Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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A comparison theorem for the Levi equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **4** (1993), n.3, p. 207–212.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1993_9_4_3_207_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1993.

Rend. Mat. Acc. Lincei
s. 9, v. 4:207-212 (1993)

Equazioni a derivate parziali. — *A comparison theorem for the Levi equation.* Nota di Giovanna Citti, presentata (*) dal Corrisp. B. Pini.

Abstract. — We prove a strong comparison principle for the solution of the Levi equation

$$L(u) = \sum_{i=1}^{n} ((1+u_t^2)(u_{x_ix_i} + u_{y_iy_i}) + (u_{x_i}^2 + u_{y_i}^2)u_{tt} + \\ + 2(u_{y_i} - u_{x_i}u_t)u_{x_it} - 2(u_{x_i} + u_{y_i}u_t)u_{y_it}) + k(x,y,t)(1+|Du|^2)^{3/2} = 0,$$

applying Bony Propagation Principle.

Key words: Maximum propagation principle; Comparison principle; Levi equation.

Riassunto. — *Un teorema di confronto per l'equazione di Levi*. Utilizzando il principio di propagazione dei massimi di Bony proviamo un principio di confronto forte per le soluzioni dell'equazione di Levi

$$L(u) = \sum_{i=1}^{n} ((1+u_t^2)(u_{x_ix_i} + u_{y_iy_i}) + (u_{x_i}^2 + u_{y_i}^2)u_{tt} + \\ + 2(u_{y_i} - u_{x_i}u_t)u_{x_it} - 2(u_{x_i} + u_{y_i}u_t)u_{y_it}) + k(x,y,t)(1+|Du|^2)^{3/2} = 0.$$

1. Introduction

Let $M \subset R^{2n+1}$ be a hypersurface of class C^2, graph of a function u. The Lervi curvature of M at a point $(x,y,t,u(x,y,t))$ with $x = (x_1, \dots, x_n) \in R^n$, $y = (y_1, \dots, y_n) \in R^n$, $t \in R$ is the real number

$$(1) \quad k = -(1+|Du|^2)^{-3/2} \sum_{i=1}^{n} ((1+u_t^2)(u_{x_ix_i} + u_{y_iy_i}) + (u_{x_i}^2 + u_{y_i}^2)u_{tt} + \\ + 2(u_{y_i} - u_{x_i}u_t)u_{x_it} - 2(u_{x_i} + u_{y_i}u_t)u_{y_it})$$

where $|Du|^2 = \sum_{i=1}^{n} (u_{x_i}^2 + u_{y_i}^2) + u_t^2$.

Viceversa, if $\Omega \subset R^{2n+1}$ is a fixed, bounded and connected open set, $k: \Omega \to R$ is a continuous function, we can look for a function $u: \Omega \to R$ of class C^2 whose graph has Levi curvature k at every point $(x,y,t,u(x,y,t))$. In other words we study the solutions of the following equation, called the *Levi equation*

$$(2) \quad L(u) = \sum_{i=1}^{n} ((1+u_t^2)(u_{x_ix_i} + u_{y_iy_i}) + (u_{x_i}^2 + u_{y_i}^2)u_{tt} + \\ + 2(u_{y_i} - u_{x_i}u_t)u_{x_it} - 2(u_{x_i} + u_{y_i}u_t)u_{y_it}) + k(x,y,t)(1+|Du|^2)^{3/2} = 0.$$

This is a quasilinear equation, whose characteristic form is positively semidefinite, but has the least eigenvalue identically 0. In particular the equation (2) is not elliptic at any point. However, suitably adapting the classical elliptic techniques, Bedford and Gaveau in [2], Debiard and Gaveau in [3], and Tomassini in [4] were able to establish some geometric properties of the solutions. In particular Debiard and Gaveau proved

(*) Nella seduta del 24 aprile 1993.

the following weak maximum principle: *if k is continuous and nonpositive, and u is a solution of $Lu = 0$ of class C^2 which satisfies $\limsup_{\xi \to \eta} u(\xi) \leq 0$ for every $\eta = (x, y, t) \in \partial\Omega$, then $u \leq 0$ in Ω.*

The strong maximum principle in general does not hold, since the set $S_u(\Omega) = \{(x, y, t) \in \Omega : u(x, y, t) = \sup_\Omega u\}$ can be different from $\emptyset$ and strictly included in Ω. In [4] Tomassini proved the following version of maximum principle: *if Ω is open and connected, $k: \Omega \to R$ is continuous and nonpositive, u is a function of class C^2 solution of the equation* (2) *in Ω and $\xi_0 = (x_0, y_0, t_0) \in S_u(\Omega)$, then $\{(x, y, t) \in \Omega : t = t_0\} \subset S_u(\Omega)$ and $k(x, y, t) = 0$ for all $(x, y, t) \in S_u(\Omega)$.*

In particular: *if k never vanishes in Ω, a regular solution of* (1) *has no local maximum in Ω.*

In this *Note* we study equation (2) with a completely different approach, using the maximum propagation principle of Bony, (see [1]) which we will now recall.

Let $\Omega \subset R^{2n+1}$ be open bounded and connected, and let L_0 be an operator of the form

$$L_0(u) = \sum_{i,j=1}^{2n+1} a_{i,j} \partial^2_{i,j} u + \sum_{i=1}^{2n+1} b_i \partial_i u \quad \text{in } \Omega, \tag{3}$$

where $a_{i,j}$ and b_i are continuous functions in Ω.

Let u be a solution of $L_0(u) \geq 0$ in Ω and let $S_u(\Omega)$ be not empty. A vector $\nu \in R^{2n+1}$ is called outer normal to $S_u(\Omega)$ at a point $\xi \in S_u(\Omega)$ if $B(\xi + \nu, |\nu|) \cap S_u(\Omega) = \emptyset$. If $S_u(\Omega) \neq \emptyset$ and $S_u(\Omega) \neq \Omega$, the set $S_u^*(\Omega) = \{\xi \in S_u(\Omega)$: there exists the outer normal at $\xi\}$ is not empty. With these notations the following properties hold:

i) (Hopf Lemma, see [1, Proposition 3.1]) *If ν is the outer normal to $S_u(\Omega)$ at a point $\xi \in S_u^*(\Omega)$, then*

$$\sum_{i,j=1}^{2n+1} a_{i,j}(\xi) \nu_i \nu_j = 0 .$$

A vector field X of class $C^1(\Omega)$ *i.e.* a function $X \in C^1(\Omega, R^{2n+1})$, is called admissible for L_0 if $\forall \nu \in R^{2n+1}$

$$\sum_{i,j=1}^{2n+1} a_{i,j}(\xi) \nu_i \nu_j = 0 \quad \text{implies } \langle X(\xi), \nu \rangle = 0 .$$

Hence, from the previous proposition it immediately follows that, if X is an ammissible vector field, X is *tangent* to $S_u(\Omega)$, in the sense that $\langle X(\xi), \nu \rangle = 0$ $\forall \nu$ outer normal to $S_u(\Omega)$ at ξ.

Now Bony propagation theorem can be stated as follows:

ii) ([1], Theorem 2.1) *Let X be a vector field of class C^1 tangent to $S_u(\Omega)$. Then for every $\xi \in S_u(\Omega)$, any integral curve of X passing through ξ is completely contained in $S_u(\Omega)$.*

Consequently, if X and Y are ammissible vector fields, any integral curve of them intersecting $S_u(\Omega)$ is completely contained in $S_u(\Omega)$. Hence, if we denote with the same symbol a vector field and the differential operator with the same coefficients, we can define $[X,Y]$, and it is tangent to $S_u(\Omega)$. Thus

iii) If $X_1,\dots,X_{2n}$ are 2n admissible vector fields which together with their commutators of order one, span all of R^{2n+1}, then $S_u(\Omega)=\Omega$.

Using these theorems we give a new, very simple proof of the Tomassini maximum principle. However the most important result is Proposition 2.1, where we show that if $k(\xi)\neq 0\ \forall\xi\in\Omega$, and $u\in C^2$, then there exist vector fields $X_1,\dots,X_{2n}$ which satisfy condition *iii*) in the preceding principle. As a consequence we prove the comparison principle for regular solutions of (2) (see Theorem 2.1).

2. Maximum and comparison principle

In the following we will always assume that $\Omega\subset R^{2n+1}$ is bounded and connected, $k\colon\Omega\to R$ is a continuous function and $u\colon\Omega\to R$ a function of class $C^2(\Omega)$. We will denote L_0 the principal part of the operator L in (2).

$$L_0(u)=\sum_{i=1}^{n}\left((1+u_t^2)(u_{x_ix_i}+u_{y_iy_i})+(u_{x_i}^2+u_{y_i}^2)u_{tt}+2(u_{y_i}-u_{x_i}u_t)u_{x_it}-2(u_{x_i}+u_{y_i}u_t)u_{y_it}\right),$$

so that the Levi operator is simply $L(u)=L_0(u)+k(x,y,t)(1+|Du|^2)^{3/2}$ and we will work on L_0 to begin with.

Remark 2.1. *Applying Bony propagation principle, we can give a new proof of Tomassini maximum principle.*

Indeed we can write L_0 in the form

$$L_0(u)=\sum_{i=1}^{n}(\partial_{x_i}^2u+\partial_{y_i}^2u)+Zu\,, \tag{4}$$

where

$$Z=\sum_{i=1}^{n}\left((u_{x_i}u_{tt}-2u_{y_it})\,\partial_{x_i}+(u_{y_i}u_{tt}+2u_{x_it})\,\partial_{y_i}+(u_{x_ix_i}u_t+u_{y_iy_i}u_t-2u_{x_i}u_{x_it}-2u_{y_i}u_{y_it})\,\partial_t\right).$$

If $k\colon\Omega\to R$ is nonpositive, u is a solution of class C^2 of $Lu=0$ in Ω, then $L_0u=-k(x,y,t)(1+|Du|^2)^{3/2}\geq 0$. The vector fields ∂_{x_i} and ∂_{y_i} are admissible for L_0, for all $i=1,\dots,n$. Hence, by the Bony propagation Theorem (see *ii*) in the Introduction), for all $(x_0,y_0,t_0)\in S_u(\Omega)$ we get $\{(x,y,t)\in\Omega\colon t=t_0\}\subset S_u(\Omega)$. In particular $u_{x_i}(\xi)=u_{y_i}(\xi)=0$ for all $i=1,\dots,p$, $\xi\in S_u(\Omega)$, and, since $L(u)=0$, we deduce that $k(\xi)=0$ for all $\xi\in S_u(\Omega)$.

Using these vector fields, we can not prove in a simple way the comparison principle for two different solutions of the equation. Thus we will look for a more appropriate choice of admissible vector fields.

In order to do this, we first note that the characteristic form of $L_0 u$ can be written

$$\sum_{i,j=1}^{2n+1} a_{i,j}(Du)\,\xi_i\xi_j = \sum_{i=1}^{n} (\langle X_i, \xi\rangle^2 + \langle Y_i, \xi\rangle^2)$$

where $\langle , \rangle$ denote the scalar product in R^{2n+1}, X_i and Y_i are the following vectors:

$$(5)\qquad X_i = \begin{bmatrix} u_t e_i \\ e_i \\ -u_{x_i} \end{bmatrix}, \qquad Y_i = \begin{bmatrix} e_i \\ -u_t e_i \\ u_{y_i} \end{bmatrix}$$

and e_i is the column vector, i-th element of the canonical basis in R^n. In order words, with the identification we have introduced, $X_i = u_t \partial_{x_i} + \partial_{y_i} - u_{x_i}\partial_t$, $Y_i = \partial_{x_i} - u_t \partial_{y_i} + u_{y_i}\partial_t$. Obviously X_i and Y_i are admissible vector fields.

The following one is our most important Proposition:

Proposition 2.1. *We will denote with*

$$\left(X_1,\dots,X_n, Y_1,\dots,Y_n, \sum_{i=1}^{n} [X_i, Y_i]\right)$$

the matrix whose columns are the components of the vector fields $X_1,\dots,X_n$, $Y_1,\dots,Y_n$, $\sum_{i=1}^{n} [X_i, Y_i]$ *respectively. Then*

$$\det\left(X_1,\dots,X_n, Y_1,\dots,Y_n, \sum_{i=1}^{n} [X_i, Y_i]\right) = (-1)^{n(n+1)/2}(1+u_t^2)^{n-1} L_0(u).$$

Proof. The proof is a simple computation, which can be made as follows: $[X_i, Y_i] = (u_t u_{y_i t} - u_{x_i t} - u_{y_i} u_{tt})\,\partial_{x_i} + (u_{x_i} u_{tt} - u_{x_i t} u_t - u_{y_i t})\,\partial_{y_i} + (u_{x_i x_i} + u_{y_i y_i} + u_{y_i} u_{x_i t} - u_{x_i} u_{y_i t})\,\partial_t$.

Hence

$$\det\left(X_1,\dots,X_n, Y_1,\dots,Y_n, \sum_{i=1}^{n} [X_i, Y_i]\right) = \sum_{i=1}^{n} \det(X_1,\dots,X_n, Y_1,\dots,Y_n, [X_i, Y_i]) =$$

$$= \sum_{i=1}^{n} (-1)^{n(n-1)/2} \det(X_1, Y_1, \dots, \hat{X}_i, \hat{Y}_i, \dots, X_n, Y_n, X_i, Y_i, [X_i, Y_i]).$$

The cap on X_i means that this index has been suppressed. This matrix is the sum of n block lower triangular matrices, and its determinant can be evaluated as follows:

$$\sum_{i=1}^{n} (-1)^{n(n-1)/2} \det(X_1, Y_1, \dots, \hat{X}_i, \hat{Y}_i, \dots, X_n, Y_n, X_i, Y_i, [X_i, Y_i]) =$$

$$= \left(\det\begin{pmatrix} u_t & 1 \\ 1 & -u_t \end{pmatrix}\right)^{n-1} \sum_{i=1}^{n} (-1)^{n(n-1)/2} \det\begin{bmatrix} u_t & 1 & u_t u_{y_i t} - u_{x_i t} - u_{y_i} u_{tt} \\ 1 & -u_t & u_{x_i} u_{tt} - u_{x_i t} u_t - u_{y_i t} \\ -u_{x_i} & u_{y_i} & u_{x_i x_i} + u_{y_i y_i} + u_{y_i} u_{x_i t} - u_{x_i} u_{y_i t} \end{bmatrix} =$$

$$= (-1)^{n(n+1)/2}(1+u_t^2)^{n-1} L_0(u).$$

From this Proposition we can deduce the following Lemma:

Lemma 2.1. *Let $\Omega \subset R^{2n+1}$ be bounded and connected and let $u\colon \Omega \to R$ a C^2 function. Let Λ be an operator of the form*

$$\Lambda v = 2 \sum_{i,j=1}^{2n+1} a_{ij}(Du)\, \partial_{ij} v + \sum_{i=1}^{2n+1} b_i \partial_i v$$

where $a_{ij}(Du)$ is the matrix of the characteristic form of L_0, and b_i is a continuous function for all $i = 1,\dots,2n+1$. Assume that $v\colon \Omega \to R$ satisfies $\Lambda v \geq 0$ in Ω and there exists $\xi_0 \in \Omega$ such that $v(\xi_0) = \max\limits_{\overline{\Omega}} v$.

If the Levi curvature k of u is always different from 0 *(see* (1) *for the definition of k), $v \equiv \max\limits_{\overline{\Omega}} v$ in Ω.*

Proof. Let $X_i = X_i(Du)$ and $Y_i = Y_i(Du)$ be the vector fields defined in (5). Since by the preceding proposition we have

$$\det\left(X_1,\dots,X_n, Y_1,\dots,Y_n, \sum_{i=1}^{n} [X_i, Y_i]\right) = (-1)^{n(n+1)/2}(1+u_t^2)^{n-1} L_0(u) =$$

$$= -(-1)^{n(n+1)/2}(1+u_t^2)^{n-1} k(x,y,t)(1+|Du|^2)^{3/2},$$

then the vector space spanned by $X_1,\dots,X_n, Y_1,\dots,Y_n$ and their brackets has dimension $2n+1$. Hence, by *iii*) in the Introduction $\{(x,y,t) \in \Omega\colon\ v(x,y,t) = \max\limits_{\overline{\Omega}} v\} \equiv \Omega$.

Theorem 2.1 (*strong comparison principle*). *Let $u, v \in C^2(\Omega)$ be such that $L(u) \geq L(v)$ in Ω, $u \leq v$ in Ω and there exists $\xi_0 \in \Omega$ such that $u(\xi_0) = v(\xi_0)$. If the Levi curvature of u is always different from 0, then $u \equiv v$ in Ω.*

Proof. The function $w = u - v$ satisfies $w \leq 0$ in Ω and is a solution of

$$\sum_{i,j=1}^{2n+1} a_{i,j}(Du)\, \partial^2_{i,j} w + \sum_{i,j=1}^{2n+1} (a_{i,j}(Du) - a_{i,j}(Dv))\, \partial^2_{i,j} v +$$

$$+ k(\xi)(1+|Du|^2)^{3/2} - k(\xi)(1+|Dv|^2)^{3/2} = L(u) - L(v) \geq 0.$$

On the other hand

$$a_{i,j}(Du) - a_{i,j}(Dv) = \int_0^1 \frac{d}{d\theta} a_{i,j}(\theta Du + (1-\theta) Dv)\, d\theta =$$

$$= \sum_{k=1}^{2n+1} \left(\int_0^1 \partial_{s_k} a_{i,j}(\theta Du + (1-\theta) Dv)\, d\theta \right) \partial_k w$$

and

$$(1+|Du|^2)^{3/2} - (1+|Dv|^2)^{3/2} = \int_0^1 \frac{d}{d\theta}(1+|\theta Du + (1-\theta) Du|^2)^{3/2} d\theta =$$

$$= 3 \sum_{k=1}^{2n+1} \left(\int_0^1 (1+|\theta Du + (1-\theta) Dv|^2)^{1/2} (\theta\, \partial_k u + (1-\theta)\, \partial_k v)\, d\theta \right) \partial_k w.$$

If we set

$$b_k = \sum_{i,j=1}^{2n+1} \int_0^1 \partial_{s_k} a_{i,j}(\theta Du + (1-\theta) Dv)\, \partial^2_{i,j} v\, d\theta +$$

$$+ 3k(\xi) \int_0^1 (1 + |\theta Du + (1-\theta) Dv|^2)^{1/2} (\theta \partial_k u + (1-\theta) \partial_k v)\, d\theta,$$

w is a solution of

$$\begin{cases} \Lambda w = \sum_{i,j=1}^{2n+1} a_{i,j}(Du)\, \partial^2_{i,j} w + \sum_{i=1}^{2n+1} b_i \partial_i w \geq 0 & \text{in } \Omega \\ w \leq 0 & \text{in } \Omega \end{cases}$$

with b_i continuous. By Lemma 2.1 we immediately conclude that $w \equiv 0$ and $u \equiv v$.

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