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A stable method for the inversion of the Fourier transform in $\mathbb{R}^N$

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Trasformazioni integrali. — *A stable method for the inversion of the Fourier transform in* $\boldsymbol{R}^N$. Nota di LEONEDE DE MICHELE e DELFINA ROUX, presentata (*) dal Socio L. Amerio.

ABSTRACT. — A general method is given for recovering a function $f\colon \boldsymbol{R}^N \to \boldsymbol{C}$, $N \geq 1$, knowing only an approximation of its Fourier transform.

KEY WORDS: Fourier transform; Inversion; Well posed.

RIASSUNTO. — *Un metodo stabile per l'inversione della trasformata di Fourier in* $\boldsymbol{R}^N$. È dato un metodo generale per ricostruire una funzione $f\colon \boldsymbol{R}^N \to \boldsymbol{C}$, $N \geq 1$ conoscendo solo un'approssimazione della sua trasformata di Fourier.

1. – In some previous papers [1-3, 5] stable inversion methods for multiple Fourier series were suggested and analized; moreover the effectiveness of the methods was discussed in [6, 7, 4]. In this paper we deal with the non compact case. The basic ideas are the same, but some technical difficulties arise from the lack of inclusion relations between the various L^p spaces. Moreover minor formal problems are due to the not satisfactory representation of Fourier transform for L^p functions with $p > 2$.

The motivations of the method are the same as in the compact case and can be found in some details in [8].

Notations and some preliminaries are contained in §2; the basic theorems are stated in §3, and as applications in §4 are given some a priori estimates for suitable classes of functions.

2. – Let us first introduce some notations.

If $N \geq 1$ let

$$B_p = \begin{cases} L^p(\boldsymbol{R}^N) & \text{if } 1 \leq p \leq 2 \\ L^p(\boldsymbol{R}^N) \cap L^2(\boldsymbol{R}^N) & \text{if } 2 < p < +\infty \\ C_0(\boldsymbol{R}^N) \cap L^2(\boldsymbol{R}^N) & \text{if } p = +\infty \end{cases}$$

with the usual L^p norm.

If $f \in B_p\,(1 \leq p \leq +\infty)$, we denote with $\hat{f}$ its usual Fourier transform, $\hat{f}(x) = \int_{\boldsymbol{R}^N} f(t)\, e^{-2\pi i x \cdot t}$, (where $x \cdot t$ is usual inner product).

Through the paper G will be a real function in $L^1(\boldsymbol{R}^N) \cap L^2(\boldsymbol{R}^N)$ with $\hat{G}(0) = 1$. For every $\sigma > 0$ let us set $G_\sigma(x) = \sigma^{-N} G(x/\sigma)$ and for every $\tau > 0$

$$R_{\sigma,\tau}(\lambda) \sim (\lambda \cdot \hat{G}_\sigma)^\vee \chi_\tau$$

(*) Nella seduta dell'11 novembre 1992.

where $\lambda \in L^q$, $2 \leqslant q \leqslant +\infty$, $(\ )^\vee$ is the inverse Fourier transform and χ_τ is the characteristic function of the interval $[-\tau/2, \tau/2]^N$.

Finally let

$$R_\sigma(\lambda) \sim (\lambda \cdot \hat{G}_\sigma)^\vee .$$

Now we relate λ and $\hat{G}_\sigma$ in such a way that the formal definitions of $R_{\sigma,\tau}$ and R_σ give us correctly defined functions.

PROPOSITION 1. *If $\lambda \in L^2(\mathbf{R}^N)$, then for every $\sigma > 0$ $R_\sigma(\lambda) \in L^2(\mathbf{R}^N) \cap C_0(\mathbf{R}^N)$ and for every p, $2 \leqslant p \leqslant +\infty$ we have*

$$\|R_\sigma \lambda\|_p \leqslant a_p \sigma^{-N|1-2/p|/2} \|\lambda\|_2 \tag{2.1}$$

where

$$a_p = \|G\|_1^{1-|1-2/p|} \cdot \|G\|_2^{|1-2/p|} . \tag{2.2}$$

PROOF. Indeed $(\lambda \cdot \hat{G}_\sigma)^\vee \in C_0$ because $\lambda \cdot \hat{G}_\sigma \in L^1$. Moreover

$$G \in L^1 \Rightarrow \hat{G}_\sigma \in C_0 \Rightarrow (\lambda \cdot \hat{G}_\sigma)^\vee \in L^2$$

then $(\lambda \cdot \hat{G}_\sigma)^\vee \in L^p$ $\forall p$, $2 \leqslant p \leqslant +\infty$.

Since

$$\|R_\sigma \lambda\|_2 = \|\lambda \cdot \hat{G}_\sigma\|_2 \leqslant \|\lambda\|_2 \cdot \|\hat{G}_\sigma\|_\infty \leqslant \|\lambda\|_2 \cdot \|G\|_1 \tag{2.3}$$

and $\|R_\sigma \lambda\|_\infty \leqslant \|\lambda \cdot \hat{G}_\sigma\|_1 \leqslant \|\lambda\|_2 \cdot \|G_\sigma\|_2 = \sigma^{-N/2} \|\lambda\|_2 \|G\|_2$ by interpolation (2.1) follows.

PROPOSITION 2. *Let $1 \leqslant p < 2$ and $\hat{G} \in L^p(\mathbf{R}^N)$. If $\lambda \in L^q(\mathbf{R}^n)$, $(1/p + 1/q = 1)$, then for every $\sigma > 0$, $R_\sigma \lambda \in C_0(\mathbf{R}^N)$ and for every $\tau > 0$ we have*

$$\|R_{\sigma,\tau} \lambda\|_p \leqslant a_p (\tau/\sigma)^{N|1-2/p|/2} \|\lambda\|_q \tag{2.4}$$

where a_p is given by (2.2).

PROOF. Since $\lambda \cdot \hat{G}_\sigma \in L^1$, then $(\lambda \cdot \hat{G}_\sigma)^\vee \in C_0$ and $R_{\sigma,\tau}\lambda \in L^p$ for every $p \geqslant 1$.

From $\|R_{\sigma,\tau}\lambda\|_1 \leqslant \|(\lambda \cdot \hat{G}_\sigma)^\vee\|_2 \cdot \|\chi_\tau\|_2 \leqslant \|\lambda\|_\infty \cdot \|\hat{G}_\sigma\|_2 \cdot \tau^{N/2} = (\tau/\sigma)^{N/2} \|G\|_2 \|\lambda\|_\infty$ and (2.3), by interpolation we have (2.4).

3. – THEOREM 1. *Let $f \in B_p$, $p \geqslant 2$, $\sigma = \sigma(\delta)$: $\mathbf{R}^+ \to \mathbf{R}^+$ such that as $\delta \to 0$*

$$\sigma(\delta) \to 0, \quad \textit{and} \quad \delta\sigma(\delta)^{-N|1-2/p|/2} \to 0 . \tag{3.1}$$

Then for every $\varepsilon > 0$ there exists $\delta_0 = \delta_0(\varepsilon, f)$ such that if $\delta < \delta_0$, for every $\lambda \in L^2(\mathbf{R}^N)$ with $\|\lambda - \hat{f}\|_2 < \delta$ we have $\|f - R_{\sigma(\delta)} \lambda\|_p < \varepsilon$.

PROOF. By Prop. 1, for every $\varepsilon > 0$ and $\delta < \delta_1(\varepsilon)$

$$\|\lambda - \hat{f}\|_2 < \delta \Rightarrow \|R_{\sigma(\delta)}(\lambda - \hat{f})\|_p < \varepsilon/2 .$$

On the other hand, since $\hat{f}$ and $\hat{G}_\sigma$ are in L^2, then

$$\|f - R_\sigma \hat{f}\|_p = \|f - f * G_\sigma\|_p .$$

Since $\{G_\sigma\}$ is an approximate unit, if $\delta < \delta_2(\varepsilon, f)$ we have

$$\|f - f * G_\sigma\|_p < \varepsilon/2.$$

Then if $\delta_0 = \min(\delta_1, \delta_2)$ the theorem follows.

The situation in the case $1 \leqslant p < 2$ is different. We have not to restrict ourselves to some subclass of L^p but as usual we have to introduce some cut functions $\{\chi_\tau\}_{\tau>0}$ in order to have an approximation of f in L^p for every λ in L^q. Obviously the choice of the family $\{\chi_\tau\}$ is quite free. Practical problems may suggest suitable families. In this paper for sake of simplicity we use characteristic functions of intervals $[-\tau/2, \tau/2]^N$.

THEOREM 2. *Let* $\hat{G} \in L^p(\mathbf{R}^N)$, $1 \leqslant p < 2$ *and* $\sigma = \sigma(\delta)$: $\mathbf{R}^+ \to \mathbf{R}^+$, $\tau = \tau(\delta)$: $\mathbf{R}^+ \to \mathbf{R}^+$ *such that as* $\delta \to 0$

$$\sigma(\delta) \to 0, \qquad \tau(\delta) \to +\infty, \qquad \delta\left(\frac{\tau(\delta)}{\sigma(\delta)}\right)^{N|1-2/p|/2} \to 0.$$

Then, if $f \in L^p(\mathbf{R}^N)$, *for every* $\varepsilon > 0$ *there exists* $\delta_0 = \delta_0(\varepsilon, f)$ *such that if* $\delta < \delta_0$ *and* $1/p + 1/q = 1$ *for every* $\lambda \in L^q(\mathbf{R}^N)$ *and* $\|\lambda - \hat{f}\|_q < \delta$ *we have* $\|f - R_{\sigma(\delta),\tau(\delta)}\lambda\|_p < \varepsilon$.

PROOF. By Prop. 2 for every $\varepsilon > 0$ and $\delta < \delta_1(\varepsilon)$

$$\|\lambda - \hat{f}\|_q < \varepsilon \Rightarrow \|R_{\sigma(\delta),\tau(\delta)}(\lambda - \hat{f})\|_p < \varepsilon/2.$$

Moreover

$$\|f - R_{\sigma(\delta),\tau(\delta)}\hat{f}\|_p = \|f - (f * G_{\sigma(\delta)})\chi_{\tau(\delta)}\|_p \leqslant \|f - f\chi_{\tau(\delta)}\|_p + \|(f - f * G_{\sigma(\delta)})\chi_{\tau(\delta)}\|_p.$$

If $\delta < \delta_2(\varepsilon, f)$ we have $\|f - f\chi_{\tau(\delta)}\|_p < \varepsilon/4$.

Since $\{G_\sigma\}_{\sigma>0}$ is an approximate unit, if $\delta < \delta_3(\varepsilon, f)$ $\|(f - f * G_{\sigma(\delta)})\chi_{\tau(\delta)}\|_p \leqslant \leqslant \varepsilon/4$.

If $\delta_0 = \min(\delta_1, \delta_2, \delta_3)$ we have the result.

For pointwise convergence, we don't need to distinguish between $p < 2$ and $p \geqslant 2$. Nevertheless, as in the compact case we need a little bit more regularity of G. Let

$$M(x) = \sup_{|y| \geqslant |x|} \operatorname{ess} |G(y)|,$$

and for every p, $1 \leqslant p \leqslant +\infty$ let $p_0 = \min(p, 2)$, $1/q_0 + 1/p_0 = 1$.

THEOREM 3. *Let* $M \in L^1(\mathbf{R}^N)$, $\hat{G} \in L^{p_0}(\mathbf{R}^N)$ *and* $\sigma = \sigma(\delta)$: $\mathbf{R}^+ \to \mathbf{R}^+$ *such that as* $\delta \to 0$

$$\sigma(\delta) \to 0, \qquad \delta\sigma(\delta)^{-N/p_0} \to 0. \tag{3.2}$$

Then if $f \in B_p$, $1 \leqslant p \leqslant +\infty$ *and* x *is a Lebesgue point of* f, *for every* $\varepsilon > 0$ *there exists* $\delta_0 = \delta_0(\varepsilon, f, x)$ *such that if* $\delta < \delta_0$

$$\lambda \in L^q(\mathbf{R}^N) \quad \textit{and} \quad \|\lambda - \hat{f}\|_q < \delta \Rightarrow |f(x) - R_{\sigma(\delta)}\lambda(x)| < \varepsilon.$$

PROOF. Of course, we can always suppose $x = 0$. We have (see *e.g.* [9], Th. 1.25 p. 13) for $\delta \leqslant \delta_1(\varepsilon, f, x)$

$$|f(0) - f * G_{\sigma(\delta)}(0)| < \varepsilon/2. \tag{3.3}$$

Moreover

$$|f * G(0) - R_{\sigma(\delta)}\lambda(0)| = |((\hat{f} - \lambda)\cdot\hat{G}_{\sigma(\delta)})^{\vee}(0)| = \tag{3.4}$$

$$= \left| \int_{R^N} (\hat{f} - \lambda)(x)\,\hat{G}_\sigma(x)\,dx \right| \leqslant \|\hat{f} - \lambda\|_{q_0}\cdot\sigma(\delta)^{-N/p_0}\|\hat{G}\|_{p_0}.$$

From (5.2), (5.3) and (5.1) we obtain the theorem.

4. – As in the compact case, we give some *a priori* estimates of $\|f - R_\sigma\lambda\|_p$ if $p \geqslant 2$ and $\|f - R_{\sigma,\tau}\lambda\|_p$ if $1 \leqslant p < 2$, for Lipschitz classes of functions $K\,\mathrm{Lip}(\alpha, B_p)$, $0 < \alpha \leqslant 1$. We recall that $f \in K\,\mathrm{Lip}(\alpha, B_p)$ if for every $y \in \mathbf{R}^N$ the function $\Delta_y f(x) = f(x+y) - - f(x)$ satisfies $\|\Delta_y f\|_p \leqslant K|y|^\alpha$.

THEOREM 4. *If* $\int_{R^N} |x|^\alpha |G(x)|\,dx = c_\alpha < +\infty$ *then for every* $\sigma > 0$, $p \geqslant 2$, *if* $f \in K\,\mathrm{Lip}(\alpha, B_p)$ *and* $\lambda \in L^2(\mathbf{R}^N)$ *we have*

$$\|f - R_\sigma\lambda\|_p \leqslant Kc_\alpha\sigma^\alpha + a_p\,\sigma^{-N|1-2/p|/2}\|\lambda - \hat{f}\|_2.$$

The proof is obtained as in the following Theorem 5 assuming $\chi_\tau \equiv 1$.

For $1 \leqslant p < 2$ it is easy to see that an analogous estimate for $f - R_{\sigma,\tau}\lambda$ it is not available; some control of the decay at infinity of f is needed.

The simplest one is the following.

Let $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ a decreasing function such that $\lim_{x\to+\infty} \psi(x) = 0$. Then we set $H\Psi_p$ the class of $L^p(\mathbf{R}^N)$ functions such that $\|f(1-\chi_\tau)\|_p \leqslant H\psi(\tau)$ $\forall\tau > 0$.

THEOREM 5. *If* $\hat{G} \in L^p(\mathbf{R}^N)$ *and* $\int_{R^N} |x|^\alpha |G(x)|\,dx = c_\alpha < +\infty$, *then for every* $\sigma > 0$, $\tau > 0$ *if* $f \in K\,\mathrm{Lip}(\alpha, B_p) \cap H\Psi_p$, $1 \leqslant p < 2$ *and* $\lambda \in L^q(\mathbf{R}^N)$ *we have*

$$\|f - R_{\sigma,\tau}\lambda\|_p \leqslant Kc_\alpha\sigma^\alpha + H\psi(\tau) + a_p(\tau/\sigma)^{N|1-2/p|/2}\|\hat{f} - \lambda\|_q.$$

PROOF. We have

$$\|f - R_{\sigma,\tau}\hat{f}\|_p \leqslant \|f(1-\chi_\tau)\|_p + \|(f - f * G_\sigma)\chi_\tau\|_p \leqslant$$

$$\leqslant H\psi(\tau) + \left\| \int_{R^N} \Delta_y f(x)\,G_\sigma(-y)\,\chi_\tau(x)\,dy \right\|_p \leqslant H\psi(\tau) + \int_{R^N} K|y|^\alpha |G_\sigma(y)|\,dy \leqslant H\psi(\tau) + Kc_\alpha\sigma^\alpha.$$

Moreover, Prop. 2 gives $\|R_{\sigma,\tau}(\lambda - \hat{f})\|_p \leqslant a_p(\tau/\sigma)^{N|1-2/p|/2}\|\lambda - \hat{f}\|_q$ and the theorem holds.

Finally we give an *a priori* estimate of the pointwise approximation. In order to do this we recall the notion of $K\,\mathrm{Leb}(\alpha, x)$ classes of functions. We say that a function

$f \in L^1_{\mathrm{loc}}$ belongs to the Lebesgue class K Leb(α, x), $0 < \alpha \leq 1$, if for every $r > 0$

$$\int_{|y-x| \leq r} |f(y) - f(x)|\, dy \leq K r^{N+\alpha}.$$

For a discussion about these classes see [1].

THEOREM 6. *Let* $\hat{G} \in L^{p_0}(\mathbf{R}^N)$ *and* $\int_{\mathbf{R}^N} |x|^\alpha \; M(x) \; dx = \gamma_\alpha < +\infty$. *Then, if* $f \in K\mathrm{Leb}(\alpha, x) \cap B_p$ *and* $\lambda \in L^{q_0}(\mathbf{R}^N)$, *for every* $\sigma > 0$ *we have*

$$|f(x) - R_\sigma \lambda(x)| \leq K\bar{c}_\alpha \sigma^\alpha + \sigma^{-N/p_0} \|\hat{G}\|_{p_0} \|\hat{f} - \lambda\|_{q_0} \tag{4.1}$$

where

$$\bar{c}_\alpha = \frac{N+\alpha}{2} \pi^{-N/2} \Gamma\left(\frac{N}{2}\right) \gamma_\alpha .$$

PROOF. We can always suppose $x = 0$. We have

$$|f(0) - R_\sigma \lambda(0)| \leq |f(0) - R_\sigma \hat{f}(0)| + |R_\sigma(\hat{f} - \lambda)(0)|.$$

From (3.4)

$$|R_\sigma(\hat{f} - \lambda)(0)| \leq \sigma^{-N/p_0} \|\hat{G}\|_{p_0} \|\lambda - \hat{f}\|_{q_0} . \tag{4.2}$$

Moreover

$$|f(0) - R_\sigma \hat{f}(0)| = \left| \int_{\mathbf{R}^N} (f(-x) - f(0))\, G_\sigma(x)\, dx \right| \leq \left| \int_{\mathbf{R}^N} M_\sigma(x) \,|f(x) - f(0)|\, dx. \right. \tag{4.3}$$

Let

$$S(r) = \int_{|x| = r} |f(x) - f(0)|\, ds$$

where ds is the surface area element of the sphere $|x| = r$ and

$$F(r) = \int_0^r S(y)\, dy .$$

From (4.3) we obtain

$$|f(0) - R_\sigma \hat{f}(0)| \leq \int_0^{+\infty} M_\sigma(r) S(r)\, dr \leq F(r) M_\sigma(r) \Big|_0^{+\infty} - \int_0^{+\infty} F(r)\, dM_\sigma(r) \leq K r^{N+\alpha} M_\sigma(r) \Big|_0^{+\infty} -$$

$$- K \int_0^{+\infty} r^{N+\alpha}\, dM_\sigma(r) \leq K(N+\alpha) \int_0^{+\infty} r^{N+\alpha-1} M_\sigma(r)\, dr = K^{(N+\alpha)} \sigma^\alpha c_\alpha / m(S_N)$$

where $m(S_N)$ is the surface measure of the unit ball of $\mathbf{R}^N$. From the above inequality and (4.2) the result follows.

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References

[1] L. De Michele - M. Di Natale - D. Roux, *Sharp properties of a regularization method for inversion of Fourier series.* Proceedings of the Meeting *Trends in Functional Analysis and Approximation Theory*. Atti Sem. Mat. Fis. Univ. Modena, 39, 1991, 253-267.

[2] L. De Michele - M. Di Natale - D. Roux, *Inverse Fourier transform.* Rend. Mat. Acc. Lincei, s. 9, 1, 1990, 305- 308.

[3] L. De Michele - M. Di Natale - D. Roux, *Inversion methods for Fourier transform.* Rend. Accad. Naz. Sci. XL, Mem. Mat., 109°, 15, 1991, 125-136.

[4] L. De Michele - M. Di Natale - D. Roux, *On the effectiveness of some inversion methods for noisy Fourier series.* In: S. P. Singh (ed.), Proc. of NATO-ASI Meeting *Approximation theory, spline functions and applications*. Kluwer Acad. Publ., 1992, 255-267.

[5] L. De Michele - M. Di Natale - D. Roux, *Féjer kernels and noisy Fourier series.* J. Comput. Appl. Math., to appear.

[6] M. Di Natale - L. Gotusso - R. Pavani - D. Roux, *Uniform approximation by Fourier polynomials with noisy coefficients.* Ist. Lombardo Accad. Sci. Lett. Rend., A, 124, 1990, 147-155.

[7] M. Di Natale - L. Gotusso - R. Pavani - D. Roux, *On the choice of the parameter in a method for inversion of Fourier series.* Math. Comp., 58, 1992, 737-744.

[8] D. Roux, *On the inversion of noisy Fourier transforms.* Proc. of 2nd International Conference in *Functional Analysis and Appproximation Theory.* Rend. Circ. Mat. Palermo, to appear.

[9] E. M. Stein - G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton Univ. Press, 1975.

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