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The exceptional sets for functions of the Bergman space in the unit ball

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Matematica. — *The exceptional sets for functions of the Bergman space in the unit ball.* Nota di Piotr Jakóbczak, presentata (*) dal Socio E. Vesentini.

Abstract. — Let D be a domain in $\boldsymbol{C}^2$. Given $w \in \boldsymbol{C}$, set $D_w = \{z \in \boldsymbol{C} \mid (z, w) \in D\}$. If f is a holomorphic and square-integrable function in D, then the set $E(D, f)$ of all w such that $f(\cdot, w)$ is not square-integrable in D_w has measure zero. We call this set the exceptional set for f. In this *Note* we prove that whenever $0 < r < 1$, there exists a holomorphic square-integrable function f in the unit ball B in $\boldsymbol{C}^2$ such that $E(B, f)$ is the circle $C(0, r) = \{z \in \boldsymbol{C} \mid |z| = r\}$.

Key words: Bergman space; Hartogs domain; Exceptional sets.

Riassunto. — *Gli insiemi eccezionali per funzioni dello spazio di Bergman nel disco unitario.* Sia D un dominio in $\boldsymbol{C}^2$. Per ogni $w \in \boldsymbol{C}$ sia $D_w = \{z \in \boldsymbol{C} \mid (z, w) \in D\}$. Se $f \in L^2$ è olomorfa in D, allora l'insieme $E(D, f)$ dei w per cui $f(\cdot, w)$ non è in $L^2(D_w)$ ha misura nulla. $E(D, f)$ denota *l'insieme eccezionale* per f. In questa *Nota* si dimostra che per ogni r, essendo $0 < r < 1$, esiste una funzione $f \in L^2$, olomorfa nel disco B di $\boldsymbol{C}^2$, per cui $E(B, f) = \{z \in \boldsymbol{C} \mid |z| = r\}$.

1. Introduction

In [1] we investigated the following problem: Let D be an open set in $\boldsymbol{C}^{n+m}$. Denote by $L^2H(D)$ the space of all functions in $L^2(D)$ (with respect to the Lebesgue measure) which are holomorphic in D. Given $w \in \boldsymbol{C}^m$, let $D_w = D \cap (\boldsymbol{C}^n \times \{w\})$, and let $p(D_w)$ be the projection of D_w onto the first coordinate space, $p(D_w) = \{z \in \boldsymbol{C}^n \mid (z, w) \in D\}$. Then given $f \in L^2H(D)$, the function $f|_{D_w}$ can be considered as the function holomorphic on the (possibly empty) open set $p(D_w)$ in $\boldsymbol{C}^n$. Let $E(D, f)$ denote the set of all $w \in \boldsymbol{C}^m$ such that $p(D_w)$ is not empty and $f|_{D_w}$ is not L^2-integrable with respect to the Lebesgue measure on $p(D_w)$. By Fubini's theorem, $E(D, f)$ is a set of Lebesgue measure zero in $\boldsymbol{C}^m$. What further properties has the set $E(D, f)$?

We have showed in [1] that if D is a Hartogs domain in $\boldsymbol{C}^2$ (we assume here that $n = m = 1$), then $E(D, f)$ is a G_δ-set, and for every G_δ-set $E \subseteq \boldsymbol{C}$ of Lebesgue measure zero there exists a Hartogs domain $D \subseteq \boldsymbol{C}^2$ (possibly with strange boundary) and a function $f \in L^2H(D)$ such that $E = E(D, f)$. If we assume that a Hartogs domain $D \subseteq \boldsymbol{C}^2$ is also a convex domain with smooth boundary, we have constructed an example for which $E(D, f)$ is a boundary of a rectangle, or a set dense in a rectangle, containing its boundary.

In this *Note* we consider the case of the unit ball B in $\boldsymbol{C}^2$. We show the following theorem:

Theorem 1. *Given r with $0 < r < 1$, there exists a function $f \in L^2H(B)$ such that $E(B, f)$ is the circle $C(0, r) = \{z \in \boldsymbol{C} \mid |z| = r\}$.*

(The question of the existence of such function was stated in [1].)

(*) Nella seduta del 12 dicembre 1992.

2. The exceptional sets for L^2H-functions in the unit ball in C^2

In this section we prove Theorem 1. Call the variables in C^2 by (z, w). Denote by U the unit disc in C. Let $g(z) = \sum_{n=1}^{\infty} a_n z^n$ be holomorphic in U. Set $G(z, w) = g(z)$, $(z, w) \in U \times C := T$. Then G is holomorphic in T. Let ϕ be any unitary mapping of C^2 onto itself. Then $G \circ \phi^{-1}$ is holomorphic in the set $\phi(T)$, containing the unit ball B. Fix r with $0 < r < 1$, and precise $\phi(z, w) = ((1 - r^2)^{1/2} z - rw, rz + (1 - r^2)^{1/2} w)$. Then

$$\phi^{-1}(z, w) = ((1 - r^2)^{1/2} z + rw, \ -rz + (1 - r^2)^{1/2} w). \tag{1}$$

Suppose that G is so chosen that $G \in L^2H(B)$. Then $G \circ \phi^{-1} \in L^2H(B)$. Let $w \in U$. Recall that

$$p(B_w) = \{z \mid (z, w) \in B\} = \{|z| < (1 - |w|^2)^{1/2}\}. \tag{2}$$

Note that for any $w \in U$ which is not of the form rp for some $p \in \partial U$, the set $\overline{B_w}$ is contained in $\phi(T)$, and so $G \circ \phi^{-1}$ is holomorphic in a neighborhood of $\overline{B_w}$; hence $w \notin E(B, G \circ \phi^{-1})$. On the other hand, if $w = rp$ for some $p \in \partial U$, we have, taking into account (2), (1), and the definition of G,

$$\int_{p(B_{rp})} |G \circ \phi^{-1}(z, rp)|^2 \, dm(z) =$$

$$= \int_{\{|z| < (1 - r^2)^{1/2}\}} |G((1 - r^2)^{1/2} z + pr^2, \ -rz + rp(1 - r^2)^{1/2})|^2 \, dm(z) =$$

$$= \int_{\{|z| < (1 - r^2)^{1/2}\}} |g((1 - r^2)^{1/2} z + pr^2)|^2 \, dm(z) = \int_{D(r^2 p, 1 - r^2)} |g(z)|^2 \, dm(z),$$

where $D(z, \varepsilon)$ denotes the disc with center at z and radius ε in C. $D(r^2 p, 1 - r^2)$ is contained in U and is innerly tangential to ∂U at the point p.

We see from this the following:

Let g be holomorphic in U, and let G, r, and ϕ be as above. Suppose that $G \in L^2(B)$. Let $E(g) = \{p \in \partial U \mid g \notin L^2(D(r^2 p, 1 - r^2))\}$. Then $E(G \circ \phi^{-1}, B) = \{rp \mid p \in E(g)\}$. In particular, if $E(g) = \partial U$, then $E(G \circ \phi^{-1}, B) = C(0, r)$.

Note that if $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in U, and $G(z, w) = g(z)$, then $G \in L^2H(B)$ is and only if

$$\sum_{n=0}^{\infty} (n + 1)^{-2} |a_n|^2 < +\infty$$

(this is well-known and can be proved by direct computation; see *e.g.* [1, 2]). Therefore it follows from the above that Theorem 1 will be proved provided that we construct the function $g(z) = \sum_{n=0}^{\infty} a_n z^n$, holomorphic in U, such that

(*i*) $\sum_{n=0}^{\infty} (n + 1)^{-2} |a_n|^2 < +\infty$, and

(*ii*) for every $p \in \partial U$ and every ρ with $0 < \rho < 1$,

$$\int_{D(\rho p, 1 - \rho)} |g(z)|^2 \, dm(z) = +\infty.$$

(It is well-known that $g \notin L^2(U)$ if and only if

$$\sum_{n=1}^{\infty} (n+1)^{-1} |a_n|^2 = +\infty; \tag{3}$$

in (*ii*) we require that g satisfies stronger condition than (3).)

We begin now with the construction of the function g satisfying (*i*) and (*ii*). The function g will be defined as the lacunary power series

$$g(z) = \sum_{n=0}^{\infty} a_n z^{k_n},$$

where $0 \leqslant k_0 < k_1 < k_2 < \ldots$ (and hence $k_n \geqslant n$ for every positive integer n). Suppose that the numbers $k_0, k_1, \ldots,$ are chosen. Then set

$$a_n = 2^{-n/2}(k_n + 1). \tag{4}$$

If we write g as $g(z) = \sum_{l=0}^{\infty} b_l z^l$, then

$$\sum_{l=0}^{\infty} (l+1)^{-2} |b_l|^2 = \sum_{n=0}^{\infty} (k_n+1)^{-2} |a_n|^2 = \sum_{n=0}^{\infty} 2^{-n} (k_n+1)^2 (k_n+1)^{-2} < +\infty,$$

which proves (*i*). Therefore it remains to choose $k_0, k_1, \ldots$ so that (*ii*) is satisfied.

Fix ρ with $0 < \rho < 1$. Consider an arbitrary point $p \in \partial U$, $p = e^{i\vartheta}$, where $\vartheta \in \boldsymbol{R}$. Let $\Psi_p(r, \phi) = (r\cos(\phi + \vartheta), r\sin(\phi + \vartheta))$, $0 < r < +\infty$, $\vartheta - \pi < \phi < \vartheta + \pi$. Then there exists s with $0 < s < 1$, and $b > 0$, both independent of $p \in \partial U$, such that

$$L_p = \{(r, \phi) \,|\, s < r < 1, \, -(1-r)^{1/2} b^{-1/2} < \phi < (1-r)^{1/2} b^{-1/2}\} \subseteq \subseteq \Psi_p^{-1}(D(\rho p, 1-\rho) \cap \{z \,|\, |z| > s\}) \tag{5}$$

(the set in the left-hand side of (5) is a part of the interior of the parabola given by the equation $r = -b\phi^{-2} + 1$, which is tangent to the line $r = 1$ at the point $(0, 0) = = \Psi_p^{-1}(p)$). Therefore, for every $p \in \partial U$ and every positive integer k, we have by (5)

$$\int_{D(\rho p, 1-\rho) \cap \{z| \, |z| > s\}} |z^k|^2 dm(z) \geqslant \int_{\Psi_p(L_p)} |z^k|^2 \, dm(z) = \tag{6}$$

$$= 2\int_s^1 r\,dr \int_0^{(1-r)^{1/2} b^{-1/2}} r^{2k} d\phi = 2b^{-1/2} \int_s^1 r^{2k+1}(1-r)^{1/2}\, dr.$$

We claim that there exists a positive constant $c = c(s)$ such that for each $k = 1, 2, \ldots,$

$$\int_s^1 r^{2k+1}(1-r)^{1/2} dr \geqslant c\,k^{-3/2}. \tag{7}$$

In fact the above integral is equal to

$$\int_0^s (1-u)^{2k+1} u^{1/2}\, du.$$

By a straightforward calculation we obtain that the function $f_k(u) = (1-u)^{2k+1}u^{1/2}$ attains its maximum on the interval $[0, 1]$ at the point $u = (4k+3)^{-1}$. Therefore, for k so great that $(4k+3)^{-1} \leqslant s$, we have $f_k(u) \geqslant g_k(u)$, where

$$g_k(u) = \begin{cases} (1-(4k+3)^{-1})^{2k+1}u^{1/2}, & 0 \leqslant u \leqslant (4k+3)^{-1} \\ (4k+3)^{-1/2}(1-u)^{2k+1}, & (4k+3)^{-1} \leqslant u \leqslant s. \end{cases}$$

Hence

$$(8) \quad \int_0^s (1-u)^{2k+1}u^{1/2}\,du \geqslant \int_0^{(4k+3)^{-1}} (1-(4k+3)^{-1})^{2k+1}u^{1/2}\,du +$$

$$+ \int_{(4k+3)^{-1}}^{s} (4k+3)^{-1/2}(1-u)^{2k+1}\,du = (2/3)(1-(4k+3)^{-1})^{2k+1}(4k+3)^{-3/2} +$$

$$+ (2k+2)^{-1}(4k+3)^{-1/2}\big(-(1-s)^{2k+2} + (1-(4k+3)^{-1})^{2k+2}\big) =$$

$$= \big((2/3)(4k+3)^{-3/2} + (1/2)(k+1)^{-1}(4k+3)^{-1/2}(1-(4k+3)^{-1})\big) \times$$

$$\times \big((1-(4k+3)^{-1})^{4k+3}\big)^{(2^{-1}-(8k+6)^{-1})} - (1/2)(k+1)^{-1}(4k+3)^{-1/2}(1-s)^{2(k+1)}.$$

Since

$$\big((1-(4k+3)^{-1})^{4k+3}\big)^{(2^{-1}-(8k+6)^{-1})} \to e^{-1/2}$$

and $(1-s)^{2k+1} \to 0$ quickly as $k \to +\infty$, we see that the right-hand side of (8) behaves like $k^{-3/2}$. Therefore the constant $c > 0$ in (7) exists.

Moreover, for every $0 < t < 1$ we have

$$(9) \quad (k+1)^2 \int_{D(0,t)} |z^k|^2\,dm(z) = (k+1)^2 \int_0^{2\pi}\int_0^t r^{2k+1}\,dr = (k+1)^2\pi(k+1)^{-1}t^{2(k+1)} \to 0$$

as $k \to \infty$.

We will show inductively that there exist a sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers and a sequence $\{k_n\}_{n=1}^{\infty}$ of positive integers such that

$$(10) \quad 0 < s_1 < s_2 < \ldots, \quad \lim_{n\to\infty} s_n = 1, \quad 1 - s_1 < \rho, \quad 1 - s_n < 2^{-n},$$

$$n = 1, 2, \ldots, \qquad 1 \leqslant k_1 < k_2 < \ldots,$$

and such that for each $p \in \partial D$, and for each $n = 1, 2, \ldots$,

$$(11) \quad 2^{-n}(k_n+1)^2 \int_{D(\rho p, 1-\rho) \cap \{s_n < |z| < s_{n+1}\}} |z^{k_n}|^2\,dm(z) \geqslant (n+1)^2,$$

$$(12) \quad 2^{-m}(k_m+1)^2 \int_{D(0, s_{n+1})} |z^{k_m}|^2\,dm(z) \leqslant 2^{-m}(k_m+1)^2 \int_{D(0, s_m)} |z^{k_m}|^2\,dm(z) \leqslant 4^{-m}$$

for $m > n$, and

$$(13)\quad 2^{-l}(k_l + 1)^2 \int_{D(\rho p,\, 1-\rho) \cap \{s_{n+1} < |z|\}} |z^{k_l}|^2\, dm(z) \leqslant$$

$$\leqslant 2^{-l}(k_l + 1)^2 \int_{D(\rho p,\, 1-\rho) \cap \{s_{l+1} < |z|\}} |z^{k_l}|^2\, dm(z) \leqslant 4^{-l}$$

for $l \leqslant n$.

Take any s_1 with $0 < s_1 < 1$, $1 - s_1 < \rho$, and such that $1 - s_1 < 2^{-1}$. It follows from (6), (7), and (9) that there exists $c_1 > 0$ and a positive integer k_1 such that for every $p \in \partial U$,

$$(14)\quad 2^{-1}(k_1 + 1)^2 \int_{D(\rho p,\, 1-\rho) \cap \{s_1 < |z|\}} |z^{k_1}|^2\, dm(z) \geqslant$$

$$\geqslant 2^{-1}(k_1 + 1)^2 \cdot c_1 (k_1 + 1)^{-3/2} \geqslant (1 + 2)^2,$$

and

$$2^{-1}(k_1 + 1)^2 \int_{D(0, s_1)} |z^{k_1}|^2\, dm(z) \leqslant 4^{-1}.$$

Then, by (14), there exists s_2 with $s_1 < s_2 < 1$, so near 1, that $1 - s_2 < 2^{-2}$ and for every $p \in \partial D$,

$$2^{-1}(k_1 + 1)^2 \int_{D(\rho p,\, 1-\rho) \cap \{s_2 < |z| < 1\}} |z^{k_1}|^2\, dm(z) \leqslant 4^{-1}$$

and

$$2^{-1}(k_1 + 1)^2 \int_{D(\rho p,\, 1-\rho) \cap \{s_1 < |z| < s_2\}} |z^{k_1}|^2\, dm(z) \geqslant (1 + 1)^2.$$

Assume that we have constructed the numbers $s_1 < s_2 < \ldots < s_{n+1} < 1$ and positive integers $k_1 < k_2 < \ldots < k_n$ such that $1 - s_r > 2^{-r}$, $r = 1, \ldots, n + 1$, and for every $p \in \partial D$

$$2^{-r}(k_r + 1)^2 \int_{D(\rho p,\, 1-\rho) \cap \{s_r < |z| < s_{r+1}\}} |z^{k_r}|^2\, dm(z) \geqslant (r + 1)^2,$$

$$2^{-r}(k_r + 1)^2 \int_{D(\rho p,\, 1-\rho) \cap \{s_{r+1} < |z|\}} |z^{k_r}|^2\, dm(z) \leqslant 4^{-r},$$

and

$$2^{-r}(k_r + 1)^2 \int_{D(0, s_r)} |z^{k_r}|^2\, dm(z) \leqslant 4^{-r},$$

$r = 1,\ldots,n$. Then, by (6), (7), and (9), there exists $c_{n+1} > 0$ and a positive integer $k_{n+1} > k_n$ such that for every $p \in \partial D$

$$2^{-(n+1)}(k_{n+1} + 1)^2 \int_{D(\rho p,\, 1-\rho) \cap \{s_{n+1} < |z|\}} |z^{k_{n+1}}|^2\, dm(z) \geqslant$$

$$\geqslant 2^{-(n+1)}(k_{n+1} + 1)^2 \cdot c_{n+1}(k_{n+1} + 1)^{-3/2} \geqslant ((n + 1) + 2)^2,$$

and

$$2^{-(n+1)}(k_{n+1}+1)^2 \int_{D(0,s_{n+1})} |z^{k_{n+1}}|^2\, dm(z) \leqslant 4^{-(n+1)}.$$

Hence there exists s_{n+2} with $s_{n+1} < s_{n+2} < 1$, so near to 1, that $1 - s_{n+2} > 2^{-(n+2)}$, and for every $p \in \partial D$,

$$2^{-(n+1)}(k_{n+1}+1)^2 \int_{D(\rho p,\, 1-\rho) \cap \{s_{n+1} < |z| < s_{n+2}\}} |z^{k_{n+1}}|^2\, dm(z) \geqslant ((n+1)+1)^2,$$

and

$$2^{-(n+1)}(k_{n+1}+1)^2 \int_{D(\rho p,\, 1-\rho) \cap \{s_{n+2} < |z|\}} |z^{k_{n+1}}|^2\, dm(z) \leqslant 4^{-(n+1)}.$$

The sequences $\{s_n\}$ and $\{k_n\}$, constructed above, satisfy the conditions (10), (11), (12), and (13).

Having constructed $\{k_n\}$, denote now, according to (4),

$$a_n = 2^{-n/2}(k_n + 1), \qquad n = 1, 2, \ldots,$$

and let

$$g(z) = \sum_{n=1}^{\infty} a_n z^{k_n}.$$

Then $a_n^{1/k_n} = 2^{-n/2k_n}(k_n+1)^{1/k_n}$. Since $k_n \to +\infty$, we have $(k_n+1)^{1/k_n} \to 1$, and by construction, $k_n \geqslant n$, $n = 1, 2, \ldots$. Therefore, by Hadamard's test, the function g is holomorphic in the whole disc U. Set $g_n = a_n z^{k_n} = 2^{-n/2}(k_n+1)z^{k_n}$, $n = 1, 2, \ldots$. It follows from (11), (12) and (13) that for every $n = 1, 2, \ldots$, and for every $p \in \partial U$,

$$\text{(15)} \qquad \int_{D(\rho p,\, 1-\rho) \cap \{s_n < |z| < s_{n+1}\}} |g_n|^2\, dm(z) \geqslant (n+1)^2,$$

$$\text{(16)} \qquad \int_{D(0,\, s_{n+1})} |g_m|^2\, dm(z) \leqslant 4^{-m}, \qquad m > n,$$

and

$$\text{(17)} \qquad \int_{D(\rho p,\, 1-\rho) \cap \{s_{l+1} < |z|\}} |g_l|^2\, dm(z) \leqslant 4^{-l}, \qquad 1 \leqslant n.$$

Suppose, contrary to (*ii*), that for some $p \in \partial U$,

$$\int_{D(\rho p,\, 1-\rho)} |g(z)|^2\, dm(z) < +\infty.$$

Then there exists $M > 0$ such that for every $n = 1, 2, \ldots$,

$$\text{(18)} \qquad \int_{D(\rho p,\, 1-\rho) \cap \{s_n < |z| < s_{n+1}\}} |g(z)|^2\, dm(z) \leqslant M.$$

Set $L_n := D(\rho p, 1-\rho) \cap \{s_n < |z| < s_{n+1}\}$. Since $g = \sum_{m=1}^{\infty} g_m$, then in virtue of (15),

(16) and (17) for every $n = 1,2,\ldots,$

$$\|g\|_{L_n} \geqslant \|g_n\|_{L_n} - \sum_{m \neq n} \|g_m\|_{L_n} = \|g_n\|_{L_n} - \sum_{m=1}^{n-1} \|g_m\|_{L_n} - \sum_{m=n+1}^{\infty} \|g_m\|_{L_n} \geqslant$$

$$\geqslant (n+1) - \sum_{m=1}^{n-1} 2^{-m} - \sum_{m=n+1}^{\infty} 2^{-m} \geqslant n.$$

This contradicts (18).

This ends the proof of (*ii*).

References

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