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A uniqueness theorem for the approximable solutions of the stationary Navier-Stokes equations

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Analisi matematica. — *A uniqueness theorem for the approximable solutions of the stationary Navier-Stokes equations.* Nota (*) del Corrisp. Giovanni Prouse.

Abstract. — It is proved that there can exist at most one solution of the homogeneous Dirichlet problem for the stationary Navier-Stokes equations in 3-dimensional space which is approximable by a given consistent and regular approximation scheme.

Key words: Fluid dynamics; Approximation schemes; Weak solutions.

Riassunto. — *Un teorema di unicità per le soluzioni approssimabili delle equazioni di Navier-Stokes.* Si dimostra che esiste al più una soluzione del problema di Dirichlet omogeneo per le equazioni stazionarie di Navier-Stokes in 3 dimensioni che sia approssimabile mediante uno schema di approssimazione consistente e regolare.

1. – The aim of this paper is to prove a uniqueness theorem relative to the solution of the homogeneous Dirichlet problem for the stationary Navier-Stokes equations in 3-dimensional space. This will be done, basically, by introducing the definition of «approximable» weak solution and of the corresponding «approximation scheme» and showing that *there cannot exist two weak solutions which are approximable by the same approximation scheme*; hence, for instance, a certain finite difference scheme leads to only one solution, which however can be different from the one obtained by another approximation scheme.

Let Ω be a bounded, open set $\subset R^3$ and denote by N, N^s $(s \geq 0)$ respectively the space of vectors $\boldsymbol{v} = \{v_1, v_2, v_3\} \in \mathcal{D}(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = 0$ and the closure of N in $H^s(\Omega)$; we shall, in particular, consider the Hilbert spaces N^0 and N^1, in which the scalar product is defined by

$$(\boldsymbol{u}, \boldsymbol{v})_{N^0} = (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)}, \qquad (\boldsymbol{u}, \boldsymbol{v})_{N^1} = (\boldsymbol{u}, \boldsymbol{v})_{H_0^1(\Omega)}. \tag{1.1}$$

Setting

$$A\boldsymbol{v} = -\mu \Delta \boldsymbol{v}, \qquad B(\boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \sum_{j,k=1}^{3} u_j \frac{\partial u_k}{\partial x_j},$$

we shall say, following a well known definition (see, for instance [1, 2]) *that* $\boldsymbol{u}$ *is a weak solution in* Ω *of the stationary Navier-Stokes equations satisfying homogeneous Dirichlet boundary conditions if*

i) $\boldsymbol{u} \in N^1$;

ii) $\forall \boldsymbol{\varphi} \in N^1$, $\langle A\boldsymbol{u} + B(\boldsymbol{u}) - \boldsymbol{f}, \boldsymbol{\varphi}\rangle = 0$, having assumed that $\boldsymbol{f} \in (N^1)'$ and denoted by $\langle , \rangle$ the duality between $(N^1)'$ and N^1.

In order to introduce the concept of approximable weak solution, we must recall some definitions, due essentially to Temam [1], regarding approximation schemes

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relative to Navier-Stokes equations; in what follows, h will denote a real index which will, eventually, $\to 0$ (1).

Let W be a Banach space $\subset L^2(\Omega)$; we shall call *approximation of* W a set consisting of:

a_1) A Banach space $F \subset L^2(\Omega)$ and an isomorphism ω of W into F;

b_1) A family of triples $\{W_h, p_h, r_h\}$ such that, $\forall h$, W_h is a Banach space $\subset L^2(\Omega)$, p_h is a continuous linear mapping of W_h into F, r_h is a mapping of W into W_h.

An approximation of W is said to be *stable* if

$$\|p_h\| = \sup_{\|u_h\|_{W_h}=1} \|p_h u_h\|_F \le M \quad (M \text{ independent of } h). \tag{1.2}$$

An approximation of W is said to be *convergent* if:

a_2) $\lim_{h\to 0} p_h r_h u = \omega u$ in F, $\forall u \in W$;

b_2) $\forall$ sequence $\{u_{h'}\}$ of elements of $W_{h'}$ such that $\lim_{h'\to 0} p_{h'} u_{h'} = \psi$ in F, there exists $u \in W$ such that $\psi = \omega u$.

Let now $\{(N_h^1, p_h, r_h), (\omega, F)\}$ be a stable and convergent approximation of N^1; we introduce the forms $a_h(u_h, v_h)$, $b_h(u_h, v_h, w_h)$ defined in the following way.

$a_h(u_h, v_h)$ is a bilinear, continuous form on $N_h^1 \times N_h^1$ such that, $\forall h$,

$$|a_h(u_h, v_h)| \le c_1 \|u_h\|_{N_h^1} \|v_h\|_{N_h^1}, \tag{1.3}$$

$$a_h(u_h, u_h) \ge \sigma \|u_h\|^2_{N_h^1}, \tag{1.4}$$

with c_1, σ positive constants, independent of h.

$b_h(u_h, v_h, w_h)$ is a trilinear, continuous form on $N_h^1 \times N_h^1 \times N_h^1$ such that, $\forall h$,

$$b_h(u_h, v_h, w_h) = -b_h(u_h, w_h, v_h). \tag{1.5}$$

$$|b_h(u_h, v_h, w_h) \le c_2 \|u_h\|_{L^q} \|v_h\|_{N_h^1} \|w_h\|_{L^p} \tag{1.6}$$

with $1/p + 1/q = 1/2$, c_2 independent of h.

We shall, moreover, assume that

a_3) If $\lim_{h\to 0} p_h u_h = \omega u$, $\lim_{h\to 0} p_h v_h = \omega v$ respectively in the strong and weak topology of F, then

$$\lim_{h\to 0} a_h(u_h, v_h) = (u, v)_{N^1}; \tag{1.7}$$

b_3) If $\lim_{h\to 0} u_h = u$ in L^2, $\lim_{h\to 0} p_h v_h = \omega v$ in the weak topology of F, $\lim_{h\to 0} w_h = w$ in L^∞, then

$$\lim_{h\to 0} b_h(u_h, v_h, w_h) = ((u\cdot\nabla) v, w)_{N^0}. \tag{1.8}$$

The set $\Gamma_h = \{(N_h^1, p_h, \Gamma_h), (\omega, F), (a_h, b_h)\}$ will be called an *approximation scheme* of the problem considered if $\{(N_h^1, p_h, \Gamma_h), (\omega, F)\}$ is a stable and convergent approximation of N^1 and if (1.3), (1.4), (1.5), (1.6), (1,7) (1.8) hold.

On the basis of the definitions given above, it is possible to define approximate and approximable solutions. We shall say that u_h is an *approximate solution*, relative to

(1) For other approximation schemes, using a different approach, see [3].

the scheme Γ_h, of the homogeneous Dirichlet problem for the stationary Navier-Stokes equations, with known term $f_h \in (N_h^1)'$, if:

i_h) $\boldsymbol{u}_h \in N_h^1$;

ii_h) $\forall \boldsymbol{\varphi}_h \in N_h^1,\ \mu\, a_h(\boldsymbol{u}_h, \boldsymbol{\varphi}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{\varphi}_h) = \langle f_h, \boldsymbol{\varphi}_h \rangle$,

where $\langle\, , \rangle$ denotes the duality between $(N_h^1)'$ and N_h^1.

Let f be an arbitrary function $\in (N^1)'$ and $\{f_h\}$ a set such that $f_h \in (N_h^1)'$ and

(1.9) $$\lim_{h \to 0} \langle f_h, \boldsymbol{\varphi}_h \rangle = \langle f, \boldsymbol{\varphi} \rangle$$

$\forall$ set $\{\boldsymbol{\varphi}_h\}$ such that $\boldsymbol{\varphi}_h \in N_h^1$ $\lim\limits_{h \to 0} p_h \boldsymbol{\varphi}_h = \omega \boldsymbol{\varphi}$ in F.

We shall say that *the approximation scheme Γ_h is consistent and regular if*:

α) $\forall h$ there exists one, and only one, approximate solution $\boldsymbol{u}_h$ corresponding to f_h;

β) It is possible to select, from any sequence $\{\boldsymbol{u}_{h_n}\}$ a subsequence $\{\boldsymbol{u}_{h_{n'}}\}$ such that

(1.10) $$p_{h_{n'}} \boldsymbol{u}_{h_{n'}} \to \omega \boldsymbol{u} \text{ weakly in } F, \qquad \boldsymbol{u}_{h_{n'}} \to \boldsymbol{u} \text{ weakly in } L^2,$$

where $\boldsymbol{u}$ is a weak solution corresponding to f;

γ) $\forall$ fixed $\bar{h} > 0$, $\lim\limits_{h \to \bar{h}} \boldsymbol{u}_h = \boldsymbol{u}_{\bar{h}}$ weakly in L^2.

We shall, finally, say that the weak solution $\boldsymbol{u}$, corresponding to the known term f, is *approximable* by the consistent and regular scheme Γ_h if there exists a sequence $\{\boldsymbol{u}_{h_n}\}$ of approximate solutions, corresponding to the known terms f_{h_n} satisfying (1.9), such that

(1.11) $$\lim_{n \to \infty} p_{h_n} \boldsymbol{u}_{h_n} = \omega \boldsymbol{u} \qquad \text{weakly in } F.$$

In the sections which follow, we shall prove the following

THEOREM. *Let Γ_h be a consistent and regular approximation scheme. There exists at most one weak solution which is approximable by such a scheme.*

The guideline of the proof is the following.

a) If there exist two weak solutions which are approximable by the same consistent and regular approximation scheme Γ_h, then there exist infinitely many weak solutions, with the power of the continuum, which are approximable by Γ_h. The set of these solutions will in the sequel be denoted by U (Lemma 1).

b) The set U is compact in L^4 (Lemma 2).

c) Let $\{\boldsymbol{u}_n\}$ be any sequence $\subset U$; $\{\boldsymbol{u}_n\}$ cannot then be a basis of U (Lemma 3). The proof of the Theorem will then consist in showing that

α) It is not possible that there exist a sequence $\{\boldsymbol{v}_n\}$ of linearly independent solutions (since, in this case, $\{\boldsymbol{v}_n\}$ would be a basis, against *c*));

β) it is not possible that there exist only a finite number p of linearly independent solutions (since, in this case, U would contain only a finite number of elements, against *a*)).

2. – Let $\{g_j\}$ be an orthonormal basis in N^1 and denote by $\tilde{g}_j$ the (eventual) elements of $\{g_j\}$ such that $(u, g_j)_{N^1} = 0 \ \forall u \in U$; the remaining elements of $\{g_j\}$ will form a sequence which we shall denote by $\{g_j^*\}$ (eventually $\{g_j^*\} = \{g_j\}$). We shall, moreover, denote by N^{*1} be the subspace of N^1 spanned by $\{g_j^*\}$.

We now state and prove the Lemmas mentioned in the preceding section.

LEMMA 1. *Let u, v be two weak solutions, corresponding to the same known term f, approximable by the same consistent and regular approximation scheme Γ_h. There exist then infinitely many (with the power of the continuum) weak solutions which are approximable by Γ_h.*

Let u_h be the approximate solution corresponding to the scheme Γ_h; by the assumptions made, there exist a function $\varphi^* \in N^0$ and two sequences $\{u_{h_j'}\}$, $\{u_{h_j''}\}$ such that

(2.1) $$\lim_{h_j' \to 0} (u_{h_j'}, \varphi^*)_{L^2} = (u, \varphi^*)_{L^2} = \alpha; \qquad \lim_{h_j'' \to 0} (u_{h_j''}, \varphi^*)_{L^2} = (v, \varphi^*)_{L^2} = \beta > \alpha .$$

Hence, since Γ_h is regular ($\Rightarrow h \to (u_h, \varphi^*)_{L^2}$ is continuous $\forall h > 0$), fixed an arbitrary number $\gamma \in (\alpha, \beta)$, there exists a sequence $\{u_{h_j'''}\}$ such that

(2.2) $$\lim_{h_j''' \to 0} (u_{h_j'''}, \varphi^*)_{L^2} = \gamma .$$

Since Γ_h is a consistent scheme, we may, on the other hand, assume that

(2.3) $$\lim_{h_j''' \to 0} u_{h_j'''} = z \qquad \text{weakly in } L^2,$$

with $z \in U$; consequently, $\gamma = (z, \varphi^*)_{L^2}$ and, since γ is arbitrarily chosen in (α, β), there exist infinitely many solutions with the power of the continuum, all approximable by Γ_h.

LEMMA 2. *The set U is compact in L^4.*

The statement is obviously true if U consists of only a finite number of elements. Suppose now that $\{u_n\}$ is a sequence $\subset U$; setting in *ii*) of section 1 $\varphi = u$, we obtain directly (since $\langle B(u), u \rangle = 0 \ \forall u \in N^1$)

(2.4) $$\|u_n\|_{N^1} \le M \qquad (M \text{ independent of } n).$$

It is then possible to select from $\{u_n\}$ a subsequence $\{u_{n'}\}$ such that

(2.5) $$\lim_{n' \to 0} u_{n'} = u$$

in the weak topology of N^1 and in the strong topology of L^4. Bearing in mind that $\forall u$, $v \in N^1$, $\langle (u \cdot \nabla)\, u, \varphi \rangle = -\langle (u \cdot \nabla)\, \varphi, u \rangle$, it follows then, by (2.5), that $u \in U$. Hence, U is L^4 compact.

LEMMA 3. *Let $\{u_n\}$ be a sequence $\subset U$; then $\{u_n\}$ cannot be a basis in N^{*1}*

By Lemma 2, we may assume, in fact, that $\lim_{n \to \infty} u_n = u \in U$. On the other hand, it is well known that, if a sequence $\{z_n\}$ is a basis in a Banach space and if $z_n \to z$, then necessarily $z = 0$. If therefore $\{u_n\}$ were a basis, $u = 0$ would be a solution, which, by the uniqueness theorem of «small» solutions (see, for instance [1]), is unique. Hence, U would consist of only one element and could not contain a sequence. The Lemma is thus proved.

3. – Let us now prove the uniqueness Theorem stated in section 1.

We show, to begin with, that, if $\{u_k\}$ is a sequence of solutions, then the elements of $\{u_k\}$ cannot all be linearly independent in N^{*1}. Let, in fact,

$$u_k = \sum_{j=1}^{\infty} \zeta_{kj}\, g_j^* \qquad (k = 1, 2, \dots) \tag{3.1}$$

where, by the assumptions made, the linear combinations $\sum_{j=1}^{\infty} \zeta_{kj}\, g_j^*$ are all linearly independent in N^{*1}. Since $\{g_j^*\}$ is a basis in U, it follows that also $\{u_k\}$ is a basis in U, and this, by Lemma 3, is not possible.

Finally, we must consider the case in which all solutions u are linear combinations of a finite number p of solutions: $u = \sum_{k=1}^{p} \alpha_k v_k$. Since, by the assumptions made, there exists an infinite number of solutions, all satisfying (2.4), there must exist a solution, u^*, which is a limit point for U; we shall then denote by σ the neighbourhood of u^* defined by

$$\sigma = \{v \in L^4 \colon \|u^* - v\|_{L^4} < 1\}\,. \tag{3.2}$$

By construction, σ contains an infinte number of solutions; let $q \leqslant p$ be the number of linearly independent solutions $\in \sigma$; these will be denoted by v_k $(k = 1, 2, \dots, q)$, with $v_1 = u^*$. By (3.2) we have

$$\|v_1 - v_k\|_{L^k} < 1 \qquad \forall k = 1, 2, \dots, q\,. \tag{3.3}$$

Let now $u = \sum_{k=1}^{q} \alpha_k v_k$ be a solution $\in \sigma$; the coefficients α_k must then satisfy the equations

$$\begin{cases} \displaystyle\sum_{k=1}^{q} \alpha_k A v_k + \sum_{k=1}^{q} \alpha_k B(v_k) - f \sum_{k=1}^{q} \alpha_k = 0\,, \\ \displaystyle A \sum_{k=1}^{q} \alpha_k v_k + B\Big(\sum_{k=1}^{q} \alpha_k v_k\Big) - f = 0 \end{cases} \tag{3.4}$$

and, consequently, $\forall \varphi \in N^1$,

$$\psi_\varphi(\alpha_1, \dots, \alpha_q) = \sum_{k=1}^{q} \alpha_k ((v_k \cdot \nabla) v_k, \varphi)_{L^2} - \tag{3.5}$$

$$- \left(\left(\sum_{k=1}^{q} \alpha_k v_k \cdot \nabla\right) \sum_{k=1}^{q} \alpha_k v_k, \varphi\right)_{L^2} + \left(1 - \sum_{k=1}^{q} \alpha_k\right) \langle f, \varphi \rangle = 0\,.$$

Equations (3.5) constitute an algebraic system of degree 2 in the unknowns $\alpha_1, \alpha_2, \dots, \alpha_q$, which, obviously, admits the q solutions $P_1(1, 0, \dots, 0)$, $P_2(0, 1, \dots, 0)$, $\dots$, $P_q(0, 0, \dots, 1)$. According to Lemmas 1 and 2, on the other hand, if we assume that there exists more than one solution, in every neighbourhood of v_1 must be contained infinitely many solutions, *i.e.* the system (3.5) must admit infinitely many solutions in every neighbourhood of P_1. We shall show that this is absurd and, consequently, that the solution is unique. The proof is given in the Appendix.

4. – The uniqueness Theorem proved in the preceding Section holds for solutions which are approximable by a consistent and regular approximation scheme.

Examples of such schemes are the finite difference and finite elements schemes described by Temam. It is, in fact, proved in [1] that these schemes are consistent and from Temam's analysis it also follows that they are regular (*i.e.* that condition γ) of section 1 is satisfied) provided that, when $h \to 0$, the sequence of grids is chosen in an appropriate way (2).

It can also easily be shown that another consistent and regular approximation scheme can be obtained by considering as approximate solutions the solutions of the Navier-Stokes equations with an «artificial viscosity» term, *i.e.*

$$A\boldsymbol{u}_h + B(\boldsymbol{u}_h) + h\Delta^2\boldsymbol{u}_h + \nabla p = f, \qquad \operatorname{div}\ \boldsymbol{u}_h = 0 \tag{4.1}$$

with the boundary conditions

$$\boldsymbol{u}_h\,\big|_{\partial\Omega} = \frac{\partial \boldsymbol{u}_h}{\partial n}\bigg|_{\partial\Omega} = 0. \tag{4.2}$$

This approximation scheme falls into the general pattern described in Sect. 1 by setting $F = W = N^1$, $N_h^1 = N^1 \cap H^2$, with norm defined by

$$\|\boldsymbol{v}\|^2_{N_h^1} = \|\boldsymbol{v}\|^2_{N^1} + h\|\Delta \boldsymbol{u}\|^2_{L^2}, \tag{4.3}$$

$r_h = G$ (Green's operator, from $(N^1)'$ to N^1, associated to $-\Delta$), p_h embedding operator from N_h^1 into N^1. For a more detailed description of this scheme see [2, 4].

Appendix

Let us prove the following statement:

Assume that all the solutions belonging to the neighbourhood σ of $\boldsymbol{v}_1$ introduced in Section 3 depend linearly on the q independent solutions $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_q$; there exists then a neighbourhood of $\boldsymbol{v}_1$ which contains no other solutions apart from $\boldsymbol{v}_1$.

The proof will be carried out considering separately the following cases:

i) It is possible to find in σ q linearly independent solutions $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_q$ such that the functions

$$z_k = (\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_k + (\boldsymbol{v}_k \cdot \nabla)\boldsymbol{v}_1 + A\boldsymbol{v}_k \qquad (k = 1, \ldots, q) \tag{5.1}$$

are linearly independent.

ii) Whatever choice is made of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_q$, the corresponding functions $z_1, \ldots, z_q$ are not linearly independent, *i.e.* there exist $(\lambda_1, \ldots, \lambda_q) \not\equiv 0$ such that

$$\sum_{k=1}^{q} \lambda_k z_k = 0. \tag{5.2}$$

This case can be divided into two subcases:

(2) In the finite difference scheme, for example, it is sufficient to assume that, if the net points are the centers of the grid, one net point remains fixed $\forall h$.

ii$_1$) There exist $v_1, \ldots, v_q$ such that (5.2) holds with

$$\sum_{k=1}^{q} \lambda_k \neq 0, \tag{5.3}$$

or

ii$_2$) For every choice of $v_1, \ldots, v_q$, (5.2) holds with

$$\sum_{k=1}^{q} \lambda_k = 0. \tag{5.4}$$

Let us start by considering case i).

Observe that the tangent hyperplanes in P_1 to the hypersurfaces (3.5) are given by

$$\sum_{k=1}^{q} \alpha_k \langle z_k, \varphi \rangle = \langle z_1, \varphi \rangle. \tag{5.5}$$

Observe, moreover, that, if $z_1, \ldots, z_q$ are linearly independent, it is possible to choose $\varphi_1, \ldots, \varphi_q$ such that the corresponding hyperplanes are linearly independent.

Assume, in fact, that $z_1, \ldots. z_q$ are linearly independent and set

$$\varphi_j = G z_j \tag{5.6}$$

where G is Green's operator, from $(N^1)'$ to N^1 relative to A; by (5.5) we obtain

$$\sum_{k=1}^{q} \alpha_k \langle z_k, G z_j \rangle = \langle z_1, G z_j \rangle \qquad (j = 1, \ldots, q). \tag{5.7}$$

If the hyperplanes (5.7) were not linearly independent, there would exist $(\lambda_1, \ldots, \lambda_q) \not\equiv 0$ such that

$$\sum_{j=1}^{q} \lambda_j \langle z_k, G z_j \rangle = \sum_{j=1}^{q} \lambda_j (z_k, z_j)_{(N^1)'} = 0 \qquad (k = 1, \ldots, q), \tag{5.8}$$

which is absurd, since $z_1, \ldots, z_q$ are linearly independent. Hence, the hypersurfaces $\psi_{\varphi_j} = 0$ $(j = 1, 2, \ldots, q)$ have, in a neighbourhood of P_1, in common only the point P_1. Our statement is therefore true in this case.

Consider now case ii$_1$). By (5.1), (5.2), we have

$$(v_1 \cdot \nabla) \sum_{k=1}^{q} \lambda_k v_k + \left(\sum_{k=1}^{q} \lambda_k v_k \cdot \nabla \right) v_1 + A \sum_{k=1}^{q} \lambda_k v_k = 0 \tag{5.9}$$

and, consequently, multiplying by $w = \sum_{k=1}^{q} \lambda_k v_k$ and observing that, by (5.3), we can always assume that $\sum_{k=1}^{q} \lambda_k = 1$,

$$\langle (v_1 \cdot \nabla) w, w \rangle + \langle (w \cdot \nabla) \sum_{k=1}^{q} \lambda_k v_1, w \rangle + \| w \|_{N^1}^2 = 0. \tag{5.10}$$

On the other hand, bearing in mind (3.2) (3),

$$\begin{aligned} |\langle (w \cdot \nabla) \sum_{k=1}^{q} \lambda_k v_1, w \rangle &\leq |\langle (w \cdot \nabla) w, w \rangle| + \\ &+ |\langle (w \cdot \nabla) \sum_{k=1}^{q} \lambda_k (v_1 - v_k), w \rangle| \leq \| w \|_{N^1}^2 \max_{k=1,\ldots,q} \| v_1 - v_k \|_{L^4} < \| w \|_{N^1}^2 . \end{aligned} \tag{5.11}$$

(3) For the sake of simplicity, we shall assume that the embedding constants are $= 1$.

Hence, by (5.10),

$$w = \sum_{k=1}^{q} \lambda_k v_k = 0\,, \tag{5.12}$$

which is absurd, since $v_1, \ldots, v_q$ are linearly independent.

We now consider case ii$_2$). Let us choose arbitrarily $v_2, \ldots, v_q$ such that $v_1, \ldots, v_q$ are linearly independent solutions $\in \sigma$ and let

$$u = \sum_{k=1}^{q} \alpha_k v_k \tag{5.13}$$

be any other solution $\in \sigma$. Setting

$$z_u = (v_1 \cdot \nabla)\, u + (u \cdot \nabla)\, v_1 + A\, u\,, \tag{5.14}$$

there exist then $(\lambda_1, \lambda_2, \ldots, \lambda_q) \not\equiv 0$ with $\lambda_q \neq 0$ such that

$$\lambda_q z_u + \sum_{k=1}^{q-1} \lambda_k z_k = 0\,, \qquad \sum_{k=1}^{q} \lambda_k = 0\,, \tag{5.15}$$

that is

$$\sum_{k=1}^{q-1} \lambda_k [(v_1 \cdot \nabla)\, v_k + (v_k \cdot \nabla)\, v_1 + A v_k] + \tag{5.16}$$

$$+ \lambda_q \left[(v_1 \cdot \nabla) \sum_{k=1}^{q} \alpha_k v_k + \left(\sum_{k=1}^{q} \alpha_k v_k \cdot \nabla \right) v_1 + A \sum_{k=1}^{q} \alpha_k v_k \right] = 0\,.$$

It follows, setting $w = \sum_{k=1}^{q-1} (\lambda_k + \lambda_q \alpha_k)\, v_k + \lambda_q \alpha_q v_q$, that

$$(v_1 \cdot \nabla)\, w + (w \cdot \nabla)\, v_1 + A\, w = 0 \tag{5.17}$$

and, consequently, multiplying by w,

$$\langle (w \cdot \nabla)\, v_1, w \rangle + \| w \|_{N^1}^2 = 0\,. \tag{5.18}$$

Hence, repeating the procedure followed in formulas (5.9) to (5.12), we may conclude that if

$$\sum_{k=1}^{q-1} (\lambda_k + \lambda_q)\, \alpha_k + \lambda_q \alpha_q \neq 0\,, \tag{5.19}$$

i.e. if $\sum_{k=1}^{q} \alpha_k \neq 1$, then (5.18) implies that $w = 0$, which is absurd.

Consequently, (5.15) can hold only if u is given by (5.13), with

$$\sum_{k=1}^{q} \alpha_k = 1\,, \tag{5.20}$$

that is if all the solutions of (3.5) belong to the hyperplane Π of equation (5.28).

Observe now that, repeating the proof given in formulas (5.10) to (5.12), it can be shown that $z_k \neq 0$ ($k = 1, 2, \ldots, q$); by (5.4) it is then obviously possible to choose $\varphi = \tilde{\varphi}$ in such a way that the hypersurface $\psi_{\tilde{\varphi}} = 0$ has, in P_1, a tangent hyperplane which coincides with Π. Hence, bearing in mind that $\psi_{\tilde{\varphi}}$ is a polynomial of degree 2, the hypersurface $\psi_{\bar{\varphi}} = 0$ has in common with Π either only the point P_1, or a set which contains a straight line Λ.

On the other hand, if we assume that the number of solutions is infinite and denote by Γ the set of points of Π which represent these solutions, it is obvious that Γ must have infinite points in common with Λ, *i.e.* that all the second grade hypersurfaces $\psi_{\varphi} = 0$ must contain Λ. This however is absurd, because there would then exist solutions for which (2.4) does not hold.

Hence, also in this case these cannot by any other solutions in a neighbourhood of P_1. This completes the proof of our statement.

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References

[1] R. Temam, *Navier-Stokes equations.* North Holland, 1977.

[2] O. A. Ladyzhenskaya, *The mathematical theory of viscous, incompressible fluids.* Gordon and Breach, 1969.

[3] F. Brezzi - J. Rappaz - P. A. Raviart, *Finite dimensional approximation of nonlinear problems.* Num. Math., 36, 1980, 1-25.

[4] O. A. Ladyzhenskaya, *On modifications of the Navier-Stokes equations for large velocity gradients.* Sem. Inst. Steklov, Leningrad 1968 (in russian).

[5] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, 1969.

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