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Maximum principle for viscosity sub solutions and viscosity sub solutions of the Laplacian

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Equazioni a derivate parziali. — *Maximum principle for viscosity sub solutions and viscosity sub solutions of the Laplacian.* Nota di SUNDARARAJA RAMASWAMY, presentata (*) dal Socio E. Magenes.

ABSTRACT. — The aim of this paper is to characterize the u.s.c. (resp. l.s.c.) viscosity sub (resp. super) solutions of the Laplacian which do not take the value $+\infty$ (resp. $-\infty$) as precisely the sub (resp. super) harmonic functions.

KEY WORDS: Viscosity solutions; Harmonic functions; Maximum principle.

RIASSUNTO. — *Principio di massimo per sotto soluzioni viscose e sotto soluzioni viscose del Laplaciano.* Lo scopo del lavoro è quello di caratterizzare le sopra (risp. sotto) soluzioni semicontinue inferiormente (risp. superiormente) di tipo «viscoso» del Laplaciano, le quali non prendano il valore $+\infty$ (risp. $-\infty$), come funzione sub (risp. super) armoniche.

1. BASIC DEFINITIONS

Let Ω be a nonempty open set in $\boldsymbol{R}^n$ and let M_n be the space of all symmetric $n \times n$ matrices. Let F be a mapping from $\Omega \times M_n$ to $\boldsymbol{R}^1$.

DEFINITION 1. F is said *uniformly elliptic* if there exists a $\lambda > 0$ such that for all A, $B \in M_n$, B positive definite and for all $x \in \Omega$, one has $F(x, A+B) - F(x, A) \geqslant \lambda \|B\|$ where $\|\cdot\|$ is any norm on M_n.

Any second order linear elliptic differential operator with second order terms only is an example of F satisfying Definition 1.

DEFINITION 2. An extended real-valued function u defined on Ω is said to be a *viscosity sub* (resp. *super*) solution of F if for all $\phi \in C^2(\Omega)$ with $u - \phi$ having a local maximum at a point $x_0 \in \Omega$ implies that $F(x_0, D^2\phi(x_0)) \geqslant 0$, where $D^2\phi(x_0)$ stands for the Hessian of ϕ at x_0, that is the matrix $((\partial^2\phi/\partial x_i \partial x_j(x_0)))$ (resp. $u - \phi$ having a local minimum at a point $x_0 \in \Omega$ implies that $F(x_0, D^2\phi(x_0)) \leqslant 0$).

DEFINITION 3. A real-valued function u is said to be a *viscosity solution of* $F = 0$ if it is both a viscosity sub and super solution of F.

2. A MAXIMUM PRINCIPLE

PROPOSITION 1. *Let F be uniformly elliptic and let us assume that $F(x, 0) = 0$ for all $x \in \Omega$. Then, we have $F(x, A) < 0$ for all matrix $A = ((a_{ij}))$ which is negative-definite in the sense that*

$$\sum_i \sum_j a_{ij}\alpha_i\alpha_j < 0, \qquad \forall \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \neq 0 .$$

(*) Nella seduta del 24 aprile 1993.

PROOF. $0 = F(x,0) = F(x, A + (-A))$. Therefore,

$$0 - F(x,A) = F(x, A + (-A)) - F(x,A) \geqslant \lambda \|(-A)\|$$

as $-A$ is positive definite and F is uniformly elliptic. Therefore,

$$F(x,A) \leqslant -\lambda\|(-A)\| < 0 .$$

THEOREM 1. *Let Ω be bounded. Let F be uniformly elliptic with $F(x,0) = 0$, $\forall x \in \Omega$. Let u be a viscosity sub solution of F such that $u(x) < \infty$, $\forall x \in \Omega$. If u is upper semi continuous on $\overline{\Omega}$, then*

$$\sup_{x \in \partial\Omega} u(x) = \sup_{x \in \overline{\Omega}} u(x) .$$

PROOF. It is obvious that

$$\sup_{x \in \partial\Omega} u(x) \leqslant \sup_{x \in \overline{\Omega}} u(x) .$$

Suppose that

$$\sup_{x \in \partial\Omega} u(x) < \sup_{x \in \overline{\Omega}} u(x) . \tag{1}$$

u being upper semi continuous on $\overline{\Omega}$ and $\overline{\Omega}$ being compact, $\sup_{x \in \overline{\Omega}} u(x)$ is attained at some point $x_0 \in \overline{\Omega}$. Equation (1) implies that $x_0 \notin \partial\Omega$. Hence $x_0 \in \Omega$. Thus u has a local maximum at x_0.

CLAIM. The function $u_\varepsilon = u + \varepsilon|x - x_0|^2$ also has a local maximum in Ω for small values of $\varepsilon > 0$.

PROOF OF THE CLAIM. Suppose for some $\varepsilon > 0$, u_ε attains its maximum only on $\partial\Omega$. Let $X \in \partial\Omega$ be such that $u_\varepsilon(X) \geqslant u_\varepsilon(x)$, $(\forall x \in \overline{\Omega})$.

In particular, $u_\varepsilon(X) \geqslant u_\varepsilon(x_0) = u(x_0)$. That is $u(X) + \varepsilon|X - x_0|^2 \geqslant u(x_0)$.

$$\Rightarrow \varepsilon|X - x_0|^2 \geqslant u(x_0) - u(X) \geqslant u(x_0) - \sup_{x \in \partial\Omega} u(x)$$

$$\Rightarrow \varepsilon \geqslant \frac{u(x_0) - \sup_{x \in \partial\Omega} u(x)}{|X - x_0|^2} \geqslant \frac{u(x_0) - \sup_{x \in \partial\Omega} u(x)}{\sup_{y \in \partial\Omega} |y - x_0|^2} .$$

Let us observe that $u(x_0) - \sup_{x \in \partial\Omega} u(x) > 0$. Hence if $\varepsilon > 0$ is

$$< \frac{u(x_0) - \sup_{x \in \partial\Omega} u(x)}{\sup_{y \in \partial\Omega} |y - x_0|^2} ,$$

u_ε has a local maximum in Ω and thus the claim is proved.

Fix one such $\varepsilon > 0$. Let u_ε have a local maximum at a point $y \in \Omega$. As u is a viscosity subsolution, applying the definition taking ϕ to be $-\varepsilon|x - x_0|^2$, we see that $F(y, -2\varepsilon I_n) \geqslant 0$ where I_n is the $n \times n$ identity matrix. As $\varepsilon > 0$, $-2\varepsilon I_n$ is negative-definite and hence, Proposition 1 is contradicted. □

3. Viscosity sub (resp. super) solutions of the Laplacian Δ: a characterization

Theorem 2. *Let u be an upper semi continuous (resp. lower semi continuous) function u such that $u < \infty$ (resp. $u > -\infty$). Then u is a viscosity sub (resp. super) solution for Δ, if and only if u is subharmonic (resp. superharmonic).*

Proof. Sufficient to prove the characterization in the subharmonic case. Let us recall the definition of a subharmonic function.

Definition 4. An extended real-valued function defined on an open set $\Omega \neq \emptyset$ is said to be *subharmonic* if

i) u is upper semi continuous,

ii) $u(x) < \infty \ \forall x \in \Omega$ and

iii) $\forall x_0 \in \Omega$, $\exists r_0 > 0$ such that

$$u(x_0) \leqslant \int_{\partial B(x_0;r)} u(x)\, d\sigma_r(x), \qquad \forall r \leqslant r_0,$$

where $d\sigma_r$ is the unit surface measure on $\partial B(x_0;r)$, the boundary of $B(x_0;r)$.

Proof of Theorem 2. (*i*) *If part*: Let us assume that u is subharmonic. Let $\phi \in C^2(\Omega)$ be such that $u - \phi$ has a local maximum at a point $x_0 \in \Omega$. Let us assume that $u(x_0) - \phi(x_0)$ is a maximum of $u - \phi$ in a ball $B(x_0;\delta)$ for some $\delta > 0$.

If $\Delta\phi(x_0) < 0$, then $\Delta\phi(x) < 0 \ \forall x$ in some neighbourhood of x_0, say for example in $B(x_0;\eta)$ for some $\eta \in (0,\delta)$. Therefore, $u - \phi$ is subharmonic in $B(x_0;\eta)$, as u is subharmonic and ϕ is super harmonic in $B(x_0;\eta)$. Therefore, by the classical maximum principle for subharmonic functions, $u - \phi$ must be equal to $u(x_0) - \phi(x_0)$ in $B(x_0;\eta)$.

That is $\phi = u - u(x_0) + \phi(x_0)$ in $B(x_0;\eta)$. Therefore ϕ is subharmonic in $B(x_0;\eta) \Rightarrow$ $\Rightarrow \Delta\phi \geqslant 0$ in $B(x_0;\eta)$.

In particular, $\Delta\phi(x_0) \geqslant 0$.

This contradicts that $\Delta\phi(x_0) < 0$ proving that u is a viscosity subsolution.

(*ii*) *Only if part*: Before we start the proof, let us make the following remark, which is an easy consequence of the definitions.

Remark. If u is a viscosity subsolution for Δ, and if h is any harmonic function, then $u + h$ is also a viscosity subsolution.

Let u be upper semi continuous and let $u(x) < \infty \ \forall x \in \Omega$. Let u be a viscosity subsolution for Δ. Let $x_0 \in \Omega$. Let $R > 0$ be less than $d(x_0, \partial\Omega)$ so that $\overline{B(x_0;R)} \subset \Omega$. Let $r \leqslant R$.

Since u is upper semi continuous, $\exists$ a decreasing of sequence $\{f_m\}_{m=1}^{\infty}$ of real-valued continuous functions on $\partial B(x_0;r)$ such that $f_m(x) \downarrow u(x)$, $\forall x \in \partial B(x_0;r)$.

Consider the Poisson integral,

$$I_r^{f_m}(x) = r^{n-2} \int_{\partial B(x_0;r)} f_m(X) \frac{r^2 - |x - x_0|^2}{|x - X|^n} \, d\sigma_r(X)$$

in $B(x_0;r)$.

Then, it is well known that $I_r^{f_m}$ is a harmonic function in $B(x_0;r)$ and that

$$\forall X \in \partial B(x_0;r), \qquad I_r^{f_m}(x) \to f_m(X), \qquad \text{as } x \to X, \ x \in B(x_0;r).$$

Consider $u - I_r^{f_m}$ in $B(x_0;r)$. As u is a viscosity subsolution of Δ in $B(x_0;r)$ and $I_r^{f_m}$ is harmonic, by the remark above, $u - I_r^{f_m}$ is also a viscosity subsolution of Δ in $B(x_0;r)$.

Define v in $\overline{B(x_0;r)}$ as

$$v(x) = \limsup_{\substack{y \to x \\ y \in \Omega}} \{u(y) - I_r^{f_m}(y)\}.$$

Then v is upper semi-continuous in $\overline{\Omega}$, $= u - I_r^{f_m}$ in $B(x_0;r)$ and $\forall X \in \partial B(x_0;r)$

$$v(X) = \limsup_{\substack{y \to X \\ y \in \Omega}} \{u(y) - f_m(X)\}.$$

$$\leqslant u(X) - f_m(X) \text{ as } u \text{ is upper semicontinuous}.$$

$$u(X) - f_m(X) \leqslant 0, \qquad \forall X \in \partial B(x_0;r).$$

Therefore, $v(X) \leqslant 0$, $\forall X \in \partial B(x_0;r)$. By the maximum principle proved in Theorem 2,

$$\sup_{X \in \partial B(x_0;r)} v(X) = \sup_{x \in \overline{B(x_0;r)}} v(x).$$

The L.H.S. is $\leqslant 0$. Hence $v(x) \leqslant 0$ $\forall x \in B(x_0;r)$. In particular, $v(x_0) \leqslant 0$.

$$v(x_0) = u(x_0) - \int_{\partial B(x_0;r)} f_m(X) \, d\sigma_r(X).$$

Therefore,

$$u(x_0) \leqslant \int_{\partial B(x_0;r)} f_m(X) \, d\sigma_r(X), \qquad \forall m \in \mathbf{N}.$$

Hence

$$u(x_0) \leqslant \int_{\partial B(x_0;r)} u(X) \, d\sigma_r(X),$$

proving that u is subharmonic.

COROLLARY. *A real-valued continuous function on Ω is a viscosity solution for Δ if and only if it is a harmonic function.*

4. Concluding remarks

The definition of uniformly elliptic second order non-linear differential operators given here is taken from L. A. Caffarelli [1] and the definitions of viscosity sub and super solutions are taken from Ishii and Lions [2]. The definitions of viscosity sub and super solutions given in [1] are apparently not the same as given in [2]. The equivalence of the definitions in [1] and [2] are proved in [3], for some class of uniformly elliptic operators.

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