \langle a \rangle$ $h_{ii} = -1$ and for $i, j \in \langle a \rangle$, $i \neq j$, if $h_{ij} = +1$ then $h_{ji} = -1$.

The constructed regular Hadamard matrix Z of Sect. 3 is diagonal-skew if for all pairs (i, e, b), $(l, r, c) \in \langle a \rangle \otimes \langle a \rangle \otimes \langle v \rangle$, $(i, e, b) \neq (l, r, c)$ if $f^i_{el} g^l_{ir} m^{il}_{bc} = +1$ then $f^l_{ri} g^i_{le} m^{li}_{cb} = -1$ and for $(i, e, b) \in \langle a \rangle \otimes \langle a \rangle \otimes \langle v \rangle$ $f^i_{ei} g^i_{ir} m^{ii}_{bb} = -1$. This condition is satisfied when the ingredients are chosen so that $F^i = (G^i)^T$ and $M^{il} = -(M^{li})^T$, $(i, l \in \in \langle a \rangle, i \neq l)$, and the regular Hadamard matrices M^{ii} are themselves diagonal-skew.

The regular Hadamard matrix (-1) of order 1 is diagonal-skew so we have a construction for diagonal-skew regular Hadamard matrices of order a^2 whenever a Hadamard matrix of order a exists. Szekeres [6] gives such a construction. This corresponds to a case of our construction with $v = 1$ and the matrices $F^i = (G^i)^T$, $(i \in \in \langle a \rangle)$, all equivalent to one another.

6. The construction with $a = 2$

In the case $a = 2$ the matrix constructed by the Hadamard matrix construction of Sect. 2 depends up to equivalence only on the four Hadamard matrices of order v. In any case the Hadamard matrix constructed from ingredients M^{il}, $(i, l = 1, 2)$, is equivalent to the partition matrix

$$Z_2(M^{il} \mid i, l = 1, 2) = \begin{array}{c} \\ A'1 \\ \\ A'2 \\ \\ B'1 \\ \\ B'2 \end{array} \begin{array}{c} \begin{array}{cccc} A1 & A2 & B1 & B2 \end{array} \\ \left[\begin{array}{c|c|c|c} M^{11} & M^{11} & M^{12} & -M^{12} \\ \hline M^{11} & M^{11} & -M^{12} & M^{12} \\ \hline M^{21} & -M^{21} & M^{22} & M^{22} \\ \hline -M^{21} & M^{21} & M^{22} & M^{22} \end{array}\right] \end{array}$$

which is regular if the matrices M^{11} and M^{22} are regular.

It can be seen immediately that Z_2 is invariant under the two permutations of the rows and columns defined by

(1) Swap the rows between partitions $A'1$ and $A'2$ and the columns between partitions $B1$ and $B2$

(2) Swap the rows between partitions $B'1$ and $B'2$ and the columns between partitons $A1$ and $A2$.

When Z_2 is regular these operations correspond to involutions of the corresponding Menon design which fix exactly half the points and half the blocks of the design. W. Feit and H. Wilbrink have proved (see [3]) that Menon designs are the only symmetric 2-designs which admit automorphisms fixing half their points.

In fact the constructed Hadamard matrix is equivalent to a Hadamard matrix which can be constructed using repeated applications of a construction for Hadamard matrices given by Din and Mavron [2]. This construction takes two Hadamard matrices H_1 and H_2 of the same order a to construct a third of order $2a$ given by

$$Q(H_1, H_2) = \begin{pmatrix} H_1 & H_1 \\ H_2 & -H_2 \end{pmatrix}.$$

We also need the dual of this construction which is given by

$$Q^T(H_1, H_2) = \begin{pmatrix} H_1 & H_2 \\ H_1 & -H_2 \end{pmatrix}.$$

We note that $Q(H_1, H_1) = Q^T(H_1, H_1)$. It is easily checked that the constructed Hadamard matrix $Z_2(M^{il} \mid i, l = 1, 2)$ is equivalent to $Q^T(Q(M^{11}, M^{12}), Q(M^{21}, M^{22}))$.

Din and Mavron [2] provided a method for distinguishing between inequivalent Hadamard matrices of the same order by considering the number of certain configurations in the matrices. They call this number the *characteristic number*. We summarize their results in the following.

RESULT 2. *Let H be a Hadamard matrix of order $v \neq 1, 2$ and characteristic number θ. Any Hadamard matrix equivalent to H has the same characteristic number θ. The characteristic number of $Q(H, H)$ is $8\theta + 4v(v-1)$. If $v > 4$ and $\theta \neq 0$ then there is a Hadamard matrix H^1, obtained from H by a permutation of rows, such that the characteristic number κ of $Q(H, H')$ satisfies $0 < \kappa < 8\theta + 4v(v-1)$.*

We can apply this result with particular reference to the constructed regular Hadamard matrices $Z_2(M^{il} \mid i, l = 1, 2)$ where M^{11} and M^{22} are regular Hadamard matrices of order $v \neq 1$. The following Theorem shows that the number of inequivalent regular Hadamard matrices of order $2^{2n}v$ increases with n. This Theorem can be considered to be a result about the number of pairwise non-isomorphic Menon designs on $2^{2n}v$ points.

THEOREM 3. *Suppose that there exist $s > 0$ regular Hadamard matrices of order $v \neq 1$ with pairwise distinct characteristic numbers. Then there are at least $s + n - 1$ pairwise inequivalent regular Hadamard matrices of order $2^{2n}v$, $(n \geqslant 1)$.*

PROOF. We use Result 2 to give the characteristic number of some particular regular Hadamard matrices $Z_2(M^{il} \mid i, l = 1, 2)$ formed by our construction. Since $Q(H, H) = Q^T(H, H)$ $Z_2(M^{il} \mid i, l = 1, 2)$ is equivalent to, and hence has the same characteristic number as, $Q(Q(M^{11}, M^{12}), Q(M^{11}, M^{12}))$ in the case $M^{11} = M^{21}$ and $M^{12} = M^{22}$. By Result 2 the characteristic number of $Q(Q(M, M), Q(M, M))$ is $8(8\theta + 4w(w-1)) + 8w(2w-1)$ where M is a Hadamard matrix of order w with characteristic number θ. Hence if there are u regular Hadamard matrices of order w with distinct characteristic numbers then there are at least u regular Hadamard matrices of order $4w$ with pairwise distinct non-zero characteristic numbers.

Now we consider the case when there exist u regular Hadamard matrices M^1, $M^2, \ldots, M^u$ of order $w \neq 4$ with pairwise distinct non-zero characteristic numbers of which matrix M^1 has the smallest characteristic number $\hat{\theta}$. By Result 2 there exists a regular Hadamard matrix H' obtained from M^1 by permutation of the rows, such that $Q(M^1, H')$ has characteristic number κ satisfying $0 < \kappa < 8\hat{\theta} + 4w(w-1)$. Hence $Z_2(M^{11} = M^{21} = M^1, M^{12} = M^{22} = H')$, which has the same characteristic number as $Q(Q(M^1, H'), Q(M^1, H'))$, has characteristic number $8\kappa + 8w(2w-1)$ which is less than the characteristic numbers of all the regular Hadamard matrices $Z_2(M^{11} = = M^{21} = M^{12} = M^{22} = M^d)$, $(d = 1, 2, \ldots, u)$ by the choice of M^1. Thus there are at least $u + 1$ regular Hadamard matrices of order $4w$ with pairwise distinct non-zero characteristic numbers in this case.

Since regular Hadamard matrices with distinct characteristic numbers are necessarily inequivalent by Results 3 the Theorem follows.

Clearly Theorem 3 gives only a loose lower bound for the number of inequivalent regular Hadamard matrices which can be constructed using the construction $Z_2(M^{il} \mid i, l = 1, 2)$ with various ingredients. For example there is exactly one regular Hadamard matrix of order 4 up to equivalence which can be used to construct all three inequivalent regular Hadamard matrices of order 16 [2] although the Theorem tells us only that there is at least one. However Theorem 3 does indicate the usefulness of the construction in creating pairwise inequivalent regular Hadamard matrices.

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