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The theorems of Glicksberg and Hurwitz for holomorphic maps in complex Banach spaces

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Matematica. — *The theorems of Glicksberg and Hurwitz for holomorphic maps in complex Banach spaces.* Nota di KAZIMIER WŁODARCZYK, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — General versions of Glicksberg's theorem concerning zeros of holomorphic maps and of Hurwitz's theorem on sequences of analytic functions is extended to infinite dimensional Banach spaces.

KEY WORDS: Fréchet holomorphic map; Local uniform convergence; Compact open topology.

RIASSUNTO. — *I teoremi di Glicksberg e di Hurwitz per applicazioni olomorfe in spazi di Banach complessi.* I teoremi di Glicksberg e di Hurwitz sugli zeri di funzioni e sulle successioni di funzioni olomorfe sono generalizzati ad applicazioni olomorfe di spazi di Banach complessi di dimensione infinita.

1. INTRODUCTION

E. Rouché's theorem [9], its different statements discovered in $\boldsymbol{C}$ by L. Fejér [5] and I. Glicksberg [3] and their applications (*e.g.* A. Hurwitz's theorem [4], Fundamental Theorem of Algebra and others) are important tools of investigations of zeros and their multiplicities of holomorphic maps in finite-dimensional complex analysis. For details, see *e.g.* J. B. Conway [1], M. Marden [7] and N. G. Lloyd [6].

The main results of this paper are of the above two types (see §2). We first prove a general version of Glicksberg's theorem for (Fréchet-) holomorphic maps in infinite-dimensional complex Banach spaces. We next prove, as an application of this, an extension of Hurwitz's theorem to sequences of holomorphic maps, convergent in the compact-open topologies. In particular, our results imply versions of the theorems of Glicksberg and Hurwitz in $\boldsymbol{C}^n$ (see §5). Our paper is a continuation of [11] and [12].

2. NOTATIONS AND MAIN RESULTS

Let U be a subset of a complex Banach space E. Denote by $\mathcal{C}(U; E)$ the vector space of continuous maps $f\colon U \to E$. When U is open, we define in $\mathcal{C}(U; E)$ the compact-open topology, *i.e.* the Hausdorff locally convex topology defined in $\mathcal{C}(U; E)$ by the seminorms p_K,

$$p_K\colon f \in \mathcal{C}(U; E) \to p_K(f) = \sup_{x \in K} \|f(x)\| \in \boldsymbol{R},$$

as K ranges over the family $\mathcal{K}$ of compact subsets of U.

A map f is called holomorphic if the Fréchet derivative of f at x (denoted

(*) Nella seduta del 12 dicembre 1992.

by $Df(x)$) exists as a bounded complex linear map for each x in the domain of definition of f.

Let U be an open subset of E. It is well known (see *e.g.* L. Nachbin [8, Proposition 4, p. 23]) that the complex vector space $\mathcal{H}(U; E)$ of maps $f\colon U \to E$ holomorphic in U is a closed vector subspace of $\mathcal{C}(U; E)$ for the compact-open topology, complete in the induced topology.

For subsets U of E we use the usual symbols ∂U, $\overline{U}$ and $\operatorname{int}(U)$ to denote the boundary, the closure and the interior of U, respectively.

Definition 2.1. Let Ω be a bounded open subset of E. Let $f \in \mathcal{C}(\overline{\Omega}; E) \cap \mathcal{H}(\Omega; E)$ and let the set $(I - f)(\Omega)$ be contained in a compact subset of E. If $q \in \Omega$ and a neighbourhood V of q in Ω are such that $\overline{V} \subset \Omega$ and $f(q) = 0$ but $f(x) \neq 0$ for all $x \in \overline{V} \setminus \{q\}$, then the positive integer k defined by $k = \deg(f, V, 0)$ is called the multiplicity of zero of f at q.

Remark 2.1. It is worth noticing that if $1 \notin \sigma[D(1 - f)(q)]$, then $k = 1$ and if $1 \in \sigma[D(I - f)(q)]$, then $k \geqslant 2$; here $\sigma[A]$ denotes the spectrum of a compact linear operator A on E.

We shall use these notations and the definition in proving the following Glicksberg-type theorem for E.

Theorem 2.1. *Let Ω be a bounded open subset of a complex Banach space E. Let f, $g \in \mathcal{C}(\overline{\Omega}; E) \cap \mathcal{H}(\Omega; E)$, and let the sets $(I - f)(\Omega)$ and $(I - g)(\Omega)$ be contained in compact subsets of E. If*

$$\|f - g\| < \|f\| + \|g\| \quad \text{on } \partial\Omega, \tag{2.1}$$

then f has finitely many zeros in Ω and, by counting multiplicity, f and g have the same number of zeros in Ω.

We can also use Theorem 2.1 to obtain a Hurwitz-type theorem for E.

Theorem 2.2. *Let Ω be a bounded open subset of a complex Banach space E, let the maps $f_n \in \mathcal{H}(\Omega; E)$ be such that the sets $(I - f_n)(\Omega)$ are contained in compact subsets of E, $n = 1, 2, \ldots$, and let the sequence $\{f_n\}$ converge on Ω in the compact-open topology to some map f. If $\|f(x)\| > 0$ on ∂W where W is some closed subset of Ω such that ∂W is compact and* $\operatorname{int}(W) \neq \emptyset$, *then f has finitely many zeros in* $\operatorname{int}(W)$ *and, for all sufficiently large n, by counting multiplicity, f_n and f have the same number of zeros in* $\operatorname{int}(W)$.

For the special case when $E = \boldsymbol{C}^n$, see §5.

3. Proof of Theorem 2.1

Inequality (2.1) implies

$$\|f\| > 0 \quad \text{and} \quad \|g\| > 0 \quad \text{on } \partial\Omega. \tag{3.1}$$

Let $H(\cdot,\cdot)\colon [0, 1] \times \overline{\Omega} \to E$ denote a map defined by $H(t, x) = x - tF(x) - (1-t)\,G(x)$ where $F = I - f$ and $G = I - g$. Let us observe that

$$\|H(t, x)\| > 0 \quad \text{for all} \quad (t, x) \in [0, 1] \times \partial\Omega.$$

Indeed, if $t = 0$ and $x \in \partial\Omega$ then $H(0, x) = g(x) \neq 0$ by (3.1). Now, let $H(\eta, y) = 0$ for some $0 < \eta < 1$ and $y \in \partial\Omega$. Then $H(\eta, y) = \eta f(y) + (1-\eta) g(y) = 0$ and, simultaneously, by (2.1), $\|\eta f(x) - \eta g(x)\| < \eta \|f(y)\| + \eta\|g(y)\|$. Thus we conclude that $\|g(y)\| = \|\eta f(y) - \eta g(y)\| < \eta\|f(y)\| + \eta\|g(y)\| = (1-\eta)\|g(y)\| + \eta\|g(y)\| = \|g(y)\|$, a contradiction. Moreover, $H(1, x) = f(x) \neq 0$ on $\partial\Omega$ by (3.1).

From the assumptions and the above considerations we infer that there exists an open set V such that $\overline{V} \subset \Omega$, $\|f(x)\| > 0$ and $\|g(x)\| > 0$ on $\overline{\Omega} \setminus V$ and $\|H(t, x)\| > 0$ for all $(t, x) \in [0, 1] \times (\overline{\Omega} \setminus V)$.

Now, note that, by using [10], the following statements can be obtained:

$$\deg(f, V, 0) = \deg(g, V, 0),$$

$V \cap (I-F)^{-1}(0) = \Omega \cap f^{-1}(0) = \{x_1, \dots, x_n\}$ and $V \cap (I - G)^{-1}(0) = \Omega \cap g^{-1}(0) = \{y_1, \dots, y_m\}$ for some $m, n \in \mathbf{N}$ and, consequently,

$$\sum_{i=1}^{n} \deg(f, A_i, 0) = \sum_{j=1}^{m} \deg(g, B_j, 0);$$

here A_i (B_j) are small neighbourhoods of x_i (y_j) such that the sets $\overline{A}_i$ ($\overline{B}_j$) are pairwise disjoint and $\overline{A}_i \subset V$ ($\overline{B}_j \subset V$). From this, by Definition 2.1, we get the desired conclusion.

4. Proof of Theorem 2.2

We have that $f \in \mathcal{H}(\Omega; E)$ (see [8, Proposition 4, p. 23]). Moreover, the set $(I - f)(\Omega)$ is contained in a compact subset of E, $\operatorname{int}(W) \cap f^{-1}(0) = \{x_1, \dots, x_n\}$ for some $n \in \mathbf{N}$ (see [10, Lemma 2(c)]) and $\delta = \inf\{\|f(x)\|\colon x \in \partial W\} > 0$.

On the other hand, in particular,

$$\|f_n(x) - f(x)\| \leqslant p_{\partial W}(f_n - f) < \frac{1}{2}\delta < \|f(x)\| \leqslant \|f(x)\| + \|f_n(x)\| \quad \text{on } \partial W$$

for all sufficiently large n. From this and Theorem 2.1 we get the desired conclusion.

5. Concluding remarks

If $U \subset E$ is open and $\mathcal{K}$ is the family of all finite unions of closed balls completely interior in U (in norm), then the corresponding topology in $\mathcal{C}(U; E)$ is called the topology of local uniform convergence. It is well known (see *e.g.* T. Franzoni and E. Vesentini [2, Chapt. IV, §3]) that the topology of local uniform convergence is finer than the compact-open topology and the two topologies are equivalent if and only if $\dim_{\mathbf{C}} E < \infty$. When $\dim_{\mathbf{C}} E < \infty$, the compactness assumptions in Theorems 2.1 and 2.2 may be omitted.

As corollaries from our main results and their proofs we get the following theorems of Glicksberg and Hurwitz for C^n with norm $\|\cdot\|$.

Theorem 5.1. *Let Ω be a bounded open subset of C^n and let $f, g \in \mathcal{C}(\overline{\Omega}; E) \cap \mathcal{H}(\Omega; E)$. If $\|f - g\| < \|f\| + \|g\|$ on $\partial\Omega$ then f has finitely many zeros in Ω and, by counting multiplicity, f and g have the same number of zeros in Ω.*

Theorem 5.2. *Let Ω be a bounded open subset of C^n. If the sequence $\{f_n\}$ of maps $f_n \in \mathcal{H}(\Omega; C^n)$, $n = 1, 2, \ldots$, converges on Ω in the topology of local uniform convergence to some map f, and $\|f\| > 0$ on ∂B for some ball $B = B(a; r) = \{x \in C^n : \|x - a\| < r\}$ such that $\overline{B} \subset \Omega$, then f has finitely many zeros in B and, for all sufficiently large n, by counting multiplicity, f_n and f have the same number of zeros in B.*

Remark 5.1. If $n = 1$, Theorems 5.1 and 5.2 imply the results of I. Glicksberg [3], [1, p. 125] and A. Hurwitz [4], [1, p. 152], respectively.

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