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Foliations with complex leaves

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Geometria. — *Foliations with complex leaves.* Nota di GIULIANA GIGANTE e GIUSEPPE TOMASSINI, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — Let X be a smooth foliation with complex leaves and let $\mathcal{O}$ be the sheaf of germs of smooth functions, holomorphic along the leaves. We study the ringed space $(X, \mathcal{O})$. In particular we concentrate on the following two themes: function theory for the algebra $\mathcal{O}(X)$ and cohomology with values in $\mathcal{O}$.

KEY WORDS: Foliations; Several complex variables and analytic spaces; CR-structures.

RIASSUNTO. — *Foliazioni con foglie complesse.* Sia X una varietà differenziabile fogliata con foglie complesse e sia $\mathcal{O}$ il fascio dei germi delle funzioni differenziabili su X, olomorfe lungo le foglie. Si studia lo spazio anellato $(X, \mathcal{O})$; in particolare la teoria delle funzioni per l'algebra $\mathcal{O}(X)$ e la coomologia a valori in $\mathcal{O}$.

1. PRELIMINARIES

1. In the following new results on foliations with complex leaves are announced. Complete proofs will appear elsewhere.

A foliation with complex leaves is a (smooth) foliation X of dimension $2n+k$ whose local models are domains $U = V \times B$ of $\boldsymbol{C}^n \times \boldsymbol{R}^k$, $V \subset \boldsymbol{C}^n$, $B \subset \boldsymbol{R}^k$ and whose local transformations are of the form

$$(*)\qquad \begin{cases} z' = f(z,t) \\ t' = h(t) \end{cases}$$

where f is holomorphic with respect to z.

A domain U as above is said to be a *distinguished coordinate domain* of X and $z = (z_1, \ldots, z_n)$, $t = (t_1, \ldots, t_k)$ are said to be *distinguished local coordinates.*

k is called the real codimension of X.

As an example of such foliations we have the Levi flat hypersurfaces of $\boldsymbol{C}^n$ [13, 4, 11].

Let X be a smooth foliation as above. Then the leaves are complex manifolds of dimension n. Let $\mathcal{O}$ be the sheaf of germs of smooth functions, holomorphic along the leaves (namely the germs of CR-functions on X).

$\mathcal{O}$ is a Fréchet sheaf and we denote by $\mathcal{O}(X)$ the Fréchet algebra $\Gamma(X, \mathcal{O})$.

X is said to be a *q-complete foliation* if there is an exhaustive, smooth function $\Phi: X \to \boldsymbol{R}$ which is strictly q-pseudoconvex along the leaves.

X is a *Stein foliation* if

(*a*) $\mathcal{O}(X)$ separates points of X;

(*b*) X is $\mathcal{O}$-convex;

(*) Nella seduta del 13 febbraio 1993.

(*c*) for every $x \in X$ there exist $f_1, \ldots, f_n, h_1, \ldots, h_k \in \mathcal{O}(X)$ such that

$$\operatorname{rank} \frac{\partial(f_1, \ldots, f_n, h_1, \ldots, h_k)}{\partial(z_1, \ldots, z_n, t_1, \ldots, t_k)} = n + k$$

($z_1, \ldots, z_n, t_1, \ldots, t_k$ distinguished local coordinates at x).

It is possible to prove that a Stein foliation is 1-complete.

REMARK. If we replace $\mathbf{R}^k$ by $\mathbf{C}^k$ and in (∗) we assume $t \in \mathbf{C}^k$ and that f, h are holomorphic with respect to z, t then we obtain the notion of *complex foliation* of (*complex*) *codimension* k.

2. Every real analytic foliation can be complexified. Precisely we have the following

THEOREM 1. *Let* X *be a real analytic foliation with complex leaves, of codimension* k. *Then there exists a complex foliation* $\tilde{X}$ *of codimension* k *such that*:

(1) $X \hookrightarrow \tilde{X}$ *by a closed real analytic embedding which is holomorphic along the leaves*;

(2) *every real analytic* CR-*function* $f\colon X \to \mathbf{R}$ *extends holomorphically on a neighbourhood*;

(3) *if* X *is a* q-*complete foliation with exhaustive function* Φ *then for every* $c \in \mathbf{R}$, $\overline{X}_c = \{\Phi \le c\}$ *has a fundamental system of neighbourhoods which are* q-*complete manifolds*.

REMARK. $\tilde{X}$ with the properties (1), (2), (3) is essentially unique.

As a corollary, using the approximation theorem of M. Freeman [5] we prove the following

THEOREM 2. *Under the assumptions of Th.* 1. *if* X *is* 1-*complete, a smooth* CR-*function on a neighbourhood of* $\overline{X}_c$ *can be approximated by smooth global* CR-*functions*.

REMARK. A similar argument can be applied to prove that in the previous statement $\overline{X}_c$ can be replaced by an arbitrary $\mathcal{O}$-convex compact K (*i.e.* $\hat{K} = K$).

2. APPLICATIONS

1. The approximation theorem allows us to prove an embedding theorem for real analytic Stein foliations [7].

Let X be a smooth foliation with complex leaves of dimension n and of codimension k. Let us denote by $\mathcal{A}(X; \mathbf{C}^N)$ the set of the smooth CR-maps $X \to \mathbf{C}^N \cdot \mathcal{A}(X; \mathbf{C}^N)$ is Fréchet.

We have the following

THEOREM 3. *Assume* X *is a real analytic Stein foliation. Then there exists* *a smooth* CR-*map* $X \to \boldsymbol{C}^N$, $N = 2n + k + 1$ *which is one-to-one, proper and regular.*

2. We apply the above theorem to obtain information about the topology of X.

THEOREM 4. *Let X be a real analytic Stein foliation. Then $H_j(X,\boldsymbol{Z}) = 0$ for $j \geq n+k+1$ and $H_{n+k}(X, \boldsymbol{Z})$ has no torsion.*

SKETCH OF PROOF. Embed X in $\boldsymbol{C}^N$ and consider on X the distance function ρ from a point $z^0 \in \boldsymbol{C}^N \setminus X$. z^0 can be chosen in such a way that ρ is a Morse function.

Next we show that ρ has no critical point of index $j \geq n + k + 1$ [14].

COROLLARY 5. *Let $X \subset \boldsymbol{P}^N(\boldsymbol{C})$ be a closed oriented real analytic foliation and let W be a smooth algebraic hypersurface which does not contain X. Then the homomorphism $H^j(X, \boldsymbol{Z}) \to H^j(X \cap W, \boldsymbol{Z})$ induced by $X \cap W \to X$ is bijective for $j < n - 1$ and injective for $j = n - 1$. Moreover the quotient group $H^{n-1}(X \cap W, \boldsymbol{Z})/H^{n-1}(X, \boldsymbol{Z})$ has no torsion.*

3. COHOMOLOGY

1. Given a q-complete smooth foliation X, according to the Andreotti and Grauert theory for complex spaces it is natural to expect that the cohomology groups $H^j(X, \mathcal{O})$ vanish for $j \geq q$.

This is actually true for domains in $\boldsymbol{C}^n \times \boldsymbol{R}^k$ [1]. More generally we prove the following:

THEOREM 6. *Let X be a* 1-*complete real analytic foliation. Then $H^j(X, \mathcal{O}) = 0$ for $j \geq 1$.*

SKETCH OF PROOF. Let us assume $k=1$ and let Φ be an exhaustive function for X. Then the vanishing theorem for domains in $\boldsymbol{C}^n \times \boldsymbol{R}^k$, bumps lemma and Mayer-Vietoris sequence [1] yield the following: for every $c > 0$ there is $\varepsilon > 0$ such that

$$H^j(X_{c+\varepsilon}, \mathcal{O}) \to H^j(X_c, \mathcal{O}) \tag{1}$$

is onto for $j \geq 1$ (and this holds true for $j \geq q$ whenever X is a q-complete smooth foliation). Now let $\tilde{X}$ be the complexification of X and let us consider the compact $\overline{X}_c = \{\Phi \leq c\}$. In view of theorem 1, $\overline{X}_c$ has in $\tilde{X}$ a fundamental system of Stein neighbourhoods U. X is oriented around $\overline{X}_c$ and consequently $U \setminus X$ has two connected components U_+, U_- (U connected).

Denote by O_+ (resp. O_-) the sheaf of germs of holomorphic functions on U_+ (resp. U_-) that are smooth on $U_+ \cup (U_+ \cap X)$ (resp. $U_- \cup (U_- \cap X)$).

Then we have the exact sequence

$$0 \to O \to O_+ \oplus O_- \xrightarrow{\text{re}} \mathcal{O} \to 0 \tag{2}$$

[2]; (here O_+ (resp. O_-)) is a sheaf on $\overline{U}_+$ (resp. $\overline{U}_-$) extended by 0 on all U and $\mathrm{re}(f \oplus g) = f_{|X} - g_{|X}$.

Since U is Stein we derive from (2) that

$$H^j(\overline{U}_+, O_+) \oplus H^j(\overline{U}_-, O_-) \xrightarrow{\sim} H^j(U \cap X, \mathcal{O}) \tag{3}$$

for $j \geq 1$ (and this holds true for $j \geq q$ whenever X is a q-complete real-analytic foliation of codimension 1).

Let ξ be a j-cocycle of $\mathcal{O}$ on a neighbourhood of $\overline{X}_c$. In view of (2) we have $\xi = \xi_+ - \xi_-$ where ξ_+ and ξ_- are represented by two $(0, j)$-forms ω_+, ω_- on U_+, U_- respectively which are smooth up to X.

Moreover according to [6] it is possible to construct pseudoconvex domains U'_+ and U'_- satisfying the following conditions: $U'_+ \subset U_+$, $U'_- \subset U_-$, $\partial U'_+$, $\partial U'_-$ are smooth and $\partial U'_+ \cap X$, $\partial U'_- \cap X$ contain a neighbourhood of $\overline{X}_c$.

Then Kohn's theorem [10] implies that on U'_+ and U'_- respectively we have $\omega_+ = \bar{\partial} v_+$, $\omega_- = \partial v_-$ where $v_+ \in C^\infty(\overline{U}'_+)$, $v_- \in C^\infty(\overline{U}'_-)$. It follows that $H^j(\overline{X}_c, \mathcal{O}) = 0$ for $j \geq 1$ and from (1) we deduce that $H^j(X_c, \mathcal{O}) = 0$ for every $c \in \boldsymbol{R}$ and $j \geq 1$.

At this point, in order to conclude our proof we can repeat step by step the proof of the Andreotti-Grauert vanishing theorem for q-complete complex spaces [1].

If $k \geq 2$ the situation is much more involved. Using the Nirenberg Extension Lemma [10] it is possible to reduce the cohomology $H^*(X, \mathcal{O})$ to the $\bar{\partial}$-cohomology of $\tilde{X}$ with respect to the differential forms on $\tilde{X}$ which are flat on X and to conclude invoking a theorem of existence proved by J. Chaumant and A. M. Chollet [3].

Assume that X is a real analytic and let O' be the sheaf of germs of real analytic CR-functions. Then an analogous statement for O' is not true. Andreotti and Nacinovich [2] showed that $H^1(X, O')$ is never zero. However by virtue of Th. 1 we have for arbitrary k, $H^j(\overline{X}_c, O') = 0$ for $j > 0$ whenever X is q-complete.

2. Using the same method of proof, under the hypothesis of Theorem 6, we have the following

THEOREM 7. *Let $A = \{x_v\}$ be a discrete subset of X and let $\{c_v\}$ be a sequence of complex numbers. Then there exists $f \in \mathcal{O}(X)$ such that $f(x_v) = c_v$, $v = 1, 2, \ldots$. In particular X is $\mathcal{O}$-convex and $\mathcal{O}(X)$ separates points of X.*

4. The Kobayashi metric

1. Let X be a foliation with complex leaves, of codimension k and let $T(X) \xrightarrow{\pi} X$ be the tangent bundle of X. The collection of all tangent spaces to the leaves of X forms a complex subbundle $T_H(X)$ of $T(X)$. Let D be the unit disc in $\boldsymbol{C}$ and let us denote by $\mathrm{CR}(D, X)$ the set of all CR-maps $D \to X$.

Given $\zeta \in T_H(X)$ with $x = \pi(\zeta)$ we define the function $F = F_X$ on $X \times T_H(X)$ by $F(x, \zeta) = \inf\{s \in \boldsymbol{R}: s \geq 0, s\varphi'(0) = \zeta\}$ where $\varphi \in \mathrm{CR}(D, X)$ and $\varphi(0) = x$.

When $k = 0$, F reduces to the Kobayashi «infinitesimal metric» of the complex manifold X[8]. In particular if $X = \boldsymbol{C}^n \times \boldsymbol{R}^k$, then $F = 0$.

If X' is another foliation as above and $\phi\colon X \to X'$ is a CR-map then $d\phi\colon T_H(X) \to T_H(X')$ and

$$F_{X'}(\phi(x), d\phi\zeta) \leq F_X(x, \zeta).$$

THEOREM 7. *F_X is upper semicontinuous.*

According to the complex case [8], X is said to be *hyperbolic* if $F(x, \zeta) > 0$ for every $x \in X$ and $\zeta \in T_H(X)$, $\zeta \neq 0$.

REMARKS 1). The fact that all the leaves are hyperbolic does not imply that X itself is hyperbolic;

2) every bounded domain in $\boldsymbol{C}^n \times \boldsymbol{R}^k$ is hyperbolic;

3) following [12] it can be proved that if X admits a continuous bounded function u, p.s.h. along the leaves and strictly p.s.h. in a neighbourhood of x, then X is hyperbolic at x.

2. Now consider a riemannian metric on X and let V be a smooth distribution of transversal tangent k-spaces. Then every $\zeta \in T(X)$ splits into $\zeta_0 + \zeta_c$ where $\zeta_0 \in V$, $\zeta_c \in T_H(X)$ and we denote by $\tau(\zeta_0)$ the length of ζ_0.

Let F be the infinitesimal Kobayashi metric on X and for $\zeta \in T_x(X)$ set $g(x, \zeta) = F(x, \zeta_c) + \tau(x, \zeta_0)$.

Then g is an upper semicontinuous pseudometric.

If $\gamma = \gamma(s)$, $0 \leq s \leq 1$ is a smooth curve joining $x, y \in X$, the *pseudo length* of γ with respect to g is

$$L(\gamma) = \int_0^1 g(\gamma(s), \dot{\gamma})\, ds$$

and the *pseudo-distance* between x, y is

$$d(x, y) = \inf_{\gamma} L(\gamma).$$

d is a real distance on X inducing the topology of X if X is hyperbolic.

X is said to be *complete* if a field V can be chosen making X complete with respect to d.

For example the unit ball in $\boldsymbol{C} \times \boldsymbol{R}$ is complete for the choice

$$V = \lambda(t)(x\, \partial/\partial x + y\, \partial/\partial y) + (1 + t^2)^{-1}\partial/\partial t$$

where $\lambda(t) = 2 \arctang t[(1 + t^2)^{-1}(1 - \arctang^2 t)^{-3/2}]$.

The interest of this construction is due to the following

THEOREM 8. *Let $\Omega \subset \boldsymbol{C}^n \times \boldsymbol{R}^k$ be with the riemannian structure induced by $\boldsymbol{C}^n \times \boldsymbol{R}^k$. If Ω is hyperbolic and complete then Ω is $\mathcal{O}$-convex.*

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