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On the automorphisms of surfaces of general type in positive characteristic

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Geometria algebrica. — *On the automorphisms of surfaces of general type in positive characteristic.* Nota di EDOARDO BALLICO, presentata (*) dal Corrisp. E. Arbarello.

ABSTRACT. — Here we give an explicit polynomial bound (in term of K_X^2 and not depending on the prime p) for the order of the automorphism group of a minimal surface X of general type defined over a field of characteristic $p > 0$.

KEY WORDS: Algebraic surface; Automorphism group; Chern numbers; Surfaces of general type.

RIASSUNTO. — *Sui gruppi di automorfismi delle superfici in caratteristica p.* In questa *Nota* diamo una limitazione superiore esplicita di tipo polinomiale (in funzione solo di K_X^2 e non dipendente dal primo p) per l'ordine del gruppo degli automorfismi di una superficie generale X definita su un campo di caratteristica $p > 0$.

Recently several mathematicians (see [3], announcement in the introduction after the statement of 3.14, [4, 11, 16, 17]) considered the problem of bounding (in terms of suitable numerical invariants, *e.g.* the Chern numbers) the order of the automorphism group Aut (X) of a smooth projective manifold X of general type or with K_X ample. Here «bounding» means «find a good polynomial bound». Except for the work in progress mentioned in the introduction of [3], all the quoted papers consider the case in which X is a surface of general type. All the quoted papers uses in an essential way the fact that the algebraically closed base field $\boldsymbol{K}$ has char $(\boldsymbol{K}) = 0$. We think that the problem is interesting even if $p := \text{char}(\boldsymbol{K}) > 0$. Although this paper is signed only by an author, it is only part of a joint project (hence joint ideas and trick!); however we cannot promise anything (joint or not) for the near future. In the body of this paper we will consider the case of surfaces of general type, using heavily [11]. We will show in the introduction to this paper that in positive characteristic this problem ramifies into several different natural problems, each of them interesting in its own, which as far as we know require completely different techniques; sometimes different subproblems could be linked since «not too many bad things can happen simultaneously» (*i.e.* the failure of an estimate has often geometric consequences).

The first point is: bound independent of the choice of the prime p or not? An example will make clear the difference. Let C be a smooth curve of genus $g \geq 2$ over $\boldsymbol{K}$. If char $(\boldsymbol{K}) = 0$, it is known that $\#(\text{Aut}(C)) \leq 84(g-1)$ and that this bound is sharp for infinitely many g. Now assume $p := \text{char}(\boldsymbol{K}) > 0$. If $p > g$, it is known that this bound holds true. However, it is also known that there is no linear upper bound on g if we allow all primes p; for this problem it is known an upper bound of order g^3. This splitting of the general problem into two problems with, sometimes, different answers, can be made for each of the following natural subproblems of the «bounding the authomorphism group problem». First of all, for any algebraic object X, Aut (X) is

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not only a variety (or a finite set in the cases considered here or in the quoted papers) but a group scheme. There is no difference if char$(\boldsymbol{K}) = 0$ because if char$(\boldsymbol{K}) = 0$ every group scheme is reduced (a theorem of Cartier and Nishi [6, SGA 3, Exp. $\mathrm{VII_B}$, Cor. 3.3.2]). But if char$(\boldsymbol{K}) > 0$, this is not true and it is this scheme which is the natural object to study. In this case the problem ramifies into bounding the order of the connected components of the scheme Aut(X) (*i.e.* bounding the set of automorphisms of X if this set, as will be the case when «K_X is quite ample», is finite), and describing the connected component $\mathrm{Aut}^0(X)$ of Aut(X). It is known (see *e.g.* [6, SGA 3, Exp. II] for a quite general case) that the tangent space to Aut(X) (*i.e.* to $\mathrm{Aut}^0(X)$) is $H^0(X, TX)$, where TX is the tangent bundle (or tangent sheaf) of X. It is known (see [13, 9, 14]) that in positive characteristic there are smooth surfaces, X, of general type with $H^0(X, TX) \neq 0$. Hence it arises the problem of finding an upper bound for $h^0(X, TX)$ (which, anyway, is useful also to give estimates for the dimensions of the irreducible components of the moduli scheme of surfaces of general type). Furthermore, the condition «$h^0(X, TX) = 0$» is exactly the condition needed to apply [5, Th. 1.10], to be sure of the fact that the set Aut(X), when X moves in any family, has a very strong semicontinuity property (a semicontinuity not only for its order, but also for its group-theoretic structure). In [1, 3.12], it was shown as easy side remark that in any characteristic for a minimal surface X of general type we have $h^0(X, TX) \leq \leq 18(K_X)^2$ (with the coefficient 18 given only as example, since with a few more lines of text one could lower it).

After computing the tangent space of the group scheme of automorphisms, the next problem is the computation of its structure, or at least its length (when as in the case of surfaces of general type it is a finite scheme over $\boldsymbol{K}$).

The problem of bounding or computing the order of the group $G := \mathrm{Aut}(X)_{\mathrm{red}}$ seems to split in a natural way into two subproblems, for which different techniques are available. Setting $p :=$ char$(\boldsymbol{K})$, one should bound separately the order of the p-Sylow subgroup of G and the order of every subgroup G with order prime to p (*i.e.* the non modular subgroups). For non modular subgroups the best is to find «reasonable» (*e.g.* polynomial, or explicit or ....) bounds which do not depend on p, but are «universal». This would be done for surfaces of general type in sect. 1 and 2. We will prove in particular the following result.

THEOREM 0.1. Let X be a minimal surface of general type defined over an algebraically closed field $\boldsymbol{K}$; set $c := K_X^2$. Then there is a universal constant C (which does not depend on char$(\boldsymbol{K})$) such that for every subgroup G with order prime to char$(\boldsymbol{K})$ we have $\#(G) \leq C \log(c)\, c^{(45/2)}$.

The bound in 0.1 should be far from optimal. The main problem to improve it comes from the bound obtained in 1.1 in the case in which G is Abelian. In 1.8 we will state 0.1 making explicit the dependence of the exponent from the corresponding result for Abelian non modular subgroups and from the bound on the order of non modular subgroups of Aut(C) for a curve C of genus g. In 1 we will fix the notations, prove a few key remarks and then prove the case of an Abelian non modular subgroup. In 2 we will list the bound on $\#(G)$ which is obtained for our non modular subgroup using the bound on Abelian subgroups and following step by step [11]. We think that the group theoretic approach of [11] may be useful even in higher dimensions (after a non trivial input completely unknown to us, at least now).

1. The Abelian modular subgroups

Let Y be an integral curve; the genus $p_g(Y)$ of the normalization of Y will be called (as usual) the geometric genus of Y. From now on in this paper we fix a minimal surface of general type X over $\boldsymbol{K}$, and set $K := K_X$ and $c := K^2$. For simplicity we will write $\mathrm{Aut}(X)$ instead of $\mathrm{Aut}(X)_{\mathrm{red}}$. The notation $\Phi \propto \Gamma$ means that there is a universal constant D (not depending on the characteristic of the base field) such that $\Phi \leqslant D\Gamma$; the notation $\propto \Gamma$ means that there is a universal constant D such that the object considered in that sentence has order at most $D\Gamma$. A group with order prime to the characteristic of the base field is usually called non modular. In this section we will bound $\#(A)$ for every non modular Abelian subgroup of $\mathrm{Aut}(X)$, proving the following result.

Proposition 1.1. *Let X be a minimal surface of general type defined over an algebraically closed field $\boldsymbol{K}$; set $c := K_X^2$. Then there is a universal constant D (which does not depend on $\mathrm{char}(\boldsymbol{K})$) such that for every Abelian subgroup A of $\mathrm{Aut}(X)$ with order prime to $\mathrm{char}(\boldsymbol{K})$ we have $\#(A) \leqslant Dc^4$.*

We need the next 2 (well-known) lemmas.

Lemma 1.2. *Let $P \in X$ be a fixed point for the non modular group H. Assume that H acts trivially on the tangent space $T_P X$. Then H is trivial.*

Proof. Fix $g \in H$ of order r prime to p, and consider the action of g on the completion $\boldsymbol{K}[\![x, y]\!]$ of the local ring of X at P. By assumption $g(x) - x$ and $g(y) - y$ vanish to order at least 2. Take y general and set $y = 0$. Then g induces h: $\boldsymbol{K}[\![x]\!] \to \boldsymbol{K}[\![x]\!]$; if $g \neq$ $\neq \mathrm{Id}$, then $h \neq \mathrm{Id}$ for a general choice of y. Hence $h(x) = x + ax^t + o(x^t)$ with $t \geqslant 2$ and $a \neq 0$. Thus $h^r(x) = x + rax^t + o(x^t) \neq x$, contradiction. ♦

The result corresponding to 1.2 may be false (even for smooth curves) for groups with order divisible by $\mathrm{char}(\boldsymbol{K})$. For instance if $\mathrm{char}(\boldsymbol{K}) = 2$ the automorphism σ: $\boldsymbol{K}[\![x]\!] \to \boldsymbol{K}[\![x]\!]$ with $\sigma(x) = x + x^2 + x^4 + x^8 + \dots$ has order 2.

Lemma 1.3. *Let $H \subset \mathrm{Aut}(X)$ be a non modular group. If H fixes pointwise a reduced singular curve $C \subset X$ (C may be reducible), then $H = \{\mathrm{Id}\}$.*

Proof. Fix $P \in \mathrm{Sing}(C)$. Since C contains the first infinitesimal neighborhood of 0 in the Zariski tangent space $T_P X$, the result follows from Lemma 1.2. ♦

Remark 1.4. Let Y be a smooth connected curve of genus 1. Fix $P \in Y$. In every characteristic there are at most 24 automorphisms of Y fixing P [15, Ch. III, Th. 10.1, and Appendix A, Prop. 1.2(c)].

REMARK 1.5. Let C be a smooth connected curve of genus $g \geqslant 2$. Then every Abelian non modular subgroup A of $\mathrm{Aut}(C)$ has order $\propto g$.

PROOF. Fix a cyclic subgroup A' of A and consider $A'' := A/A'$ as a subgroup of $\mathrm{Aut}(C'')$ with $C'' := C/A'$. Apply Riemann-Hurwitz formula (with all ramification which is tame since A' is cyclic and non modular). If C'' has genus at least 2, continue with (C'', A'') instead of (C, A). Assume that C'' has genus 1. Since $g > 1$ the covering $C \to C''$ is not étale. Hence we conclude by Remark 1.4. Assume $C'' \cong \boldsymbol{P}^1$. By Riemann-Hurwitz and the fact that A' is cyclic, say of order t, we find $2g - 2 = -2t + r(t-1)$ with r the number of branch points. Since every automorphism of $\boldsymbol{P}^1$ fixing pointwise 3 points is the identity, we conclude. ◆

REMARK 1.6. By a result of Igusa [12] the rank, ρ, of the Neron-Severi group of X is bounded by $B_2(X)$. In turn $B_2(X)$ is bounded by the sum of the Hodge numbers h^{ij} with $i + j = 2$, *i.e.* by $2p_g + h^1(X, \Omega^1)$. Using the Theorem in the introduction of [8], *i.e.* the fact that $|5K|$ is very ample on the canonical model of X plus standard exact sequences on normal bundles of divisors, we see that $h^1(X, \Omega^1) \propto c$; hence $B_2(X) \propto c$. Since the (-2) smooth rational smooth curves on X are numerically independent, we see that their number is $\propto c$.

PROOF OF 1.1: (*a*) By [8, Main theorem, p. 97], $|5K|$ is very ample on the canonical model of X (any fixed integer t independent of p with $|tK|$ with that property would be sufficient for us). $\mathrm{Aut}(X)$ acts on $|5K|$ and in any representation each element of A (having order prime to p) may be diagonalized. Since A is Abelian all the elements of A can be simultaneously diagonalized; take a corresponding basis $\{D(i)\}_{1 \leqslant i \leqslant z}$ with $z := h^0(X, 5K) = 10c + \chi$ (by [8], part (*i*) of the Theorem in the introduction). Let S be the set of irreducible curves whose union is the support of the union of all divisors $D(i)$'s.

(*b*1) Since K is ample outside the (-2) smooth rational curves, every element of $|5K|$ has at most $5c$ irreducible components (even counting multiplicities) plus perhaps some (-2) rational curves, hence by Remark 1.6 $\propto c$ irreducible components.

(*b*2) By the adjuction formula every curve $C \in S$ has $p_a(C) \propto c$.

(*c*) Suppose there is a subgroup A' of A fixing pointwise one of the following configurations of curves:

(*c*1) configuration (λ): a singular curve $C \in S$;

(*c*2) configuration (μ): the union of two curves, say $C' \cup C''$, in S with $C' \cap C'' \neq \emptyset$. Then A' is trivial by Lemma 1.3. Hence it is sufficient to find one such A' and bound the index of A' in A.

(*d*) By (*b*1) for every curve $C \in S$ we find $A' \subseteq A$ stabilizing C and with index $\propto c$ in A. Hence by (*b*2) and 1.5 we obtain $\#(G) \propto c^2$ if there is some $Y \in S$ with $p_g(Y) \geqslant 2$. Similarly, we find a suitable configuration (λ) and conclude by 1.3 and 1.4 if S contains singular curves, say C, with $p_g(C) \geqslant 1$ or whose normalization is $\boldsymbol{P}^1$ but such that at least 3 points of $\boldsymbol{P}^1$ are mapped by the normalization map into $\mathrm{Sing}(C)$ (remember: $p_a(C) \propto c$ by ($b2$)); in the latter case we find $\#(A) \propto c^4$. In particular from now on we may (and will) assume that the smooth curves in $\boldsymbol{S}$ are rational or elliptic and that the only singular curves in $\boldsymbol{S}$ are rational curves for which the condition on the normalization just considered fails.

(e) Note that by the classification of surfaces for no index i $D(i)_{\mathrm{red}}$ may be integral, smooth and rational. If $D(i)_{\mathrm{red}}$ is integral for an index i, then $\#(G) \propto c$ by ($b1$). Hence we may assume that every $D(i)_{\mathrm{red}}$ is reducible. Furthermore, in $|15K|$ we may take as part of a invariant basis the divisors $\{D(j) + D(j') + D(j'')\}$. Hence working in $|15K|$ (or any $|tK|$ with $t \geqslant 15$ and t fixed and independent from the characteristic of the base field) instead of $|5K|$ we may be sure that for «many» (to be specified soon) indices i $D(i)_{\mathrm{red}}$ contains at least, say, 6 curves. From now on we will assume that we are working in some $|tK|$ with $t \geqslant 15$ and such that the condition on $D(i)_{\mathrm{red}}$ is satisfied for at least D' indices with $D' - 1$ much larger than the universal bound of order c for the number of (-2) smooth rational curves given in the last assertion of 1.6. We fix any such t and apply the notations $D(i)$, $\boldsymbol{S}$, and so on, introduced for $|5K|$ to the linear system $|tK|$.

(f) Fix a configuration $C' \cup C''$ of type (μ). Since each $D(i)$ has $\propto c$ irreducible components, there is a subgroup A'' of A of index $\propto c^2$ and stabilizing C' and C''. Assume C' is elliptic and C'' is smooth and rational; as in the proofs given below in part (h) and (i), we will be able to fix two points of C'' at a cost $\propto c^2$ (below we will need and prove better bounds) and obtain $\propto c^4$ as bound by 1.4. Now assume the existence of a singular rational curve C. Take a subgroup A'' of index $\propto c$ in A and stabilizing C. Again, we will see below that there is a subgroup of A'' of index $\propto c^2$ and fixing 3 points of C, one of them being a singular one (note that $p_a(C'') \propto c$). In this case we obtain $\propto c^3$ as bound. Now assume that C'' is smooth of genus $\geqslant 1$ but that to find a configuration of type μ we are compelled to choose C' smooth and rational; using the same proofs we obtain $\propto c^4$ as bound.

(g) From now on we will handle the case in which every curve in $\boldsymbol{S}$ is smooth and rational, with proofs which work (with better bounds than needed) in the two missing cases of part (f). Take $C' \in \boldsymbol{S}$, say $C' \in D(i)$, with minimal $C' \cdot K$; set $u := C' \cdot K$; for every $j \neq i$ there are at most u curves in $D(j)$ which intersect C', while $D(i)$ contains only $\propto (c/u)$ curves. Let v be the number of the curves in $D(i)_{\mathrm{red}}$.

(h) We distinguish two cases: $u \leqslant (c)^{1/2}$ and $u \geqslant (c)^{1/2}$ which will be handled using two different configurations of type λ.

($h1$) First we will describe the two configurations and shows the bound on $\#(G)$ they give (using again Lemma 1.2). Then we will show that indeed we may find such configurations (under our assumption). Note that $uv \propto c$. If $u \geqslant (c)^{1/2}$ we fix (not pointwise) a curve C' in $D(1)$ (adding $\propto v$ to the bound), a curve C'' in $D(1)$ with $C' \cap C'' \neq \emptyset$ (again not pointwise, hence adding $\propto v$ to the bound). If $\mathrm{card}(C' \cap C'') = 1$ we fix two points on $C' \setminus C''$ and two points on $C''\setminus C'$ among the special points (*e.g.* $\mathrm{Sing}(D(i)_{red})$ and the isolated points of $D(i)\cap D(j)$ with $i\neq j$) (adding $\propto u^4$ to the bound); we obtain $\#(A)\propto c^3$.

($h2$) If $\#(C'\cap C'')\neq 1$ but small (*i.e.* bounded by a universal constant) to fix pointwise $C'\cup C''$ we need to choose less points on $(C'\cup C'')\setminus(C'\cap C'')$; hence we have a better bound. If $\#(C'\cap C'')\geqslant 3$ and we have no restriction on this number (except the bound u coming from the definition of u), we have an even better bound.

($h3$) Now assume $u\geqslant(c)^{1/2}$. In $D(1)$ we fix (not pointwise) curves C', C'' with $C'\cap C''\neq\emptyset$; then in several $D(j)$'s ($j=1$ is allowed) we stabilize other curves (different from C', C'') in such a way that both C' and C'' contains at least 3 points in the union of $C'\cap C''$ and these curves. Since the choice of each curve costs $\propto v$, we get a good bound.

(i) Now we check that indeed we may find such configurations. Since these smooth rational curves cannot have 0 as self-intersection, each $D(i)_{\mathrm{red}}$ is reducible. If in, say, $D(i)$, all these curves pass through a point, $P(i)$, this is fixed by A. Assume that this case occurs for almost all indices i. Since the hyperplane sections $D(j)$'s form a basis, we see that $\mathrm{card}(\{P(i)\})>1$. Hence this trouble cannot arise (for too many indices) in $|2tK|$. Alternatively a very sharp bound is again obtained if there is a «small» number of points on $D(1)$ such that all the curves of $D(1)$ contains one of these points or if there is a «small» set $S(1i)\subset D(1)$, $i\neq 1$, (or a small set $S'(1i)$) such that every curve in $D(i)$ (resp. every curve in $D(i)$ which is not contained in $D(1)$) contains at least a point of $S(1i)$ (resp. in $S'(1i)$). Taking $|10tK|$ instead of $|2tK|$ we may assume the existence of a set $\boldsymbol{T}$ of indices with cardinality a suitable multiple of c and such that for every pair $(i,j)\in\boldsymbol{T}\times\boldsymbol{T}$ there is no subset of cardinality 5 in $D(i)$ intersecting every curve of $D(j)$. Since each $D(i)$ is an ample divisor, it is clear that we may find the configurations needed working in $|10tK|$. ♦

Note that if we distinguish the two cases «$u\leqslant(c)^{1/2}$» and «$u\geqslant(c)^{1/2}$» in the discussion with curves with geometric genus at least 2 (or 1 but singular curves) we would obtain much better bounds in those cases; the main trouble to improve in a substantial way the bound given here in 1.1 comes (in the proof given!) from the possibility of having almost only «bad curves» as natural candidates to be stabilized by subgroups of small index.

We were unable to use the proof of [10, Th. 4.2], in positive characteristic because in positive characteristic there are non isotrivial morphism from a surface to $\boldsymbol{P}^1$ with only one or two singular fibers (see [2, p. 98]).

To make explicit the dependence of the bound in 0.1 from the bound obtained in 1.1 and the bound on the order of the non modular subgroups of $\mathrm{Aut}(C)$ for a smooth curve C of genus g, we will introduce the following notations α, β and γ which will be used heavily in the next section, too.

DEFINITION 1.7. Let α be an integer such that $\#(A)\propto c^{\alpha}$ for all abelian non modular subgroups of $\mathrm{Aut}(X)$. Let β be an integer such that $\#(H)\propto g^{\beta}$

for all non modular subgroups of $\mathrm{Aut}(C)$ with C any smooth curves of genus g. Set $\gamma := \max(\alpha, \beta)$.

Note that it is known that $\beta \leqslant 3$ and by 1.1 we may take $\alpha = 4$; hence, unless one improves in a substantial way 1.1, we have to use $\gamma = \alpha$. The Proof of 0.1 given in §2 will prove the following result.

THEOREM 1.8. Let X be a minimal surface of general type. Then there is a universal constant C such that for every subgroup G with order prime to char $(\boldsymbol{K})$ we have $\#(G) \leqslant C \log(c) c^{5(\alpha+\gamma+1)/2}$.

2. The general case

Here we list the bound on $\#(G)$ which is obtained for our non modular subgroup following step by step [11]. We use $c := K^2$ instead of c_2 and use the notations α, β and $\gamma := \max(\alpha, \beta)$ introduced in 1.7. Propositions 1 and 2 of [11] work with the same bound. The exponents of the bounds in [11, §2, Prop. 3, Cor. 4, Prop. 5 (and hence Cor. 6)] are respectively: α, $\alpha + 1$, 2α, $\alpha + \gamma$, thanks to the following two remarks:

(*i*) Use 1.6 to bound by $\propto c$ the set of (-2) smooth rational curves on X in the proof of Cor. 4.

(*ii*) An outline of the proof of the result quoted from ref. [HO] of [11] in the proof of Prop. 3 can be reconstructed from the proof of Prop. 5 of [11], §2.

We note explicitely that during the proofs in [11] it is used several times an induction taking the quotient of the surface by subgroups of $\mathrm{Aut}(X)$ acting freely on X. Hence the possibility of existence of rational or elliptic fixed curves (which makes worst the bound obtained) cannot be avoided (apparently) assuming some natural geometric conditions. Propositions 7 and 8 of [11], §2, works for our non modular G. Note that the quotient of X by a group acting freely is again of general type (*e.g.* by the classification of surfaces and the fact that no rational smooth surface has an étale cover).

We claim that in the very important «Centralizer theorem» in [11], the exponent is $\alpha + \gamma + 1$ (assuming of course $\alpha \geqslant 1$ and $\beta \geqslant 1$). To check the claim we use two remarks. First, even in characteristic $p > 0$ the number of fixed points for a non trivial action of an Abelian non modular group on a smooth curve of genus g has an upper bound of the order g (*e.g.* by a Lefschetz type fixed point formula or one can copy the proof of remark 1.5 plus the fact that any tame étale covering of $\boldsymbol{P}^1 \setminus \{0, 1\}$ is cyclic). The second remark is concerned with Case 2 of the proof of the Centralizer theorem in [11]; in that part there is a blowing-up, $\widetilde{X}$, of X and a certain morphism $\widetilde{f}\colon \widetilde{X} \to \boldsymbol{P}^1$; although we cannot claim in positive characteristic that the general fiber of $\widetilde{f}$ is smooth, the computations there use essentially only that the integral fiber $\widehat{F}$ in their situation is the normalization of a certain curve, F; here instead we make the computations on the normalization of $\widehat{F}$. Here are the new exponents for the results corresponding to [11], §3 (with no change in the proofs); in Prop. 1, Prop. 2, Cor. 3, Prop. 4, Prop. 5, Prop. 7, the exponents now are respectively $\alpha + \gamma + 1$, $\alpha + \gamma + 1$, $(\alpha + \gamma + 1)^{(k/(k-1))}$, $5(\alpha + \gamma + 1)/2$. The bound in [11,§3, Prop. 8] now is $\log(c)\, c^{5(\alpha+\gamma+1)/2}$. Then we carry over with no change the «Proof of the estimate...» at the end of [11]; now we got as exponents $2\alpha + \gamma + 1$ if G is not semi-simple, and case (1) and case (2) of loc. cit. have respectively $2\alpha + 2\gamma + 1$ and (if a certain integer k is at least 10), $10(\alpha + \gamma + 1)/9$, otherwise the extimates for the order is $\log(c)\, c^{5(\alpha+\gamma+1)/2}$, proving Theorem 1.8, hence Theorem 0.1.

REMARK 2.1. It is easy to obtain better bounds with either stronger geometric conditions on S (as in [11, 4.5, 4.6 and 4.7]) or under group theoretic assumptions on the non modular group G (*e.g.* nilpotent or solvable) as in [11].

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