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On linearly normal strange curves

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Geometria algebrica. — *On linearly normal strange curves.* Nota di EDOARDO BALLICO, presentata (*) dal Corrisp. M. Cornalba.

ABSTRACT. — Here we prove a numerical bound implying that, except for smooth plane conics in characteristic 2, no complete linear system maps birationally a smooth curve into a projective space with a strange curve as image.

KEY WORDS: Strange curve; Projective curve; Linear series; Strange point; Inseparable degree.

RIASSUNTO. — *Sulle curve «strane» linearmente normali.* Si dimostra qui una diseguaglianza numerica che ha come applicazione il fatto che le coniche piane in caratteristica 2 corrispondono agli unici sistemi lineari completi su una curva liscia X che determinano un morfismo birazionale h da X in uno spazio proiettivo con $h(X)$ curva «strana».

Let $C \subset \boldsymbol{P}^n$ be an integral curve. Recall that C is called *strange* if there is a point $v \in \boldsymbol{P}^n$ (called the *strange point of* C) such that for every smooth point x of C the tangent line $T_x C$ contains v. It is known (and easy) that if C is a strange curve which is not a line, then the base field $\boldsymbol{K}$ has positive characteristic. The strangeness of C gives very strong conditions on C. One of them is a lovely theorem of Lluis (see the original paper [13, p. 51], or, for instance [12, Prop. 3]) which states that such a curve is not smooth (and even must have «cusps») unless C is a conic in characteristic 2. The aim of this *Note* is to give an easy proof of the following results.

THEOREM 0.1. *Let X be a smooth connected complete curve and $L \in \mathrm{Pic}(X)$ such that the associated complete linear system $|L|$ is base point free; let $h_{|L|}: X \to \boldsymbol{P}^n$ be the morphism associated to $|L|$. Assume that $h_{|L|}$ is birational and that the curve $C := h_{|L|}(X)$ is strange but not a line. Then we are in characteristic 2 and C is a plane conic.*

THEOREM 0.2. *Let X be a smooth connected complete curve defined over an algebraically closed field $\boldsymbol{K}$ with $p := \mathrm{char}(\boldsymbol{K}) > 0$. Take $L \in \mathrm{Pic}(X)$ and a vector space $V \subseteq H^0(X, L)$ such that the associated linear system $|V|$ is base point free; let $h_{|V|}: X \to \boldsymbol{P}^n$ be the morphism associated to $|V|$. Assume that $h_{|V|}$ is birational and that the curve $C := h_{|V|}(X)$ is strange but not a line; let $q = p^e$ (with $e > 0$) be the inseparability degree of the projection of C from the strange point. Then*

$$h^0(X, L) \geq q(n-1) + 1. \tag{1}$$

Furthermore, if C contains its strange point, then:

$$h^0(X, L) \geq q(n-1) + 2. \tag{2}$$

MOTIVATIONS. The main motivation comes from a result of Castelnuovo's theory [11, ch. III] and results in [14]. Recall from [4] that an integral non degenerate curve $C \subset \boldsymbol{P}^n$ is called *very strange* if the generalized trisecant lemma fails for C, *i.e.* if

(*) Nella seduta dell'11 novembre 1992.

any $n-1$ general points of C span a linear space containing at least another point of C. In [12, Lemma 4], it was proved that a very strange curve is strange. The usual upper bound for $p_a(C)$ in terms of $\deg(C)$ and n (and some of its generalizations contained in [11, ch. III], [3, ch. 3], [7]) is proven verbatim in any characteristic for curves which are not very strange. For very strange curves this bound (and even more) was checked in [4] using [13, §2]. Often this bound on the arithmetic genus is far from being sharp for (very) strange curves. In recent years many papers on curve theory were motivated by arithmetic questions (and used at some points big theorems on the arithmetic of varieties). It seems that for the curve theoretic part of the proofs Castelnuovo's theory is a key tool, but that it is often applied only to complete linear systems. Hence:

(*a*) Theorem 0.1 justifies in positive characteristic all the uses of the uniform position principle in [2] and the second chapter of [1];

(*b*) Theorem 0.1 shows that [8] holds in positive characteristic (although it would be easy to justify in positive characteristic [8] without using 0.1).

Another motivation (and application of Castelnuovo's theory) comes from the computations of possible dimensions of certain Chow families of curves in $\boldsymbol{P}^n$ (in the sense of [10]), *i.e.* of families of integral curves in $\boldsymbol{P}^n$ with the same degree and geometric genus. By Theorem 0.1 every strange curve, C, in any such family (and in particular the very strange ones) comes as a birational linear projection of a curve, D, in a higher dimensional projective space, with D not strange. Since it is easy to check that the general projection of D into $\boldsymbol{P}^n$ is not strange, we see that the set of strange curves forms only a proper family in every irreducible component of such Chow varieties.

Remark 0.3. Theorem 0.2 gives a strong upper bound for the geometric genus of a strange curve C (applying Castelnuovo's theory to the corresponding complete linear system on the normalization of C) in terms of $\deg(C)$, n and p. However, other methods gives other bounds (and as far as I know none of them covers completely the other ones).

We feel it is important that the specialists on curve theory check if their proofs work cheaply in positive characteristic using the statements of [14] (*e.g.* [14, 0.1, 1.8, 2.4, 2.5] (and 2.2 or equivalently [5]) and 0.1. If this trivial check does not suffice and the matter seems not to be meaningless in positive characteristic, they could try to use 0.2, 0.3 and [4] (and the proofs in [4]). This job seems to us important (although very easy), because if this type of job will not be done now, perhaps there will be a too huge amount of material and references for which the positive characteristic status is not known (and this can be very bad for arithmetical applications).

1. The proofs

Both theorems stated in the introduction will be proven simultaneously.

Proof of 0.1 and 0.2. Assume that C is strange with strange point v. Set $d := \deg(C)$. Take a hyperplane H of $\boldsymbol{P}^n$ with $v \notin H$ and let $D \subset H$ be the image of C

under the projection, τ, from the point v; let $\pi\colon X \to D$ be the induced morphism. Since v is the strange point of C, π is not separable; let $q = p^e$ be its inseparable degree and t its separable degree. Note that if m is the multiplicity of C at v, then $d = m + tq$. Let $v\colon D' \to D$ be the normalization of D and $\pi'\colon X \to D'$ the morphism induced by π. Let U be the base point free part of the linear system on X corresponding to the linear system V' induced by the family of hyperplanes of $\boldsymbol{P}^n$ passing through v; let B be the base locus (as divisor) of V'. Note that B has degree m and that U induces π. By [9], Lemma 1.4, π' factors through the iterated degree q Frobenius $F_q\colon X \to X$, say $\pi' = F_q \circ u$ with $u\colon X \to D'$. In particular U is formed by the q-powers of the divisor of a linear system W (in simbols: $U = W^{(q)}$) corresponding to the pull-back u^* on X of the linear system on C induced by $H^0(H, \boldsymbol{O}_H(1))$. Hence $\dim(W) = n$. Note that the complete linear system associated to U contains $W^{\otimes q}$. Iterating $q-1$ times Hopf lemma (or on a smooth curve [3, p. 108]) we have the bound (1) given in Theorem 0.2 in the stronger form claimed by 0.3. Since $|V|$ is base point free, if $B \neq \emptyset$ we have $H^0(X, L(-B)) < H^0(X, L)$, hence the second part of 0.2 and 0.3.

At this point we have proved Theorem 0.2. Now it remains only to conclude the proof of Theorem 0.1. Hence we may assume $h^0(X, L) = n + 1$. Note that by (2) we have $q = 2$ (hence $p = 2$), $m = 0$ and $n = 2$ (hence $d = 2t$). The morphism $u\colon X \to D = D' \cong \boldsymbol{P}^1$ corresponds to a g_t^1. It is sufficient to check that $t = 1$. Assume $t > 1$ (hence, since $d > 0$ and $h^0(X, L) = 3X$ not rational). Note that if A and B are divisors in this g_t^1, then $A + B$ is a divisor of $|L|$ which does not separate the t points of a general fiber of u. Since $h_{|V|}$ is birational, we have $h^0(X, L) > 3$, contradiction. ♦

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