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The assessment of the residual post-transient stresses in elastic-perfectly plastic solids subjected to cyclic loads

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Meccanica dei solidi. — *The assessment of the residual post-transient stresses in elastic-perfectly plastic solids subjected to cyclic loads.* Nota di Castrenze Polizzotto, presentata (*) dal Corrisp. G. Maier.

Abstract. — For elastic-perfectly plastic solids (or structures) subjected to quasi-static cyclic loads, variational methods are presented for the direct evaluation of the post-transient residual stresses, that is, the residual stresses in the structure at the end of the transient response phase, consequence of the plastic strains therein produced and crucial to predict the subsequent steady structural behaviour. The problem of the evaluation of the number of cycles spanned by the transient response is also discussed.

Key words: Cyclic plasticity; Minimum principles; Residual stresses.

Riassunto. — *Determinazione delle tensioni residue post-transitorie nei solidi elasto-plastici soggetti a carichi ciclici.* Con riferimento a solidi (o strutture) di materiale elastico perfettamente plastico e soggetti a carichi ciclici di tipo quasi-statico, si presentano dei metodi variazionali per la determinazione diretta delle tensioni residue post-transitorie, cioè delle tensioni residue esistenti al momento dell'esaurimento della fase transitoria della risposta strutturale, tensioni che sono conseguenza delle deformazioni plastiche prodotte in tale fase e che sono cruciali al fine di predire il tipo di risposta stazionaria che ne segue. Viene anche discusso il problema della valutazione del numero dei cicli contenuti nella fase transitoria.

1. Introduction

A solid body (or structure) of elastic-perfectly plastic material and subjected to cyclic (*i.e.* periodic) loads is here considered in the restrictive hypothesis of small displacements and strains. For such a body, a classical result of plasticity theory [1, 2, 5, 12] states the existence of a steady-state (long-term) response characterized by stresses $\boldsymbol{\sigma}$ and plastic strain rates $\dot{\boldsymbol{\varepsilon}}^p$ periodic with the same period Δt as the loads, and preceded by a transient (short-term) response in which no periodicity features are recognizable. The steady-state response can be categorized on the basis of its own *plastic accumulation mechanism* (PAM), that is the steady cycle plastic strains that result in a compatible plastic strain field, the so-called *ratchet strain,* $\Delta\boldsymbol{\varepsilon}^p$. When in the long term plastic flow ceases everywhere in the body, the steady-state response is an elastic response and we say that *elastic shakedown* (or *adaptation*) occurs under the given loads. If on the contrary plastic flow continues in the steady-state response, but $\Delta\boldsymbol{\varepsilon}^p$ vanishes everywhere in the body, the steady-state response is characterized by *alternating* (or periodic) plastic strains $\boldsymbol{\varepsilon}^p$ and we say that *plastic shakedown*, (or alternating plasticity collapse), occurs under the given loads. Finally, if $\Delta\boldsymbol{\varepsilon}^p$ is nonvanishing even in a small part of the body, the steady-state response is characterized by progressive cycle-by-cycle accumulation of compatible plastic strains and we say that *incremental collapse,* (or *ratchetting*), occurs under the given loads. The load space region,

(*) Nella seduta dell'11 novembre 1992.

whose boundary surface is the structure's plastic collapse limit surface, can be divided into zones, each corresponding to a particular type of steady-state response, so obtaining what is usually meant as the *interaction diagram* (or generalized Bree diagram) [2, 12].

The engineering implications of the different types of the steady-state responses are quite obvious. Elastic shakedown is the most acceptable kind of structural behaviour, but also plastic shakedown is considered acceptable in some instances, *e.g.* for nuclear power plant applications; on the contrary, ratchetting is extremely dangerous because of the plastic strain growth therein induced. For this reason, criteria to *a-priori* recognize the kind of steady-state response induced by given loads are of great importance. Criteria to predict elastic shakedown have extensively developed after the pioneering work of Bleich and Melan (see *e.g.* [3, 5]). Analogous criteria to predict plastic shakedown have recently been given by the present author [6, 7] (see also [8-10]).

It is to stress that the above criteria nothing say about the transient response phase. The latter phase lasts in general a few cycles, during which a certain amount of plastic strains is produced. For preliminary design purposes, it suffices in general to have a rough estimation of the above strains, for instance by bounding techniques. Whenever an accurate evaluation of the plastic strains and of their consequences is required, a step-by-step analysis is usually considered as unavoidable.

The knowledge of the so-called *post-transient* (PT) residual stresses, that is the residual stresses in the structure at the end of the transient response phase, is of importance first because they are a direct consequence of the plastic strains produced in that phase, secondly because they enable one to predict whether plastic shakedown occurs or not. Though this point has been already addressed in [6, 7], it however deserves some further discussion, in particular with reference to related variational formulations. The purpose of the present paper is to reconsider the latter topic and to present alternative formulations together with a method leading to an estimation of the number of cycles spanned by the transient response.

A compact notation will be used. Vectors and tensors will be denoted by bold face symbols with scalar product marked by a dot or a colon, for instance, $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i$, $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij}\, \varepsilon_{ij}$, $\boldsymbol{\sigma} : \boldsymbol{A} : \boldsymbol{\sigma} = A_{ijhk}\, \sigma_{ij}\, \sigma_{hk}$, $(\boldsymbol{n} \cdot \boldsymbol{\sigma})_i = (\boldsymbol{\sigma} \cdot \boldsymbol{n})_i = \sigma_{ji} n_j$, $(\boldsymbol{A} : \boldsymbol{\sigma})_{ij} = A_{ijhk}\, \sigma_{hk}$, and the summation rule for repeated indices is used. Other symbols will be defined where they appear for the first time.

2. The steady-state response

A solid body occupies, in its reference undeformed configuration, the (open) domain V with boundary surface ∂V in the three-dimensional Euclidean space, and it is referred to a Cartesian orthogonal co-ordinate system $\boldsymbol{x} = (x_1, x_2, x_3)$. It is subject to external actions, as body forces in V, tractions on the free boundary $\partial_t V \subset \partial V$, thermal strains in V and imposed displacements on the constrained boundary $\partial_u V \subset \partial V$, $\partial_u V \cup \partial_t V = \partial V$. All these actions are all together denoted with the symbol P and represented as

$$P(t) = P^c(t) + P^0, \qquad t \geqslant 0, \quad P^0 \in \mathcal{P}^0 \tag{1}$$

where $P^c(t)$ is periodic with period Δt, *i.e.* $P^c(t + \Delta t) = P^c(t)$ for all $t \geqslant 0$. Equation (1) points out that the loading is a combination of a cyclically variable load $P^c(t)$ and a time-independent or steady load P^0. By hypothesis, $P^c(t)$ may be either a mechanical, or a kinematical load, or both, whereas P^0 can be only a mechanical load belonging to the set $\mathcal{P}^0$. The latter is meant as the entire set of loads below the plastic collapse limit (but it may on occasion be narrower). Other hypotheses are: the inertia and viscous forces are negligible, the material data is independent of temperature variations, the small displacement theory is applicable. The material is elastic-perfectly plastic with a (convex, smooth) yield surface, and the plastic potential $\phi(\boldsymbol{\sigma})$ has also the role of yield function (associated plasticity).

One of the central points in cyclic plasticity theory is the direct determination of the steady-state response, namely without a full step-by-step analysis starting at time $t = 0$. Such a task can be accomplished by considering the load history as specified in the interval $(0, \Delta t)$ and by solving the following set of field and boundary equations (see *e.g.* [2, 5-7]):

$$\phi(\boldsymbol{\sigma}) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}\,\phi(\boldsymbol{\sigma}) = 0 \quad \text{in } V \times (0, \Delta t), \tag{2}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^R, \quad \dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\,\frac{\partial \phi}{\partial \boldsymbol{\sigma}} \quad \text{in } V \times (0, \Delta t), \tag{3}$$

$$C^T \boldsymbol{\sigma}^R = \mathbf{0} \quad \text{in } V \times (0, \Delta t), \qquad \boldsymbol{\sigma}^R \cdot \boldsymbol{n} = \mathbf{0} \quad \text{on } \partial_t V \times (0, \Delta t), \tag{4a}$$

$$C\,\dot{\boldsymbol{u}}^R = \boldsymbol{A} : \dot{\boldsymbol{\sigma}}^R + \dot{\boldsymbol{\varepsilon}}^p \quad \text{in } V \times (0, \Delta t), \qquad \dot{\boldsymbol{u}}^R = \mathbf{0} \quad \text{on } \partial_u V \times (0, \Delta t), \tag{4b}$$

$$\Delta \boldsymbol{\varepsilon}^p = \int_0^{\Delta t} \dot{\boldsymbol{\varepsilon}}^p \, dt, \qquad \Delta \boldsymbol{u}^R = \int_0^{\Delta t} \dot{\boldsymbol{u}}^R \, dt, \qquad \text{in } V, \tag{5a}$$

$$C\,\Delta \boldsymbol{u}^R = \Delta \boldsymbol{\varepsilon}^p \quad \text{in } V, \qquad \Delta \boldsymbol{u}^R = \mathbf{0} \quad \text{on } \partial_u V. \tag{5b}$$

Here, the following symbology has been introduced: $\boldsymbol{\sigma}^E(\boldsymbol{x}, t)$ is the elastic stress response to the loads, $\boldsymbol{\sigma}^R(\boldsymbol{x}, t)$ the residual stress, $\dot{\lambda}$ the plastic activation coefficient, $\boldsymbol{u}^R$ the residual displacement, $\Delta\boldsymbol{\varepsilon}^p$ and $\Delta\boldsymbol{u}^R$ the net increments of $\boldsymbol{\varepsilon}^p$ and $\boldsymbol{u}^R$ in the cycle; moreover, C is the compatibility differential operator and C^T its adjoint, namely

$$C\,\boldsymbol{u} = [\operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^T]/2, \qquad C^T \boldsymbol{\sigma} = \operatorname{div} \boldsymbol{\sigma}. \tag{6}$$

Also, $\boldsymbol{A}$ denotes the elastic compliance fourth-order tensor with its usually symmetries and positive definiteness property, and $\boldsymbol{n}$ denotes the unit external normal to ∂V.

Problem (2)-(5) has the shape of a boundary value problem of plasticity theory, but with the customary initial (at $t = 0$) conditions replaced by the time-integral conditions expressing the requirement that the net increment of plastic strain, *i.e.* the ratchet strain $\Delta\boldsymbol{\varepsilon}^p$, be compatible. The solution to eqs. (2)-(5) (the steady-state solution) enjoys the following properties [5-7]:

— The plastic strain rates $\dot{\varepsilon}^p$ are uniquely determined in $V \times (0, \Delta t)$.

— The residual stresses $\boldsymbol{\sigma}^R$ are uniquely determined only within the plastic region $V_p \subseteq V$ where $\dot{\varepsilon}^p$ is not identically vanishing.

— The steady-state solution does not depend on the initial plastic strain and on the load cycle origin.

On the basis of the above properties, $\boldsymbol{\sigma}^R$ can be written as

$$\boldsymbol{\sigma}^R(\boldsymbol{x}, t) = \boldsymbol{\rho}(\boldsymbol{x}) + \boldsymbol{\tau}(\boldsymbol{x}, t) \tag{7}$$

where $\boldsymbol{\rho}$ denotes the *post-transient* (PT) residual stresses, namely the residual stresses associated with the plastic strains produced in the entire transient response phase and thus existing in the body at the stabilization time, whereas the $\boldsymbol{\tau}$ are the residual stresses associated with the increments of plastic strain in the steady cycle. Possibly with the aid of a suitable change of the load cycle origin, the $\boldsymbol{\tau}$ turn out to be uniquely determined in $V \times (0, \Delta t)$, the $\boldsymbol{\rho}$ in V_p [6]. By eq. (7), the stresses $\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^R$ of the first eq. (3) can be cast as

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \widehat{\boldsymbol{\sigma}}(\boldsymbol{x}, t) + \boldsymbol{\tau}(\boldsymbol{x}, t) \tag{8}$$

where $\widehat{\boldsymbol{\sigma}}$ denotes the *post-transient stresses* defined as

$$\widehat{\boldsymbol{\sigma}}(\boldsymbol{x}, t) = \boldsymbol{\sigma}^E(\boldsymbol{x}, t) + \boldsymbol{\rho}(\boldsymbol{x}). \tag{9}$$

3. The minimum total plastic over-potential problem

The PT residual stresses $\boldsymbol{\rho}$ not only are representative of the plastic strains produced in the transient response phase, but also characterize the state of the structure at the stabilization time such that they can be taken as a basis to predict the type of steady-state response.

At every $\boldsymbol{x} \in V$, the PT stresses $\widehat{\boldsymbol{\sigma}}$ define a stress path $\widehat{S} = \{\widehat{\boldsymbol{\sigma}}(\boldsymbol{x}, t), 0 \leqslant t \leqslant \Delta t\}$, a closed line travelled by the stress point $\widehat{\boldsymbol{\sigma}}$ at $\boldsymbol{x}$ during the cycle. $\widehat{S}$ is the result of a strain process developing in the transient response phase, in virtue of which the elastic stress path S^E associated with $\boldsymbol{\sigma}^E$ is translated to $\widehat{S}$ through the stress translation $\boldsymbol{\rho}$. Obviously $\widehat{S}$ turns out to be inside the yield surface everywhere in V only in case of elastic shakedown; otherwise, $\widehat{S}$ exceeds the yield surface at least somewhere in V, with yield violations that can be locally measured by a nonnegative scalar y defined in terms of the peak value of the plastic potential ϕ all along $\widehat{S}$. Under the reasonable hypothesis that S^E is shaped as a polygonal line with vertices $\boldsymbol{\sigma}^E_m = \boldsymbol{\sigma}^E(\boldsymbol{x}, t_m)$, $m \in I(e) = \{1, 2, \ldots, e\}$, y is given by:

$$y = \max\left\{0, \max_{m \in I(e)} \phi(\widehat{\boldsymbol{\sigma}}_m)\right\} \tag{10}$$

where $\widehat{\boldsymbol{\sigma}}_m = \boldsymbol{\sigma}^E_m + \boldsymbol{\rho}$. The scalar y is referred to as the PT *plastic over-potential* (POP) at $\boldsymbol{x}$, whereas the integral

$$Y = \int_V y \, dV \tag{11}$$

is the *total* PT-POP. It is also assumed that the t_m are the same for all points $\boldsymbol{x} \in V$.

The *initial* POP is introduced making reference to S^E instead of $\widehat{S}$, *i.e.*

$$y_0 = \max\left\{0, \max_{m \in I(e)} \phi(\boldsymbol{\sigma}_m^E)\right\}, \tag{12}$$

$$Y_0 = \int_V y_0 \, dV. \tag{13}$$

Using arguments as in [7], *a principle of nonincreasing* POP can be invoked to state that $Y_0 \geqslant Y$, what is equivalent to saying that *the total amount of yield violations cannot increase during the transient response phase*. Though a rigorous proof of the latter statement has not been provided yet, there is however enough physical evidence to admit its validity without reserve and thus to use the aforementioned principle with confidence.

The above principle of nonincreasing POP suggests the idea that the PT residual stresses, $\boldsymbol{\rho}$, may be provided by the solution of the *minimum total POP problem* cast as follows

$$\begin{cases} \min\limits_{(y,\rho)} Y = \int_V y \, dV, \qquad \text{subject to:} \\ \phi(\boldsymbol{\sigma}_m^E + \boldsymbol{\rho}) \leqslant y, \quad y \geqslant 0 \quad \text{in } V, \ \text{all } m \text{ in } I(e), \\ C^T \boldsymbol{\rho} = \boldsymbol{0} \quad \text{in } V, \qquad \boldsymbol{\rho} \cdot \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \partial_t V, \end{cases} \tag{14}$$

where y and $\boldsymbol{\rho}$ are trial scalar and tensor fields belonging to appropriate function spaces.

The mechanical implications of problem (14) can be shown by the related Euler-Lagrange equations, which read [7]:

$$\phi(\widehat{\boldsymbol{\sigma}}_m) \leqslant y, \quad l_m \geqslant 0, \quad l_m[\phi(\widehat{\boldsymbol{\sigma}}_m) - y] = 0 \quad \text{all } m \in I(e), \quad \text{in } V, \tag{15a}$$

$$\widehat{\boldsymbol{\sigma}}_m = \boldsymbol{\sigma}_m^E + \boldsymbol{\rho}, \quad \boldsymbol{p}_m = l_m \left.\frac{\partial \phi}{\partial \widehat{\boldsymbol{\sigma}}}\right|_{\widehat{\boldsymbol{\sigma}} = \widehat{\boldsymbol{\sigma}}_m} \quad \text{all } m \in I(e), \quad \textit{in } V, \tag{15b}$$

$$\Delta l \equiv \sum_{m=1}^{e} l_m \leqslant 1, \quad y \geqslant 0, \quad y(\Delta l - 1) = 0 \quad \text{in } V, \tag{16}$$

$$\Delta \boldsymbol{p} \equiv \sum_{m=1}^{e} \boldsymbol{p}_m = C\boldsymbol{v} \ \text{in } V, \qquad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \partial_u V, \tag{17}$$

$$C^T \boldsymbol{\rho} = \boldsymbol{0} \ \text{in } V, \qquad \boldsymbol{\rho} \cdot \boldsymbol{n} = \boldsymbol{0} \ \text{on } \partial_t V. \tag{18}$$

It was proved in [7] that the solution $(y, \boldsymbol{\rho})$ to problem (14) provides the POP and the PT residual stresses within the region $V_p \subseteq V$ where plastic flow occurs in the steady cycle, but the proof is not reported here for simplicity.

From equations (15), it transpires that the PT stresses obey fictitious plastic flow laws in which $\phi(\widehat{\boldsymbol{\sigma}}) - y \leqslant 0$ is the yield function and y has the role of fictitious internal variable related to a sort of isotropic hardening by which the actual yield surface is homothetically expanded. Equations (16) state that, at a point $\boldsymbol{x} \in V$ where $y > 0$, the PT stress path $\widehat{S}$ touches the expanded yield surface at least at one stress point. $\Delta \boldsymbol{p}$ is the fictitious ratchet strain, compatible with the displacements $\boldsymbol{v}$, whereas $\boldsymbol{\rho}$ is a self-stress field as required by the constraints of problem (14). It is also remarked that, at the solution, the body finds itself in a limit state of *fictitious elastic shakedown* with respect to the expanded yield surfaces.

It is worth noting that, if the minimum value of Y of problem (14) vanishes, and thus $y = 0$ in the whole V, then elastic shakedown occurs under the given loads as it is obvious from the first eq. (15*a*) and by Melan's theorem. On the contrary, if the minimum value of Y is positive, the load causes inadaptation. In the latter case, a particularly significant circumstance arises when this minimum value Y turns out to be an *absolute minimum*, that is, when the solution $(y, \boldsymbol{\rho})$ to problem (14) identifies with the solution to problem (14) modified by treating $\boldsymbol{\rho}$ as a *fully free stress variable, i.e.*

$$(19)\qquad \begin{cases} \min\limits_{(y,\rho)} Y = \int\limits_V y\, dV, \qquad \text{subject to:} \\ \phi(\boldsymbol{\sigma}_m^E + \boldsymbol{\rho}) \leqslant y, \quad y \geqslant 0 \quad \text{in } V, \ \text{all } m \text{ in } I(e). \end{cases}$$

The latter problem is particularly simple since it can be solved with mutually independent *local* operations, one at every point $\boldsymbol{x} \in V$, namely

$$(20)\qquad \min_{(y,\rho)} y \quad \text{subject to:} \quad \phi(\boldsymbol{\sigma}_m^E + \boldsymbol{\rho}) \leqslant y, \quad y \geqslant 0 \quad \text{all } m \in I(e).$$

The meaning of problem (20) is illustrated in fig. 1. The stress path S^E at $\boldsymbol{x}$ is translated by a $\boldsymbol{\rho}$ such that the POP y associated with the translated stress path $\widehat{S} = \{\boldsymbol{\sigma}^E(\boldsymbol{x}, t) + \boldsymbol{\rho}, 0 \leqslant t \leqslant \Delta t\}$, that is $\widehat{S} = S^E + \boldsymbol{\rho}$ in short, be minimized. When this con-

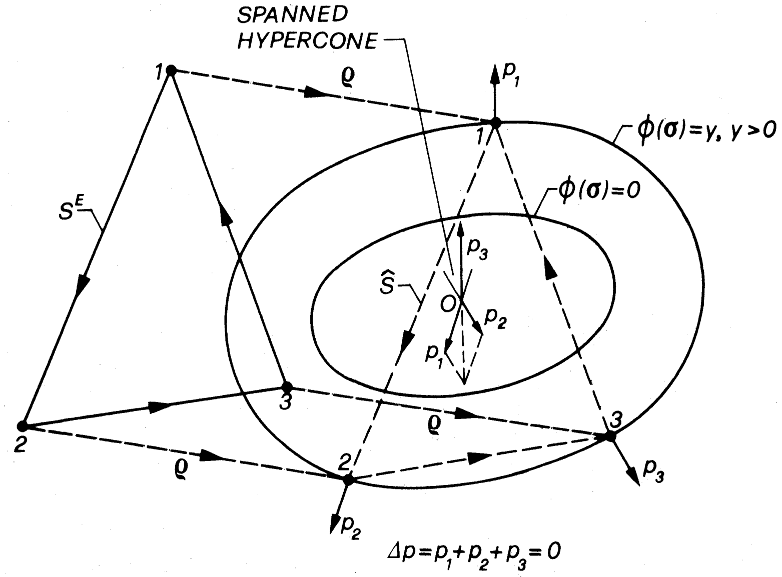

Fig. 1. – Elastic stress path and related PT stress path in a neutral configuration.

dition is reached, $\widehat{S}$ finds itself in a *neutral configuration*, characterized by the fact that the corresponding fictitious ratchet strain is vanishing. The latter condition agrees with the Kuhn-Tucker conditions of problem (20), which can be easily found to be eqs. (15*a, b*) and (16) and in addition the condition $\Delta \boldsymbol{p} = \mathbf{0}$, all written at $\boldsymbol{x}$. The vectors $\boldsymbol{p}_m$ at $\boldsymbol{x}$ constitute a linearly and positively dependent vector set, that is, there exist nonnegative coefficients l_m, not all vanishing, such that $l_1 \boldsymbol{p}_1 + l_2 \boldsymbol{p}_2 + \ldots + l_e \boldsymbol{p}_e = \mathbf{0}$; if drawn from the stress origin, the $\boldsymbol{p}_m$ span a convex hypercone whose halves contain at least one of them.

The stresses $\boldsymbol{\rho}$ of problem (20) are called *neutralizing stresses*, whereas those of problem (19) are called *minimizing self-stresses*.

Problem (20) always admits a solution $(y, \boldsymbol{\rho})$ with $\boldsymbol{\rho}$ unique at $\boldsymbol{x}$ if there $y > 0$, but not always the resulting field $\boldsymbol{\rho}(\boldsymbol{x})$ turns out to be a self-stress field. A *theorem of plastic shakedown* has been proven [7] stating that:

For a given structure under a given cyclic load exceeding the elastic shakedown limit, a *sufficient* condition of plastic shakedown is that there exists some time-independent self-stress field $\boldsymbol{\rho}$ such as on superposition of it with the elastic stress $\boldsymbol{\sigma}^E$ the resulting stress $\widehat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^E + \boldsymbol{\rho}$ is characterized by a stress path $\widehat{S}$ which finds itself in a neutral configuration everywhere it exceeds the yield surface.

As a consequence of this theorem, if the minimum POP is an absolute minimum, or, in other words, if the neutralizing stresses constitute a self-stress field, then plastic shakedown occurs with alternating plastic flow taking place within some region $V_F \subseteq V_Y$, V_Y being the region of V where $y > 0$. If, on the contrary, the neutralizing stresses are in equilibrium with some steady load P_1^0, then plastic shakedown occurs under the load $P(t) + P_1^0$.

Alternative mixed forms of problems (14) and (20) are here presented. Problem (14) is equivalent to

$$\text{(21)} \qquad \begin{cases} \min\limits_{(\rho)} \max\limits_{(l_m)} \sum\limits_{m=1}^{e} \int\limits_V l_m \phi(\boldsymbol{\sigma}_m^E + \boldsymbol{\rho})\, dV \ \text{ subject to:} \\ l_m \geqslant 0 \quad \text{all } m \in I(e), \quad \sum\limits_{m=1}^{e} l_m \leqslant 1, \ \text{ in } V, \\ C^T \boldsymbol{\rho} = \mathbf{0} \ \text{ in V}, \qquad \boldsymbol{\rho} \cdot \boldsymbol{n} = \mathbf{0} \ \text{ on } \partial_t V, \end{cases}$$

whereas problem (20) is equivalent to

$$\text{(22)} \qquad \begin{cases} \min\limits_{(\rho)} \max\limits_{(l_m)} \sum\limits_{m=1}^{e} l_m \phi(\boldsymbol{\sigma}_m^E + \boldsymbol{\rho}) \ \text{ subject to:} \\ l_m \geqslant 0 \quad \text{all } m \in I(e), \quad \sum\limits_{m=1}^{e} l_m \leqslant 1, \end{cases}$$

the latter being written at every point $\boldsymbol{x} \in V$. It is easy to shown that the Euler-Lagrange equations of problem (21) coincide with eqs. (15)-(18), and that the Kuhn-

Tuker conditions of problem (22) coincide with those of problem (20). Dual kinematical formulations are not considered here for brevity.

Finite element modelling is necessary in order to apply the above variational formulations to engineering structures. If in addition a piecewise linearization of the yield surface is operated [4], all these problems take on the shape of linear programming problems. This point will be addressed elsewhere.

4. The time length of the transient response phase

As stated before, the transient response phase constitutes a strain process producing the transfer of the elastic stress path S^E to the PT stress path $\widehat{S}$ while obeying the principle of nonincreasing total POP, namely $Y_0 \geqslant Y$. The above process, which lasts a certain number of cycles dependent on the initial conditions, can in more details be described as follows [7]. At the end of the first cycle, the increment of plastic strain, $\Delta\boldsymbol{\varepsilon}_1^p$, is likely incompatible and is thus accompanied by a nonvanishing residual stress $\Delta\boldsymbol{\rho}_1$, such that the initial stress path S^E is translated to $\widehat{S}_1 = S^E + \Delta\boldsymbol{\rho}_1$ with the POP $y_1 \not\equiv y_0$, but $Y_0 \geqslant Y_1$. At the end of the second cycle, the additional plastic strain $\Delta\boldsymbol{\varepsilon}_2^p$ is produced, which is likely incompatible too and thus is accompanied by a nonvanishing self- stress $\Delta\boldsymbol{\rho}_2$, with a further translation of the stress path from $\widehat{S}_1$ to $\widehat{S}_2 = \widehat{S}_1 + \Delta\boldsymbol{\rho}_2$ and a consequent change of the POP, from y_1 to y_2, but $Y_1 \geqslant Y_2$. And so forth till the c-th cycle. Then, at every subsequent cycle, only compatible plastic strain increments are produced and thus $\widehat{S}_c$ is the PT stress path and y_c its PT-POP, *i.e.* $\widehat{S}_c = \widehat{S}$ and $y_c = y$. The above picture is quite reasonable and has only required the application of the principle of nonincreasing POP to every individual load cycle.

Now, let one address the following series of problems:

$$(23)\qquad \begin{cases} \min\limits_{(y_\nu,\,\Delta\rho_\nu)} L_\nu = \int\limits_V y_\nu\, dV + \dfrac{1}{2}\int\limits_V \Delta\boldsymbol{\rho}_\nu : \boldsymbol{A} : \Delta\boldsymbol{\rho}_\nu\, dV, \quad \text{subject to:} \\ \phi(\widehat{\boldsymbol{\sigma}}_m^{(\nu-1)} + \Delta\boldsymbol{\rho}_\nu) \leqslant y_\nu, \quad y_\nu \geqslant 0 \quad \text{in } V, \quad \text{all } m \in I(e), \\ C^T \Delta\boldsymbol{\rho}_\nu = \boldsymbol{0} \text{ in V}, \qquad \Delta\boldsymbol{\rho}_\nu \cdot \boldsymbol{n} = \boldsymbol{0} \text{ on } \partial_t V, \end{cases}$$

where the label $\nu = 1, 2, \ldots$ refers to the cycle sequence. L_ν is defined as the sum of the total POP at the end of the ν-th cycle with the elastic strain energy associated with the self-stress increment in the same cycle. The series of problems (23) must be solved sequentially starting with $\nu = 1$, in which case $\widehat{\boldsymbol{\sigma}}_m^{(0)} = \boldsymbol{\sigma}_m^E$. Then, setting $\widehat{\boldsymbol{\sigma}}_m^{(1)} = \widehat{\boldsymbol{\sigma}}_m^{(0)} + \Delta\boldsymbol{\rho}_1$, with $\Delta\boldsymbol{\rho}_1$ pertaining to the first problem already solved, the second problem is addressed; and so on, till the $(n+1)$-th problem whose solution is characterized by $\Delta\boldsymbol{\rho}_{n+1} = \boldsymbol{0}$ in V. Then, the PT residual stresses $\boldsymbol{\rho}$ and the PT-POP y are obtained as

$$(24)\qquad \left.\begin{aligned} \boldsymbol{\rho} &= \boldsymbol{\rho}_{n+1} = \boldsymbol{\rho}_n = \Delta\boldsymbol{\rho}_1 + \Delta\boldsymbol{\rho}_2 + \ldots + \Delta\boldsymbol{\rho}_n, \\ y &= y_{n+1} = y_n, \end{aligned}\right\} \quad \text{in } V.$$

For a better understanding of the above problems, the related Euler-Lagrange equations are written [11]. Let one consider the augmented functionals

$$\tilde{L}_\nu = \int_V y_\nu dV + \frac{1}{2}\int_V \Delta\boldsymbol{\rho}_\nu : \boldsymbol{A} : \Delta\boldsymbol{\rho}_\nu dV + \sum_{m=1}^{e} \int_V l_m^{(\nu)} [\phi(\hat{\boldsymbol{\sigma}}_m^{(\nu-1)} + \Delta\boldsymbol{\rho}_\nu) - y_\nu] dV - \int_V \boldsymbol{v}_\nu \cdot C^T \Delta\boldsymbol{\rho}_\nu dV + \int_{\partial_t V} \boldsymbol{v}_\nu \cdot \Delta\boldsymbol{\rho}_\nu \cdot \boldsymbol{n}\, d\partial_t V, \tag{25}$$

where $l_m^{(\nu)}$ and $\boldsymbol{v}_\nu$ are Lagrange multipliers, and let one write the first variation of $\tilde{L}_\nu$. On application of the divergence theorem and after some easy manipulations, the first variation of $\tilde{L}_\nu$ reads:

$$\delta\tilde{L}_\nu = \int_V \delta y_\nu \left[1 - \sum_{m=1}^{e} l_m^{(\nu)}\right] dV + \int_V \delta\Delta\boldsymbol{\rho}_\nu : \left[\boldsymbol{A} : \Delta\boldsymbol{\rho}_\nu + \sum_{m=1}^{e} l_m^{(\nu)} \frac{\partial\phi}{\partial\hat{\boldsymbol{\sigma}}_m^{(\nu)}} - C\boldsymbol{v}_\nu\right] dV + \sum_{m=1}^{e} \int_V \delta l_m^{(\nu)} [\phi(\hat{\boldsymbol{\sigma}}_m^{(\nu-1)} + \Delta\boldsymbol{\rho}_\nu) - y_\nu] dV + \int_{\partial_u V} \boldsymbol{v}_\nu \cdot \delta\Delta\, \boldsymbol{\rho}_\nu \cdot \boldsymbol{n}\, d\partial_u V + \int_V \delta\boldsymbol{v}_\nu \cdot \Delta\boldsymbol{\rho}_\nu \cdot \boldsymbol{n}\, dV - \int_{\partial_t V} \delta\boldsymbol{v}_\nu \cdot \Delta\boldsymbol{\rho}_\nu \cdot \boldsymbol{n}\, d\partial_t V, \tag{26}$$

such that the desired Euler-Lagrange equations are as follows:

$$\phi(\hat{\boldsymbol{\sigma}}_m^{(\nu)}) \leqslant y_\nu, \quad l_m^{(\nu)} \geqslant 0, \quad l_m^{(\nu)} [\phi(\hat{\boldsymbol{\sigma}}_m^{(\nu)}) - y_\nu] = 0 \quad \text{in } V, \ \text{all } m \in I(e), \tag{27a}$$

$$\hat{\boldsymbol{\sigma}}_m^{(\nu)} = \hat{\boldsymbol{\sigma}}_m^{(\nu-1)} + \Delta\boldsymbol{\rho}_\nu, \qquad \Delta\boldsymbol{p}_\nu = \sum_{m=1}^{e} l_m^{(\nu)} \frac{\partial\phi}{\partial\hat{\boldsymbol{\sigma}}_m^{(\nu)}}, \tag{27b}$$

$$\Delta l^{(\nu)} \equiv \sum_{m=1}^{e} l_m^{(\nu)} \leqslant 1, \quad y_\nu \geqslant 0, \quad y_\nu(\Delta l^{(\nu)} - 1) = 0 \quad \text{in } V, \tag{28}$$

$$C\boldsymbol{v}_\nu = \boldsymbol{A} : \Delta\boldsymbol{\rho}_\nu + \Delta\boldsymbol{p}_\nu \quad \text{in } V, \ \boldsymbol{v}_\nu = \boldsymbol{0} \ \text{on } \partial_u V, \tag{29}$$

$$C^T \Delta\boldsymbol{\rho}_\nu = \boldsymbol{0} \ \text{in } V, \qquad \Delta\boldsymbol{\rho}_\nu \cdot \boldsymbol{n} = \boldsymbol{0} \ \text{on } \partial_t V, \tag{30}$$

all expressing necessary conditions for problem (23)–but also sufficient conditions due to the convexity of the problem. Observing eqs. (27*a*) and (29), one can realize that the Lagrange multiplier $l_m^{(\nu)}$ have the meanings of plastic multipliers in the fictitious plastic flow laws, and $\boldsymbol{v}_\nu$ of related residual displacements.

Problems (23) admit each a unique solution. To prove that, let the label ν be dropped for simplicity. Assuming, by absurdity, the existence of two different solutions to eqs. (27)-(30), say $(y', \Delta\boldsymbol{\rho}', l_m')$ and $(y'', \Delta\boldsymbol{\rho}'', l_m'')$, by the extended Druker's postulate we can write

$$(\Delta\boldsymbol{\rho}' - \Delta\boldsymbol{\rho}'') : (\boldsymbol{p}_m' - \boldsymbol{p}_m'') \geqslant (y' - y'')(l_m' - l_m''), \qquad \text{in } V, \ \text{all } m \in I(e). \tag{31}$$

Then, summing with respect to $m \in I(e)$ and with an integration over V, one obtains

$$\int_V (\Delta\boldsymbol{\rho}' - \Delta\boldsymbol{\rho}'') : (\Delta\boldsymbol{p}' - \Delta\boldsymbol{p}'')\, dV \geqslant \int_V (y' - y'')(\Delta l' - \Delta l'')\, dV \tag{32}$$

which, by eq. (29) and applying the virtual work principle, becomes

$$(33) \qquad -\int_V (\Delta\boldsymbol{\rho}' - \Delta\boldsymbol{\rho}'') : \boldsymbol{A} : (\Delta\boldsymbol{\rho}' - \Delta\boldsymbol{\rho}'')\, dV \geq \int_V (y' - y'')(\Delta l' - \Delta l'')\, dV .$$

But the r.h.s. of the latter equation is nonnegative, because in fact by eq. (28) one can write

$$(34) \qquad \int_V (y' - y'')(\Delta l' - \Delta l'')\, dV = \int_V (y' + y'' - y'\,\Delta l'' - y''\,\Delta l')\, dV \geq 0$$

such that eq. (33) becomes

$$(35) \qquad \int_V (\Delta\boldsymbol{\rho}' - \Delta\boldsymbol{\rho}'') : \boldsymbol{A} : (\Delta\boldsymbol{\rho}' - \Delta\boldsymbol{\rho}'')\, dV \leq 0 .$$

Since this integral is essentially positive as far as $\Delta\boldsymbol{\rho}' \neq \Delta\boldsymbol{\rho}''$ even in a small portion of V, we have that $\Delta\boldsymbol{\rho}' = \Delta\boldsymbol{\rho}''$ everywhere in V, and therefore $y' = y''$ in V, as stated.

Comparison of equations (27)-(30) with eqs. (15)-(18) reveals that, if in the $(n+1)$-th problem $\Delta\boldsymbol{\rho}_{n+1} = \mathbf{0}$ everywhere in V, the latter problem coincides with problem (14), such that eqs. (24) hold true.

Also, it follows that the series of problems (23) describes an n-cycle *fictitious transient process* leading the elastic stress path S^E to the PT stress path $\widehat{S}$ through a trajectory (not coincident with the real one, in general) along which the principle of nonincreasing total POP is obeyed. In fact, it is possible to show that the (fictitious) POP resulting from problems (23) is monotonically decreasing, just like for the real process. To this purpose, let the solutions to two subsequent problems be considered. Denoting by $\Delta\boldsymbol{\rho}_\nu$, $\boldsymbol{p}_m^{(\nu)}$, $l_m^{(\nu)}$, y_ν the solution pertaining to the ν-th problem, the following inequality holds, namely

$$(36) \qquad (\widehat{\boldsymbol{\sigma}}_m^{(\nu)} - \widehat{\boldsymbol{\sigma}}_m^{(\nu-1)}) : \boldsymbol{p}_m^{(\nu)} \geq (y_\nu - y_{\nu-1})\, l_m^{(\nu)} \quad \text{in } V, \quad \text{all } m \in I(e)$$

that is, since $\widehat{\boldsymbol{\sigma}}_m^{(\nu)} - \widehat{\boldsymbol{\sigma}}_m^{(\nu-1)} = \Delta\boldsymbol{\rho}_\nu$ and summing with respect to $m \in I(e)$,

$$(37) \qquad \Delta\boldsymbol{\rho}_\nu : \Delta\boldsymbol{p}^{(\nu)} \geq (y_\nu - y_{\nu-1})\, \Delta l^{(\nu)} \quad \text{in } V .$$

Observing that $y_\nu\, \Delta l^{(\nu)} = y_\nu$ and that $y_{\nu-1}\, \Delta l^{(\nu)} \leq y_{\nu-1}$, it follows, after integration over V,

$$(38) \qquad \int_V \Delta\boldsymbol{\rho}_\nu : \Delta\boldsymbol{p}^{(\nu)}\, dV \geq \int_V (y_\nu - y_{\nu-1})\, dV = Y_\nu - Y_{\nu-1} .$$

Since $\Delta\boldsymbol{p}^{(\nu)} = C\,\boldsymbol{v}_\nu - \boldsymbol{A} : \Delta\boldsymbol{\rho}_\nu$, applying the virtual work principle yields the inequality

$$(39) \qquad Y_{\nu-1} - Y_\nu \geq \int_V \Delta\boldsymbol{\rho}_\nu : \boldsymbol{A} : \Delta\boldsymbol{\rho}_\nu\, dV ,$$

that is, $Y_{\nu-1} > Y_\nu$ as far as $\Delta\boldsymbol{\rho}_\nu \neq \mathbf{0}$ even in a small portion of V. Since Y_ν is nonnegative, it must stop decreasing at a certain cycle $n+1$ at which $\Delta\boldsymbol{\rho}_{n+1} = \mathbf{0}$ in the whole V. So, the statement is proven.

In view of the basic similarities of the real and the fictitious transient processes, it seems quite reasonable to consider n (the time length of the fictitious transient process) as an estimation of c (the time length of the real transient process). This is for the moment just a conjecture, useful from the practical engineering standpoint, but to be properly justified.

Finally it is to remark that problems (23) could have been given different, but equivalent, shapes by writing $y_\nu = y_{\nu-1} + \Delta y_\nu$, with $y_{\nu-1}$ fixed, namely

$$\text{(40)} \quad \begin{cases} \min\limits_{(\Delta y_\nu,\, \Delta\rho_\nu)} L_\nu = \int\limits_V \Delta y_\nu \, dV + \dfrac{1}{2} \int\limits_V \Delta\boldsymbol{\rho}_\nu : \boldsymbol{A} : \Delta\boldsymbol{\rho}_\nu \, dV, \quad \text{subject to:} \\ \phi(\hat{\boldsymbol{\sigma}}_m^{(\nu-1)} + \Delta\boldsymbol{\rho}_\nu) \leqslant y_{\nu-1} + \Delta y_\nu, \quad y_{\nu-1} + \Delta y_\nu \geqslant 0 \text{ in } V, \quad \text{all } m \in I(e), \\ C^T \Delta\boldsymbol{\rho}_\nu = \mathbf{0} \text{ in } V, \quad \Delta\boldsymbol{\rho}_\nu \cdot \boldsymbol{n} = \mathbf{0} \text{ on } \partial_t V, \end{cases}$$

where obviously $y_{\nu-1}$ is the POP obtained in the previous step and equals y_0 for $\nu = 1$.

5. Conclusions

Methods for directly evaluating the so-called *post-transient* (PT) residual stresses, as well as for the estimation of the number of cycles spanned by the transient response, have been presented. These pieces of information may be useful for design purposes. The evaluation of the residual stresses is based on variational formulations in the shape of *minimum total plastic over-potential problem*, or of other alternatives. Whenever this minimum is an *absolute minimum*, then a plastic shakedown theorem given elsewhere can be invoked to state that plastic shakedown occurs under the given loads. The estimation of the number of cycles of the transient response phase is based on a procedure envisaged to evaluate the number of cycles spanned by a similar fictitious transient process. This procedure is parhaps rather cumbersome; however, it shows some conceptual value as far as the fictitious process is concerned, and discovering how much the latter process may tell about the real one is surely a notable research issue to pursue.

Methods of numerical implementation, *e.g.* by finite elements, have been considered out of the scope of this paper. A class of discrete (or discretized) structures with piecewise linear yield surfaces is of particular interest because correspondingly the problem discussed here take on the form of linear programming problems. The author believes that this paper adds a better understanding of the behaviour of elastic-perfectly plastic structures under the action of cyclic loads, and this in turn may be helpful to better understand what it is going to occur with less simple constitutive models.

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