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Quaternionic-like structures on a manifold: Note II. Automorphism groups and their interrelations

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Geometria differenziale. — *Quaternionic-like structures on a manifold: Note II. Automorphism groups and their interrelations.* Nota (*) di DMITRI V. ALEKSEEVSKY e STEFANO MARCHIAFAVA, presentata dal Socio E. Martinelli.

ABSTRACT. — We consider different types of quaternionic-like structures. The interrelations between automorphism groups of the subordinated structures and of some admissible connections are studied. A characterization of automorphisms of a quaternionic structure as some kind of projective transformations is given. General results on harmonicity of an automorphism of some G-structure are obtained and applied to the case of an almost Hermitian quaternionic structure. Different noteworthy transformations groups of quaternionic Kähler or hyperKähler manifolds and their interrelations are studied. In particular, new characterizations of quaternionic Kähler manifolds that admit a quaternionic automorphism φ different from an isometry are given. This *Note* follows a *Note I* with the same general title, published in these *Rendiconti*, [2], and it is preliminary to a wider memoir just mentioned in previous one.

KEY WORDS: G-structures; Quaternionic structures; Automorphism groups of G-structures; Quaternionic transformations; Harmonic automorphisms.

RIASSUNTO. — *Strutture di tipo quaternionale su una varietà: Nota II. Gruppi di automorfismi e loro interrelazioni.* Si considerano vari tipi di strutture di tipo quaternionale. Si studiano le interrelazioni tra i gruppi di automorfismi delle strutture subordinate e di alcune connessioni ammissibili. Si dà una caratterizzazione degli automorfismi di una struttura quaternionale come tipi di trasformazioni proiettive. Si ottengono risultati generali sulla armonicità degli automorfismi di alcune G-strutture e li si applicano al caso di una struttura quasi Hermitiana quaternionale. Si studiano diversi gruppi notevoli di trasformazioni di varietà Kähleriane quaternionali o iperKähleriane e le loro interrelazioni. In particolare, si danno nuove caratterizzazioni di varietà Kähleriane quaternionali che ammettono un automorfismo quaternionale φ diverso da una isometria. Questa *Nota* segue una *Nota I* con lo stesso titolo generale, pubblicata in questi *Rendiconti*, [2], e precede una più ampia memoria già indicata nella prima.

1. NOTATION, DEFINITIONS AND SOME BASIC RESULTS ABOUT QUATERNIONIC-LIKE STRUCTURES

Let M be a $4n$-dimensional manifold, $n > 1$.

DEFINITIONS. 1) An *almost hypercomplex structure* $H = (J_\alpha)$, $\alpha = 1, 2, 3$, on M is a triple of anticommuting almost complex structures with $J_3 = J_1 J_2$.

2) The 3-dimensional subbundle $Q = \langle H \rangle$ of the bundle of endomorphisms End TM spanned by the three almost complex structures from H is called the *almost quaternionic structure generated by* H.

General *almost quaternionic structure* Q on M is defined as a 3-dimensional subbundle of End TM which is locally generated by an almost hypercomplex structure $H = (J_\alpha)$.

(*) Pervenuta all'Accademia il 3 agosto 1992.

3) A Riemannian metric g on M is called *Hermitian* with respect to an almost hypercomplex structure $H = (J_\alpha)$ (respectively, almost quaternionic structure Q) if $g(J_\alpha X, J_\alpha X) = g(X, X)$, $\forall X \in TM$, $\alpha = 1, 2, 3$ (respectively, $g(JX, JX) = g(X, X)$, $\forall X \in TM$ for any complex structure $J \in Q$).

4) Adding an Hermitian metric g or a volume form vol to an almost hypercomplex structure H (resp., almost quaternionic structure Q) we obtain an *almost Hermitian hypercomplex structure* (H, g) (respectively, an *almost Hermitian quaternionic structure* (Q, g)) or an *almost unimodular hypercomplex structure* (H, vol) (respectively, an *almost unimodular quaternionic structure* (Q, vol)).

Hence we defined six almost quaternionic-like structures $\mathcal{S}$ (shortly, *q-like structures*) on a manifold M: $\mathcal{S} = H, Q, (H, \mathrm{vol}), (Q, \mathrm{vol}), (H, g), (Q, g)$. They may be identified with corresponding G-structures, where respectively $G = GL_n(\boldsymbol{H})$, $Sp_1 \cdot GL_n(\boldsymbol{H})$, $SL_n(\boldsymbol{H})$, $Sp_1 \cdot SL_n(\boldsymbol{H})$, Sp_n, $Sp_1 \cdot Sp_n$.

Let $\mathcal{G}$ be the Lie algebra of one of these groups G. Then there exists a natural decomposition $V \otimes \Lambda^2 V^* = \partial(\mathcal{G} \otimes V^*) \oplus \mathcal{D}(\mathcal{G})$ into the sum of G-modules, where ∂: $V \otimes V^* \otimes V^* \to V \otimes \Lambda^2 V^*$ is the Spencer operator of antisymmetrisation and $\mathcal{D}(\mathcal{G})$ is a complementary submodule to $\partial(\mathcal{G} \otimes V^*)$.

For any q-like structure $\mathcal{S}$ there exists a connection ∇ that preserves $\mathcal{S}$ and whose torsion function takes values in $\mathcal{D}(\mathcal{G})$ and hence coincides with the structure function of the G-structure. The torsion tensor $T^{\mathcal{S}}$ of the connection ∇ doesn't depend on ∇ and is called the *structure tensor* of $\mathcal{S}$. If $\mathcal{S} \neq Q$ then such connection $\nabla = \nabla^{\mathcal{S}}$ is unique and it is called the *canonical connection*. For $\mathcal{S} = Q$ we have an affine space $\mathcal{C}(Q)$ of connections that preserve Q and whose torsion function takes values in $\mathcal{D}(\mathcal{G})$. Any two of these connections are related by

$$\nabla' = \nabla + S^{\xi} \tag{1.1}$$

where ξ is a global 1-form and S^{ξ} has local expression

$$S^{\xi} = \xi \otimes \mathrm{Id} + \mathrm{Id} \otimes \xi - \sum_{\alpha} [(\xi \circ J_\alpha) \otimes J_\alpha + J_\alpha \otimes (\xi \circ J_\alpha)] \tag{1.2}$$

where $H = (J_\alpha)$ is any almost hypercomplex structure which generates Q; see [2].

Definitions. 1) An almost q-like structure $\mathcal{S}$ is called *1-integrable* if there exists a torsionless connection that preserves $\mathcal{S}$.

2) An 1-integrable almost quaternionic (respectively, almost hypercomplex) structure is called *quaternionic* (respectively, *hypercomplex*).

An 1-integrable almost Hermitian quaternionic (resp., almost Hermitian hypercomplex) structure is called *quaternionic Kähler* (resp., *hyperKähler*).

3) A connection ∇ on a manifold M that preserves an almost quaternionic structure Q is called an *almost quaternionic*. A torsionless almost quaternionic connection is called *quaternionic*.

The following diagram indicates the relations between the different q-like structures.

$$
\begin{tikzcd}
\nabla^{H} & H \arrow[l] \arrow[r] & Q \arrow[r] & \mathcal{C}(Q) \\
\nabla^{H,\mathrm{vol}} & (H,\mathrm{vol}) \arrow[l] \arrow[r] \arrow[u] & (Q,\mathrm{vol}) \arrow[r] \arrow[u] & \nabla^{Q,\mathrm{vol}} \\
\nabla^{H,g} & (H,g) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & (Q,g) \arrow[r] \arrow[u] \arrow[d] & \nabla^{Q,g} \\
\nabla^{g} \arrow[rrr, no head, bend right] & g \arrow[l] \arrow[r] & g \arrow[r] & \nabla^{g}
\end{tikzcd}
$$

Diagram 1.

Here an arrow $\mathcal{S} \to \mathcal{S}'$ indicates that an appropriate structure $\mathcal{S}'$ is associated to given structure $\mathcal{S}$. For example, arrow $(H,g) \to (H,\mathrm{vol})$ indicates that a structure (H,g) defines the structure (H,vol), where $\mathrm{vol} = \mathrm{vol}^g$ is the volume form of the metric g on an oriented manifold M. ∇^g is the Levi-Civita connection of the metric g.

From the results of [2] it follows:

THEOREM.

1) *For any almost hypercomplex structure H on a manifold M the canonical connection ∇^H belongs to the space* $\mathcal{C}(\langle H\rangle) \Leftrightarrow T^H = T^{\langle H\rangle}$.

2) *For any volume form* vol: $\nabla^{H,\mathrm{vol}} = \nabla^{\langle H\rangle,\mathrm{vol}} \Leftrightarrow T^H = T^{\langle H\rangle}$.

3) $\nabla^{H,\mathrm{vol}} = \nabla^H \Leftrightarrow V^H \mathrm{vol} = 0 \Leftrightarrow \mathrm{T}^{\mathrm{H}} = \mathrm{T}^{\mathrm{H,vol}}$.

4) $\nabla^{H,g} = \nabla^{\langle H\rangle,g} \Leftrightarrow T^H = T^{\langle H\rangle}$, $\mathrm{div}^H g = 0$ *for any Hermitian metric g, where* $\mathrm{div}^H g$ *is the divergence of g with respect to ∇^H, that is* $(\mathrm{div}^H g)(X) = \mathrm{Tr}[Y \to \to (g^{-1}\nabla^H_Y g)X]$ (Tr *denotes the trace*).

5) $\nabla^{H,g} = \nabla^{H,\mathrm{vol}g} \Leftrightarrow$ *the connection ∇^H is conformal with respect to the metric g, that is* $\nabla^H_X g = \rho(X) g$, $\rho \in \Lambda^1 M$.

6) $\nabla^{Q,g} = \nabla^{Q,\mathrm{vol}g} \Leftrightarrow T^{Q,g} = T^{Q,\mathrm{vol}g} \Leftrightarrow T^{Q,g} = T^Q \Leftrightarrow g^{-1}(\nabla^H_X g) = (1/4(n+1)) \cdot \cdot (S^\omega_X + S^{g\circ X}_{g^{-1}\omega})\, \forall X \in TM$, *where* $H = (J_\alpha)$ *locally generates* Q, $\omega := (n + 1/n)\,\mathrm{div}^H g$.

7) $\nabla^{Q,g} = \nabla^g \Leftrightarrow (Q,g)$ *is a quaternionic Kähler structure*.

8) $\nabla^{H,g} = \nabla^g \Leftrightarrow (H,g)$ *is a hyperKähler structure*.

2. CONDITIONS OF HARMONICITY OF ISOMORPHISMS OF G-STRUCTURES AND THE APPLICATION TO q-LIKE STRUCTURES

2.1. Let (M,∇) and (M',∇') be manifolds with given linear connections. We associate to any diffeomorphism $\varphi: M \to M'$ a *vector valued bilinear form* B^φ on M by

$$B^\varphi := (\varphi^{-1})^* \nabla' - \nabla \equiv \varphi_*^{-1} \circ \nabla'_{\varphi_*\cdot}\, \varphi_* \cdot - \nabla_{\cdot}\, \cdot \tag{2.1}$$

It is symmetric iff the connections $(\varphi^{-1})^* \nabla'$ and ∇ have the same torsion.

Assume now that ∇, ∇' are Levi-Civita connections of metrics g, g'. Then $\varphi_* \circ B^\varphi := \nabla'_{\varphi_*\cdot}\, \varphi_* \cdot - \varphi_* \nabla_{\cdot}\, \cdot$ is the second fundamental form of the map φ (see.[5, pag. 15]). A diffeomorphism φ is called *harmonic* if its *tension vector field* τ vanishes:

$$\tau := \mathrm{Tr}_g (B^{\varphi}) = (g^{jk} (B^{\varphi})^i_{jk}) = 0. \tag{2.2}$$

REMARKS. 1) To define the tension vector field τ we need only a metric g on M and connections ∇, ∇' on M, M' respectively.

2) If $M = M'$ and we assume that $\nabla = \nabla' = \nabla^g$ is the Levi-Civita connection we come to the standard definition of harmonic transformation φ with respect to a metric g.

2.2. Let G be a linear group and π: $P \to M$ be a G-structure, that is a principal G-subbundle of the bundle of coframes $CF(M)$ of M. Note that any transformation φ of M induces an automorphism φ^* of the bundle of coframes $CF(M)$: φ is called an *automorphism of G-structure π if $\varphi^* P = P$*. We denote the group of such automorphisms by $Aut(\pi)$. More generally, a transformation φ of M is called an *isomorphism of G-structure π: $P \to M$ onto G-structure π': $P' \to M$* if $\varphi^* P = P'$.

We may consider a connection ∇ of a G-structure π as a linear connection whose holonomy group $\mathrm{Hol}_x(\nabla)$ in a point x belongs to the group $G_x := \{A \in GL(T_x M) \mid Ap \in P \text{ for } p \in P,\ \pi(p) = x\} \cong G$.

Let π: $P \to M$ be a G-structure with a connection ∇. As before we may associate with an automorphism $\varphi \in \mathrm{Aut}(\pi)$ a tensor field $B^{\varphi} = (\varphi^*)^{-1}\nabla - \nabla$. It is symmetric iff φ preserves the torsion $\mathrm{Tor}(\nabla)$ of ∇.

PROPOSITION. *Let π: $P \to M$ be a G-structure with a connection ∇ and let $\varphi \in \mathrm{Aut}(\pi)$ be an automorphism which preserves the torsion tensor* $\mathrm{Tor}(\nabla)$ *of* ∇. (*It is the case if* $\mathrm{Tor}(\nabla) = 0$). *Then the associated symmetric 2-form B^{φ} in each point $x \in M$ belongs to the first prolongation $\mathfrak{g}_x^{(1)} \cong \mathfrak{g}^{(1)}$ of $\mathfrak{g}_x \cong \mathfrak{g}$. In particular, if $\mathfrak{g}^{(1)} = 0$ any automorphism $\varphi \in \mathrm{Aut}(\pi)$ that preserves* $\mathrm{Tor}(\nabla)$ *preserves also the connection* ∇: $\mathrm{Aut}(\pi, \mathrm{Tor}(\nabla)) \subseteq \mathrm{Aut}(\nabla)$.

REMARK. This proposition has obvious generalization to the case of an isomorphism of two G-structures.

The last statement may be applied to any almost q-like structure different from almost quaternionic one. For an almost quaternionic structure we have the following

COROLLARY. *Let (M, Q) be a manifold with an almost quaternionic structure and let ∇ be an almost quaternionic connection. If an automorphism $\varphi \in \mathrm{Aut}(Q)$ of the structure Q preserves the torsion tensor of ∇, then*

$$B^{\varphi} = S^{\xi} \tag{2.3}$$

where ξ is a globally defined 1-form and S^{ξ} is defined by (1.2). *In particular, it is true for arbitrary automorphism of a quaternionic structure Q if ∇ is quaternionic.*

Let $G \subset GL(V)$ be a linear group of an Euclidean space (V, g).

THEOREM. *Let π: $P \to M$ be a G-structure and g be a Riemannian metric on M whose Levi-Civita connection ∇^g preserves P. For a point $x \in M$ we set $V = T_x M$, $\mathcal{G} = \mathcal{G}_x \subset gl(V)$ and denote by $\mathcal{G}^{(1)} \subset V \otimes V^* \otimes V^*$ the first prolongation of $\mathcal{G}$ and by $\mathrm{Tr}_g \colon \mathcal{G}^{(1)} \to V$, $A^i_{jk} \to (g^{jk} A^i_{jk})$ the natural linear map into $V = T_x M$ defined by contraction with g^{-1}. We have*

1) *If $\mathrm{Tr}_g(\mathcal{G}^{(1)}) = 0$, then any automorphism φ of G-structure π is harmonic with respect to g.*

2) *If Tr_g is a monomorphism, then an automorphism $\varphi \in \mathrm{Aut}(\pi)$ is harmonic with respect to g iff it is affine: $\varphi^* \nabla^g = \nabla^g$.*

REMARK. This Theorem has natural generalization to the case of isomorphism of G-structures.

By applying this Theorem to different classical structures we obtain simple unifying proof of several classical results about harmonicity of some class of immersions of manifolds with these structures: for example, the harmonicity of a holomorphic immersion of a Kähler manifold into another one (see [5]).

By applying Theorem, part 2, and Corollary to the case of quaternionic structure we obtain the following

THEOREM. *Let (Q, g) be a quaternionic Kähler structure on a manifold M and let φ be an automorphism of the quaternionic structure Q. Then φ is harmonic with respect to g iff it preserves ∇^g, that is iff it is affine.*

REMARK. This Theorem has natural generalization to the case of an almost Hermitian quaternionic structure (Q, g) for automorphisms $\varphi \in \mathrm{Aut}(Q)$ which preserve the torsion tensor of the canonical connection $\nabla^{Q,g}$ (see also Remark 1) of n. 2.1).

3. CHARACTERIZATION OF AUTOMORPHISMS OF A QUATERNIONIC STRUCTURE AS PROJECTIVE TRANSFORMATIONS

Now we give a characterization of the automorphism group $\mathrm{Aut}(Q)$ of a quaternionic structure Q, which shows that such a structure can be considered as some kind of projective structure.

DEFINITION. Let (M, Q) be a manifold M with a quaternionic structure Q and let ∇ be a quaternionic connection. A curve $\gamma = \gamma(t)$ of M is called *q-planar* if the quaternionic bundle $\mathcal{H}_\gamma \equiv \{\boldsymbol{R}_{\gamma'} + Q_{\gamma'}\}$ over γ is parallel along γ, that is

$$\nabla_{\gamma'} \gamma' = a_0 \gamma' + \sum_\alpha a_\alpha J_\alpha \gamma'$$

where $\gamma(t) \to \{(J_\alpha)_{\gamma(t)} (\alpha = 1, 2, 3)\}$ is a field over γ of an almost hypercomplex structure that generates Q and the a_i are real functions over γ.

PROPOSITION (Fujimura [6]). *The class of q-planar curves doesn't depend on a particular quaternionic connection ∇.*

REMARK. Note that the geodesics of a quaternionic connection ∇ are q-planar curves.

PROPOSITION. *Locally any q-planar curve of (M, Q) coincides with a geodesic of some quaternionic connection.*

THEOREM. *Let (M, Q) be a manifold with a quaternionic structure Q. Then a transformation φ is quaternionic, that is $\varphi \in \mathrm{Aut}(Q)$, iff φ preserves the class of q-planar curves.*

The proof of Theorem is based

1) on the formula: $\nabla_{\varphi_* \gamma'} \varphi_* \gamma' = \varphi_* (\nabla_{\gamma'} \gamma' + B^{\varphi}(\gamma', \gamma'))$, where φ is any transformation of a manifold M with a linear connection ∇ and γ is any curve in M,

2) on the following

LEMMA. *Let (V, Q), (V', Q') be vector spaces of dimension $4n \geq 8$ with constant quaternionic structures Q, Q'. Then an isomorphism of vector spaces $\varphi\colon V \to V'$ is quaternionic, that is $\varphi^* Q = Q'$, iff it transforms a quaternionic line of V into a quaternionic line of V'.*

Recall that a quaternionic line of (V, Q) is a 4-dimensional Q-invariant subspace of V.

REMARK. The Lemma is true also for homomorphisms $\varphi\colon V \to V'$ under the natural definition of quaternionic map and the condition that $\dim(\mathrm{Im}\,\varphi) \geq 8$.

4. INTERRELATIONS BETWEEN GROUPS OF AUTOMORPHISMS OF DIFFERENT q-LIKE STRUCTURES

Diagram 1 implies the following diagram of interrelations between automorphism groups of different q-like structures.

$$
\begin{tikzcd}
\mathrm{Aut}(\nabla^H) & \mathrm{Aut}(H) \arrow[l, hook'] \arrow[r, hook] & \mathrm{Aut}(Q) & \mathrm{Aut}(\nabla),\ (\nabla \in \mathcal{C}(Q)) \\
\mathrm{Aut}(\nabla^{H,\mathrm{vol}}) & \mathrm{Aut}(H,\mathrm{vol}) \arrow[l, hook'] \arrow[r, hook] \arrow[u, hook] & \mathrm{Aut}(Q,\mathrm{vol}) \arrow[r, hook] \arrow[u, hook] & \mathrm{Aut}(\nabla^{Q,\mathrm{vol}}) \\
\mathrm{Aut}(\nabla^{H,g}) & \mathrm{Aut}(H,g) \arrow[l, hook'] \arrow[r, hook] \arrow[u, hook] \arrow[d, hook] & \mathrm{Aut}(Q,g) \arrow[r, hook] \arrow[u, hook] \arrow[d, hook] & \mathrm{Aut}(\nabla^{Q,g}) \\
\mathrm{Aut}(\nabla^{g}) & \mathrm{Aut}(g) \arrow[l, hook'] \arrow[r, hook] & \mathrm{Aut}(g) \arrow[r, hook] & \mathrm{Aut}(\nabla^{g})
\end{tikzcd}
$$

Diagram 2.

Here $\mathrm{Aut}(\mathcal{S})$ denotes the group of all automorphisms of the structure $\mathcal{S}$. The following Theorem states more precise results about the indicated inclusions.

THEOREM. 1) *Assume that the centralizer $Z = Z_{gl(V)}(\mathrm{Hol}(\nabla^H))$ of the holonomy* *group* $\mathrm{Hol}(\nabla^H)$ *in* $gl(V)$ *has only one ideal isomorphic to the quaternionic Lie algebra* $\{\boldsymbol{R}\,\mathrm{Id} + \sum_\alpha \boldsymbol{R} J_\alpha\}$. *Then* $\mathrm{Aut}(\nabla^H) \subseteq \mathrm{Aut}(\langle H \rangle)$.

2) *If* $\mathrm{Aut}(H)$ (*resp.*, $\mathrm{Aut}(Q)$) *preserves a Riemannian metric g, then it preserves the Hermitian metric*

$$g_0 = (1/4)\Big(g + \sum_\alpha g(J_\alpha \cdot, J_\alpha \cdot)\Big)$$

and hence $\mathrm{Aut}(H) = \mathrm{Aut}(H, g_0) = \mathrm{Aut}(H, \mathrm{vol}^{g_0})$ (*resp.*, $\mathrm{Aut}(Q) = \mathrm{Aut}(Q, g_0) = \mathrm{Aut}(Q, \mathrm{vol}^{g_0})$).

5. AUTOMORPHISMS OF HYPERKÄHLER AND QUATERNIONIC KÄHLER STRUCTURES

For a hyperKähler structure (H, g) (resp., a quaternionic Kähler structure (Q,g)) the canonical connections of appropriate q-like structures coincide: $\nabla^{H,g} = \nabla^{H,\mathrm{vol}^g} = \nabla^g = \nabla^H$ (resp., $\nabla^{Q,g} = \nabla^{Q,\mathrm{vol}^g} = \nabla^g$). Using this, we obtain

THEOREM. *Let* (H, g) (*resp.*, (Q, g)) *be a hyperKähler* (*resp.*, *quaternionic Kähler*) *structure on M. Assume that one of the following conditions holds*:

a) *The holonomy group* $\mathrm{Hol}(\nabla^g)$ *of the canonical connection* ∇^g *preserves no non zero vector.*

b) *The connection* ∇^g *is complete and the holonomy group* $\mathrm{Hol}(\nabla^g)$ *is irreducible. Then* $\mathrm{Aut}(H) = \mathrm{Aut}(H, \mathrm{vol}^g) = \mathrm{Aut}(H, g) \subseteq \mathrm{Aut}(\nabla^g) = \mathrm{Aut}(g)$ (*resp.*, $\mathrm{Aut}(Q, g) = \mathrm{Aut}(Q, \mathrm{vol}^g) \subseteq \mathrm{Aut}(\nabla^g) = \mathrm{Aut}(g)$).

Indeed, under the assumptions of the Theorem $\mathrm{Aut}(\nabla^g) = \mathrm{Aut}(g)$. Using it we have, for example, $\mathrm{Aut}(H) \subseteq \mathrm{Aut}(\nabla^H) = \mathrm{Aut}(\nabla^g) = \mathrm{Aut}(g)$. Together with obvious inclusion $\mathrm{Aut}(H, g) \subseteq \mathrm{Aut}(g)$ this gives first relations. A similar argument establishes the second ones.

REMARK. In general, inclusion $\mathrm{Aut}(H, g) \subseteq \mathrm{Aut}(g)$ is proper. The example is given by the complex Fermat surface with Calabi-Yau metric (see [1]).

To study this inclusion we need the following

DEFINITION. Two hypercomplex structures $H = (J_\alpha)$, $H' = (J'_\alpha)$ that are related by a constant orthogonal matrix $A = (A_\alpha^\beta) \in SO_3$:

$$J'_\alpha = \sum_\beta A_\alpha^\beta J_\beta \qquad (\alpha = 1, 2, 3)$$

are called *equivalent*.

LEMMA. *Let* (H, g) *be a hyperKähler structure on a manifold M such that the connected holonomy group of* ∇^g *is irreducible. Then the Levi-Civita connection* ∇^g *determines the hypercomplex structure H up to an equivalence.*

As a consequence, we obtain the following 3-dimensional linear representation R

of the isometry group $\mathrm{Aut}(g)$: $\mathrm{Aut}(g) \ni \varphi \mapsto R_\varphi \in SO_3$, where

$$\varphi^* J_\alpha = \sum_\beta (R_\alpha^\beta) J_\beta \qquad (\alpha = 1, 2, 3).$$

As a Corollary we have the following

THEOREM. *Let (H, g) be an hyperKähler structure on M. Assume that the connected isometry group* $\mathrm{Aut}_0(g)$ *doesn't preserve any complex structure* $J = a_1 J_1 + a_2 J_2 + a_3 J_3$, $H = (J_\alpha)$, $a_i = \mathrm{const}$. *Then the Lie algebra of Killing vector fields admits a decomposition into semidirect sum*

$$\mathrm{aut}(g) = so_3 + \mathcal{K} \tag{5.1}$$

where $\mathcal{K} = \mathrm{Ker}\, dR$ *is an ideal.*

If the group $\mathrm{Aut}_0(g)$ *is compact then there exists decomposition* (5.1) *into the direct sum.*

THEOREM. *Let (Q, g) be a quaternionic Kähler structure on M. Suppose that the scalar curvature K of the metric g doesn't vanish. Then* $\mathrm{Aut}(g) = \mathrm{Aut}(Q, g)$.

Actually in case $K \neq 0$ the Lie algebra $\mathcal{G}_x$ of $\mathrm{Hol}_x(\nabla^g)$ in a point $x \in M$ is irreducible and has canonical decomposition $\mathcal{G}_x = Q_x \oplus \mathcal{G}'_x$ where $\mathcal{G}'_x$ is the centralizer of the quaternionic structure Q_x in $\mathrm{so}(T_x M)$. It implies that any isometry φ preserves the quaternionic structure Q: $\varphi^*(Q_x) = Q_{\varphi(x)}$.

In general, the group $\mathrm{Aut}(Q)$ doesn't preserve any quaternionic connection. An example is the quaternionic projective space $\boldsymbol{HP}^n$.

We state the following

PROBLEM. Let Q be a quaternionic structure on a manifold. Under which conditions the group $\mathrm{Aut}(Q)$ is affine, that is it preserves a quaternionic connection?

CONJECTURE. Let (M, Q, g) be a compact quaternionic Kähler manifold with the irreducible connected holonomy group $\mathrm{Hol}_0(\nabla^g)$ different from quaternionic projective space. Then $\mathrm{Aut}(Q) = \mathrm{Aut}(g)$.

We state some partial result about this problem.

THEOREM. *Let ∇, ∇' be two quaternionic connections which have the same Ricci tensor. Suppose that the holonomy group of ∇ is irreducible. Then $\nabla' = \nabla$.*

The Theorem implies

COROLLARY. *Assume that a manifold M with a quaternionic structure Q admits a Ricci-flat quaternionic connection ∇ and the holonomy group* $\mathrm{Hol}(\nabla)$ *be irreducible. Then such connection is unique and* $\mathrm{Aut}(Q) \subseteq \mathrm{Aut}(\nabla)$.

COROLLARY. *Let (Q, g) be a quaternionic Kähler manifold. Assume that the Levi-Civita connection ∇^g is Ricci-flat and has irreducible connected holonomy group. Then* $\mathrm{Aut}_0(Q) = \mathrm{Aut}_0(g)$ *where* 0 *indicates the connected component.*

THEOREM (Piccinni [9]). *Let (M, Q, g) be a compact quaternionic Kähler manifold with negative scalar curvature. Then* $\mathrm{Aut}_0(Q) = \mathrm{Aut}_0(g)$ $(= \{1\})$.

THEOREM. *Let (M, Q, g) be a locally symmetric quaternionic Kähler manifold which is not locally isometric to* $\boldsymbol{HP}^n$ or $\boldsymbol{H}^n$. *Then* $\mathrm{Aut}(Q) = \mathrm{Aut}(g)$.

Y. S. Poon and S. Salamon recently [10] proved that any compact 8-dimensional quaternionic Kähler manifold with positive scalar curvature is symmetric. Using all these results we establish a weakened version of the conjecture for 8-dimensional manifolds:

THEOREM. *Let (Q, g) be a quaternionic Kähler structure on an 8-dimensional manifold M. In the case of zero scalar curvature we assume also that* $\mathrm{Hol}_0(\nabla^g)$ *is irreducible. If (M, Q, g) is different from the projective plane $\boldsymbol{HP}^2$ with the standard quaternionic Kähler structure, then* $\mathrm{Aut}_0(Q) = \mathrm{Aut}_0(g)$.

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