Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

# Rendiconti Lincei
# Matematica e Applicazioni

Angelo Favini, Roberto Triggiani

## New classes of analytic and Gevrey semigroups and applications

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9*, Vol. **4** (1993), n.1, p. 29–34.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1993_9_4_1_29_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1993.

*Rend. Mat. Acc. Lincei*
*s. 9, v. 4:29-34 (1993)*

**Analisi matematica.** — *New classes of analytic and Gevrey semigroups, and applications.* Nota di Angelo Favini e Roberto Triggiani, presentata (*) dal Socio R. Conti.

Abstract. — We consider the operator $-A + iB$ on a complex Hilbert space, where $A$ is positive self-adjoint and $B$ is self-adjoint, and where, moreover, «$B$ is comparable to $A^\alpha$, $\alpha \geq 1$», in a technical sense. Two applications are given.

Key words: Analytic semigroups; Gevrey class semigroups; Optimal control problems; Cauchy problems.

Riassunto. — *Nuove classi di semigruppi analitici e di tipo Gevrey, e applicazioni*. Si considera l'operatore $-A + iB$ in uno spazio di Hilbert complesso, dove $A$ è autoaggiunto positivo e $B$ è autoaggiunto, con «$B$ comparabile con $A^\alpha$, $\alpha \geq 1$». Vengono date due applicazioni.

## 1. Introduction. Statement of main results

Throughout this *Note*, $X$ is a complex Hilbert space with inner product $(\,,)$ and norm $\| \ \|$. We consider the (linear) operator

$$(1.1) \qquad \mathcal{G} = -A + iB, \qquad X \supset \mathcal{D}(\mathcal{G}) = \mathcal{D}(A) \cap \mathcal{D}(B) \rightarrow X$$

under the following standing assumptions: $\mathcal{D}(\mathcal{G})$ dense on $X$ and

(H.1) $A$: $X \supset \mathcal{D}(A) \rightarrow X$ is a strictly positive self-adjoint operator on $X$;

(H.2) $B$: $X \supset \mathcal{D}(B) \rightarrow X$ is a self-adjoint operator on $X$.

In addition, we shall assume a condition of comparison between $A$ and $B$, which in particular includes the case where «$B$ is comparable to $A^\alpha$, $\alpha \geq 1$», in the technical sense explained below. We let $B^+$ be the (unique) non-negative self-adjoint square root of the non-negative operator $B^2$: $B^+ = (B^2)^{1/2}$, so that the fractional powers $(B^+)^s$, $0 \leq s \leq 1$, are well-defined by the self-adjoint calculus. We shall distinguish two cases: $\alpha = 1$ and $\alpha > 1$.

1.1 *Case where $e^{\mathcal{G}t}$ is an analytic semi-group: $\alpha = 1$.* In this Subsection we assume further that

$$(1.2a) \quad \text{(H.3A)} \qquad \mathcal{D}(A^{1/2}) \subset \mathcal{D}((B^+)^{1/2}),$$

*equivalently* rewritten as

$$(1.2b) \qquad (B^+)^{1/2} A^{-1/2} \in \mathcal{L}(X) \Leftrightarrow \|(B^+)^{1/2} x\| \leq \sqrt{\rho_2}\, \|A^{1/2} x\| \qquad x \in \mathcal{D}(A^{1/2}),$$

(by the closed graph theorem) $\Updownarrow$

$$(1.2c) \qquad (B^+ x, x) \leq \rho_2 (Ax, x),$$

(*) Nella seduta del 12 giugno 1992.

for a constant $0 < \sqrt{\rho_2} = \|(B^+)^{1/2} A^{-1/2}\| < \infty$. Then one readily sees that the operator $\mathcal{G}$ in (1.1) with domain $\mathcal{D}(\mathcal{G})$ is the generator of a strongly continuous (s.c.) contraction semigroup $e^{\mathcal{G}t}$ on $X$. The main distinctive feature of this case is that $e^{\mathcal{G}t}$ is also analytic.

THEOREM 1.1. Under assumptions (H.1), (H.2), (H.3A) = (1.2), the operator-$\mathcal{G}$ is $m$-accretive (or $\mathcal{G}$ is $m$-dissipative) and sectorial with vertex 0 and semi-angle $< \pi/2$. Thus, the s.c. contraction semi-group $e^{\mathcal{G}t}$ is also analytic (holomorphic) on $X$. ∎

REMARK 1.1. If $\mathcal{D}(A) \subset \mathcal{D}(B) \Leftrightarrow B^2 \le \rho_2^2 A^2 \Leftrightarrow \|Bx\| \le \rho_2 \|Ax\|$, $x \in \mathcal{D}(A)$, Lowner's Theorem yields a fortiori (H.3A) = (1.2). Notice that, in this case, $B$ is $A$-bounded with an A-bound which is allowed to be arbitrarily large. Thus, standard perturbation theory for analytic semigroup [4, 7] does not cover this case. ∎

1.2 *Case where $e^{\mathcal{G}t}$ is a Gevrey class semigroup*: $\alpha > 1$. In this Subsection, in place of hypothesis (H.3A) = (1.2), we shall assume the following condition

(1.3$a$) (H.3G) $$\mathcal{D}(A^{1/2}) = \mathcal{D}((B^+)^{1/2\alpha}), \qquad \alpha > 1;$$

which setting $\theta = 1/\alpha < 1$, can be *equivalently* rewritten as

(1.3$b$) $$(B^+)^{\theta/2} A^{-1/2} \in \mathcal{L}(X) \quad \text{and} \quad A^{1/2}((B^+)^{-\theta/2}) \in \mathcal{L}(X);$$

*equivalently*

(1.3$c$) $$\sqrt{\rho_1}\,\|(B^+)^{\theta/2}x\| \le \|A^{1/2}x\| \le \sqrt{\rho_2}\,\|(B^+)^{\theta/2}x\|, \quad 0<\rho_1<\rho_2<\infty\,,\ x\in\mathcal{D}((B^+)^{\theta/2});$$

*equivalently*

(1.3$d$) $$\rho_1((B^+)^{\theta}x, x) \le (Ax, x) \le \rho_2((B^+)^{\theta}x, x), \qquad x \in \mathcal{D}((B^+)^{\theta/2});$$

when $B^+$ is positive, this is equivalently rewritten as (setting $x = (B^+)^{-\theta/2}y$)

(1.3$e$) $$\rho_1\|y\|^2 \le (S_\theta y, y) \le \rho_2\|y\|^2 \qquad y \in X,$$

where $S_\theta$ is the bounded, boundedly invertible, self-adjoint operator

(1.4) $$S_\theta \equiv (B^+)^{-\theta/2} A (B^+)^{-\theta/2}.$$

Again, the operator $\mathcal{G}$ in (1.1) with domain $\mathcal{D}(\mathcal{G})$ is the generator of a s.c. contraction semigroup $e^{\mathcal{G}t}$ on $X$. The main distinctive feature of this case is that $e^{\mathcal{G}t}$ is of Gevrey class $> \alpha$. To state the Theorem properly, we introduce (with $B^+$ positive, without loss of generality for the Theorem which follows) the spaces $[\mathcal{D}((B^+)^{\theta/2})]'$ and $\mathcal{D}((B^+)^{1-\theta/2})$, $\theta = 1/\alpha < 1$, the first being the dual of the second with respect to the $X$-topology, with norms

(1.5) $$\|y\|_{[\mathcal{D}((B^+)^{\theta/2})]'} = \|(B^+)^{-\theta/2}y\|_X; \qquad \|y\|_{\mathcal{D}((B^+)^{1-\theta/2})} = \|(B^+)^{1-\theta/2}y\|_X,$$

as well as the intermediate spaces in between

(1.6) $$X_{\theta,s} \equiv [\mathcal{D}((B^+)^{1-\theta/2}), [\mathcal{D}((B^+)^{\theta/2})]']_s \qquad 0 < s < 1.$$

We next recall that a s.c. semigroup $T(t)$ on $X$ is of Gevrey class $\delta > 1$ for $t > t_0$ if $T(t)$ is infinitely differentiable for $t \in (t_0, \infty)$ and for any compact $\mathcal{K} \subset (t_0, \infty)$ and each $k > 0$, there exists a constant $c = c(k, \mathcal{K})$ such that $\|T^{(n)}(t)\| \le ck^n (n!)^\delta$ for all $t \in \mathcal{K}$ and

$n = 0, 1, \ldots$. See [9] where a theory of Gevrey class semigroups is developed that parallels and extends the theory of differentiable semigroups in [7].

THEOREM 1.2. Under assumptions (H.1), (H.2), (H.3G) = (1.3), the resolvent operator $R(\lambda, \mathcal{G}) = (\lambda I - \mathcal{G})^{-1}$ of $\mathcal{G}$ satisfies the following uniform estimates on the imaginary axis $\lambda = i\tau$, $\tau \in \mathbf{R}$

$$(1.7) \qquad |\tau|^{1/\alpha}[\|R(i\tau, \mathcal{G})\|_{\mathcal{L}([\mathcal{D}((B^+)^{\theta/2})]')} + \|R(i\tau, \mathcal{G})\|_{\mathcal{L}(\mathcal{D}((B^+)^{1-\theta/2}))}] \le \text{Const}_\theta ,$$

and by interpolation a similar estimate holds true on each of the spaces $X_{\theta, s}$ defined in (1.6). Thus [9] $e^{\mathcal{G}t}$ extends/restricts as a s.c. semigroup of Gevrey class $> \alpha$ on each such space, in particular on $X = X_{\theta, s = 1 - \theta/2}$. ∎

REMARK 1.2. If, with $\alpha > 1$: $\mathcal{D}(A^\alpha) = \mathcal{D}(B) \Leftrightarrow \rho_1^2 B^2 \le A^{2\alpha} \le \rho_2^2 B^2$, then Lowner's theorem implies (1.3). ∎

REMARK 1.3. If $A$ and $B$ commute than the right hand side inequality $A \le \rho_2 (B^+)^\theta$ in (1.3$d$) may be dropped from the assumptions of Theorem 1.2.

In Section 2, we shall give a sketch of the proof of the more demanding Theorem 1.2.

## 2. A SKETCH OF THE PROOF OF THEOREM 1.2

It suffices to show the Theorem when $B$ is positive, for the general case of $B$ only self-adjoint can be readily reduced to this case via the operator $B^+$.

First, one shows the Theorem in the (simplest) canonical case where (essentially) $A = B^\theta$, $\theta = 1/\alpha < 1$, more precisely to the specialization of $\mathcal{G}$ in (1.1) given by

$$(2.1) \qquad \mathcal{B}_{\rho\theta} = -\rho B^\theta \pm iB \qquad \mathcal{D}(\mathcal{B}_{\rho\theta}) = \mathcal{D}(B); \quad 0 \le \theta < 1; \quad \rho > 0 .$$

Next, one consider $\mathcal{G}$ in (1.1) as perturbation of $\mathcal{B}_{\rho\theta}$ in (2.1) so that

$$(2.2) \qquad R(i\tau, \mathcal{G}) = R(i\tau, \mathcal{B}_{\rho\theta})[I + (B - \rho A)\mathcal{B}_{\rho\theta}]^{-1} .$$

A proof of Theorem 1.2 may be then given which in the general case proceeds through the following three step strategy [1]:

($i$) prove the resolvent estimate in (1.7) for $R(i\tau, \mathcal{G})$ on the larger space $[\mathcal{D}((B^+)^{\theta/2})]'$;

($ii$) prove the resolvent estimate in (1.7) for $R(i\tau, \mathcal{G})$ on the smaller space $\mathcal{D}((B^+)^{1-\theta/2})$;

($iii$) interpolate between ($i$) and ($ii$).

Once the resolvent estimates are proved, the claim of Gevrey class follows from [8]. *The key step is to show estimate* ($i$). To this end, the following result plays a crucial role.

LEMMA 2.1. Under assumptions (H.1), (H.2) and (H.3G) = (1.3), the operator

$$(2.3) \qquad I + (S_\theta - \rho I) B^\theta R(i\tau, \mathcal{B}_{\rho\theta})$$

is an isomorphism on $X$, uniformly in $\tau \in \boldsymbol{R}$, where $\rho$ is chosen as to satisfy $\rho < \rho_1 < \rho_2$ (see (1.3)); in particular

$$\|[I + (S_\theta - \rho I) B^\theta R(i\tau, \mathcal{B}_{\rho\theta})]^{-1}\|_{\mathcal{L}(X)} \leq \text{const}, \quad \forall \tau \in \boldsymbol{R}. \qquad \blacksquare \tag{2.4}$$

## 3. Applications to optimal control problems with unbounded control operators

In this Section we present an application of Theorems 1.1 and 1.2 to feedback dynamics such as they arise in the theory of optimal control problems with quadratic cost over an infinite time horizon and related algebraic Riccati (operator) equations [6]. In this reference, the original theory as in [3] was extended to include the case, important in some applications to boundary control problems for partial differential equations, where the map from the input (control) function to the solution is *unbounded*. Consider the abstract equation

$$\dot{x}(t) = Lx(t) + Mu(t); \qquad x(0) = x_0 \in X, \tag{3.1}$$

where $L$ is the generator of a s.c. semigroup on the Hilbert space $X$ and the (control) operator $M$ is continuous from another Hilbert space $U$ to $[\mathcal{D}(L^*)]'$, the dual space of $\mathcal{D}(L^*)$ with respect to the $X$-topology, where $L^*$ is the adjoint of $L$ in $X$. It is shown in [6] that (under mild assumptions which include the cases when $L$ is either self-adjoint or skew-adjoint), the unique solution $\{u^0, y^0\}$ of the optimal control problem associated with (3.1) which minimizes the quadratic cost

$$J(u, x) = \int_0^\infty \{\|Rx(t)\|_X^2 + \|u(t)\|_U^2\}\, dt \tag{3.2}$$

$R \in \mathcal{L}(X)$, $R^*R \geq cI$, $c > 0$, is given by

$$u^0(t; x_0) = -MM^*Px^0(t; x_0) \tag{3.3}$$

where $P$ is the (unique) solution of the corresponding Algebraic Riccati Equation (A.R.E.)

$$L^*P + PL + R^*R = PMM^*P \tag{3.4}$$

(in a technical sense explained in [6]). The feedback dynamics is then

$$\dot{x} = L_F x; \qquad L_F = L - MM^*P. \tag{3.5}$$

Example 1. Consider the «abstract hyperbolic» case where

$$L = iS; \qquad M = iS^\theta, \qquad 0 \leq \theta \leq 1; \qquad R = I \tag{3.6}$$

and $S$ is a positive self-adjoint operator. One then verifies that the operator $P = S^{-\theta}$ satisfies the A.R.E. (3.4). (Notice that $P^{-1} = S^\theta$ is an *unbounded* operator, a *feature* of the general theory in [6]. By contrast, unboundedness of $P^{-1}$ cannot occur within the more regular theory of [3] where the input $\to$ solution map: $u \to y$ is continuous from $L_2(0, T; U) \to C([0, T]; X)$. Thus, the feedback dynamical operator $L_F$ in (3.5) is

$$L_F = -S^\theta + iS. \tag{3.7}$$

Thus $L_F$ is of the general form as in (2.1) with $\rho = 1$, $S = B$. According to Theorems 1.1 and 1.2, we conclude that the original free dynamics described by the unitary group $e^{iLt}$ is transformed by the Riccati feedback operator into the optimal dynamics which is described by the semigroup $e^{L_F t}$: this is analytic if $\theta = 1$, and of Gevrey class $> 1/\theta$, if $0 < \theta < 1$ (and a group if $\theta = 0$). Thus, the feedback action due to the Riccati operator has a smoothing or regularizing effect on the original «hyperbolic» dynamics.

## 4. Applications to Cauchy problems

In this final Section we apply some results of the first section to the Cauchy problem

$$\text{(P)} \qquad \begin{cases} u'(t) = \mathcal{G} u(t) + f(t), & 0 < t \le T, \\ u(0) = u_0, \end{cases}$$

where the operator $\mathcal{G}$ satisfies assumption (H.3G) = (1.3). Moreover, $f$ is a given continuous function from $[0, T]$ into $X$ and $u_0 \in X$. According to [2], [5, p. 54], a *classical* or *weakened* solution of problem (P) is a function $u = u(t)$ which is continuous from $[0, T]$ into $X$, is strongly continuously differentiable on $(0, T]$, satisfies $u(t) \in \mathcal{D}(\mathcal{G})$ for $0 < t \le T$, and is such that (P) holds.

If one assumes (H.3G) = (1.3) with $\alpha > 1$, since $\mathcal{G}$ generates a strongly continuous semigroup, it is not difficult to recognize, by Taylor's expansion of the resolvent, starting from (1.7), that the estimate

$$|\lambda|^{1/\alpha} \|R(\lambda, \mathcal{G})\|_{\mathcal{L}(X)} \le \text{const} \tag{4.1}$$

holds true for any complex number $\lambda$ with $\operatorname{Re}\lambda \ge 0$, $|\lambda|$ large. Hence, (see [5, p. 135 and p. 140]), problem (P) has a unique classical solution $u$ provided that $u_0 \in \mathcal{D}(\mathcal{G}) = \mathcal{D}(B) = \mathcal{D}(A^\alpha)$ and: either $f(t) \in \mathcal{D}(B)$, $0 \le t \le T$, $Bf(t)$ being continuous, or else $f$ is continuously differentiable. However, less restrictive assumptions can be made when the number $\alpha$ is suitably restricted. To this end, we observe that estimate (4.1) remains true for all $\lambda$ in the region

$$\Sigma = \{\lambda \in C\colon \operatorname{Re}\lambda \ge -c(1 + |\operatorname{Im}\lambda|)^{1/\alpha}\}, \qquad c > 0, \tag{4.2}$$

of the resolvent set of $\mathcal{G}$ (see [2]). Hence, we are allowed to apply [2] under the assumption $1 < \alpha < 3/2$ and deduce the following

Theorem 4.1. Under assumptions (H.1), (H.2), (H.3G), let $1 < \alpha < 3/2$. Then, given $2(\alpha - 1) < \gamma \le 1$, for any $u_0 \in \mathcal{D}(B)$ and all $f \in C^\gamma([0, T]; X)$ problem (P) has a unique classical solution. ■

We also quote [5, 8] for similar results. A further existence result for (P) can be obtained by noticing that in our case the resolvent $R(\lambda, \mathcal{G})$ satisfies in fact the stronger condition

$$\|R(\lambda, \mathcal{G})\|_{\mathcal{L}(X)} \leq M(1 + \operatorname{Re}\lambda + |\operatorname{Im}\lambda|^{1/\alpha})^{-1}, \qquad \operatorname{Re}\lambda \geq 0. \tag{4.3}$$

Hence, in view of [5, Theorem 6.10, p. 142], we obtain

THEOREM 4.2. Assume (H.1), (H.2), (H.3G), $1 < \alpha < 3/2$. If $f$ is strongly continuous from $[0, T]$ into $\mathcal{D}(A^{\omega})$, with $\omega > 2(\alpha - 1)$ and $u_0 \in \mathcal{D}(A^{\gamma})$, $\gamma > 1$, then (P) has a unique classical solution. ■

Notice that $f$ satisfies a time regularity assumption in Theorem 4.1 and a space regularity assumption in Theorem 4.2.

## REFERENCES

[1] S. CHEN - R. TRIGGIANI, *Proof of two conjectures by G. Chen and D. L. Russell on structural damping for elastic systems.* Proceedings of Conference on *Approximation and Optimization* held at the University of Havana (Cuba, January 1987). Lecture notes in mathematics, n. 1354, Springer-Verlag, 1988, 234-256.

[2] A. FAVINI - A. YAGI, *Multivalued linear operator and degenerate evolution equations.* Ann. Mat. Pura Appl., to appear.

[3] F. FLANDOLI - I. LASIECKA - R. TRIGGIANI, *Algebraic Riccati equations with non-smoothing observation arising in hyperbolic and Euler-Bernoulli equations.* Ann. Mat. Pura Appl., vol. 153, 1988, 307-382.

[4] T. KATO, *Perturbation theory for linear operators.* Springer-Verlag, 1966.

[5] S. G. KREIN, *Linear differential equations in Banach space.* Translations of Math. Monographs, American Math. Soc., vol. 29, 1971.

[6] I. LASIECKA - R. TRIGGIANI, *Riccati equations arising from systems with unbounded input-solution operator: applications to boundary control problems for wave and plate problems.* J. of Nonlinear Analysis, to appear.

[7] A. PAZY, *Semigroups of operators and applications to partial differential equations.* Springer-Verlag, New York 1983.

[8] K. TAIRA, *The theory of semigroups with weak singularity and its applications to partial differential equations.* Tsakuba J. Math., 13, 1989, 513-562.

[9] S. TAYLOR, *Gevrey class semigroups.* Ph. D. thesis, School of Mathematics, University of Minnesota, 1989, Chapter 1.

A. Favini: Dipartimento di Matematica
Università degli Studi di Bologna
Piazza Porta S. Donato, 5 - 40127 BOLOGNA

R. Triggiani: Department of Applied Mathematics
Thornton Hall - University of Virginia
CHARLOTTESVILLE, VA 22903 (U.S.A.)