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Meccanica dei fluidi. — *Convection with temperature dependent viscosity in a porous medium: nonlinear stability and the Brinkman effect.* Nota di Lorna Richardson e Brian Straughan, presentata (*) dal Corrisp. S. Rionero.

Abstract. — We establish a nonlinear energy stability theory for the problem of convection in a porous medium when the viscosity depends on the temperature. This is, in fact, the situation which is true in real life and has many applications to geophysics. The nonlinear analysis presented here would appear to require the presence of a Brinkman term in the momentum equation, rather than just the normal form of Darcy's law.

Key words: Convection; Porous medium; Variable viscosity.

Riassunto. — *Convezione in un mezzo poroso con viscosità variabile con la temperatura: stabilità non lineare nel modello di Brinkman.* Si considera il problema della convezione naturale in un mezzo poroso tenendo conto – com'è nella realtà geofisica – delle variazioni della viscosità con la temperatura. Si stabiliscono condizioni che assicurano la stabilità non lineare nella norma di L_2 (stabilità in energia) nell'ambito del modello di Brinkman.

1. Introduction

For many purposes it is adequate to treat viscosity as constant. However, when a layer of fluid is subjected to thermal gradients it may be rightly argued that the variation of viscosity with temperature has to be taken into account. Weast [20] includes many tables of values of viscosity against temperature and from these one sees that the variation over a few degrees may be very large. Interest in studies of convection with temperature-dependent viscosity has been intense since the work of Tippelskirch [16], who showed that the up flow or down flow at the centre of the convection cell depends on the functional form of the viscosity-temperature relationship. A nonlinear energy stability analysis of convection in a fluid layer with a linear viscosity-temperature relationship has been presented by Richardson and Straughan [11]. However, convection in porous media when the fluid viscosity varies with temperature is also highly important in geophysical and other contexts, see *e.g.* Or [7], and the book of Nield and Bejan [6], and the references therein. Therefore, we here commence a study of nonlinear energy stability of convection in a porous layer when the viscosity depends on the temperature.

If $\mu(T)$ denotes the dynamic viscosity, then for a wide variety of situations the linear relation (employed in the fluid case in [11]),

$$\mu(T) = \mu_0 (1 - \gamma(T - T_0)), \tag{1}$$

where μ_0 and γ are positive constants and T_0 is a reference temperature, is adequate.

(*) Nella seduta del 24 aprile 1993.

Many studies of convection in porous media involve Darcy's law, see *e.g.* Nield and Bejan [6]. However, there have recently been several articles which advocate using instead the model of Brinkman [1], see *e.g.* [2, 3, 5, 8-10, 14, 17, 18, 19]. A lucid discussion on the relative merits of the Brinkman and Darcy equations, along with other alternatives, is given in chapter 1 of [6]. In the present contribution we also employ Brinkman's equation although our motivation is mainly a mathematical one in that we find this form necessary to implement the nonlinear energy stability analysis which follows: it may be possible that by some suitable selection of a generalized energy (or Lyapunov functional; such functionals are constructed in other contexts by *e.g.* [4, 15]) one may find a way to proceed without the Brinkman term but at present we have not seen such an avenue.

Hence, we now study the nonlinear stability problem of convection in a porous medium when the fluid has a temperature dependent viscosity.

The equation of motion we employ for flow in porous media is the Brinkman equation (see *e.g.* [6]), and this assumes the form

$$p_{,i} = -\frac{\mu}{k}v_i - g\rho_0\delta_{i3}(1-\alpha(T-T_0)) + \tilde{\mu}\Delta v_i. \tag{2}$$

Here μ is the viscosity of the fluid, $\tilde{\mu}$ is an effective viscosity, and $v_i, \rho_0, g, p, \alpha$ are, respectively, velocity, density, gravity, pressure, and the coefficient of thermal expansion. The reasoning behind the use of this equation is that to understand the onset of convection in a porous medium made up of a sparse distribution of particles the viscous shear must be taken into account no matter how small it may be in relation to the Darcy resistance.

As mentioned above, we here find using the Brinkman model makes sense mathematically, as it enables us to prove an energy stability theorem. The reason for this is that the Δv_i term in the momentum equation is necessary in order to control the nonlinear term that arises in the energy analysis. Without it we do not see how to proceed in order to derive a result.

Nield [5] deduces that the Brinkman equation could be used successfully for problems where the velocity of the fluid within the porous medium is constant except in regions close to the boundary. He also concludes that it is useful for porous media whose porosity is close to unity, which is indeed the case that it was designed to deal with. However, he also states that the Brinkman equation is not generally applicable to flow in porous media. Notwithstanding this statement we feel that mathematically using the equation is justified and as there are some physical situations where it is relevant the analysis is not entirely without use.

We shall use an energy argument (cf. [12, 13, 15]) in order to establish a conditional nonlinear stability criterion. It transpires that the linear and nonlinear problems coincide and so we have an optimum result. Numerical calculations appear in §3.

2. Basic equations and energy stability analysis

For our viscosity-temperature relation we use a linear law as in eq. (1). The geometry studied is a horizontal plane layer, $z \in [0, d]$, with gravity taken to be pointing downwards. The layer is filled with a porous medium but we shall assume that the porosity is approaching unity, so that our use of Brinkman's equation is justified (according to Nield [5]).

The governing equations of motion are given by putting (1) in (2), noting the fluid is incompressible, and an equation for the evolution of temperature, thus

$$p_{,i} = -\frac{\mu_0}{k}(1-\gamma[T-T_0])v_i + \tilde{\mu}\Delta v_i - g\rho_0\,\delta_{i3}(1-\alpha[T-T_0]), \tag{3}$$

$$v_{i,i} = 0, \tag{4}$$

$$T_{,t} + v_i T_{,i} = \kappa\Delta T. \tag{5}$$

We now consider a steady solution, $\bar{v}_i \equiv 0$, $\bar{T}(z)$, $\bar{p}$. The temperature is assumed assigned on the horizontal boundaries, as follows,

$$T(0) = T_1, \qquad \text{and} \qquad T(d) = T_2, \qquad \text{with } T_1 > T_2,$$

where T_1, T_2 are constants; the layer is thus heated from below.

There is no loss in generality in taking $T_0 = T_1$ and it simplifies the mathematics; hence we adopt this selection.

Equations (3) and (5) reveal that $\bar{T}(z) = -\beta z + T_1$, where $\beta = (T_1 - T_2)/d$, and $\bar{p}$ is $\bar{p} = -\rho_0 g(z + (\alpha/2)\beta z^2) + p_0$, p_0 being a reference pressure.

The next step is to introduce perturbations u_i, θ, π via $v_i = \bar{v}_i + u_i$, $T = \bar{T} + \theta$, $p = \bar{p} + \pi$.

We substitute these into (3)-(5) and non-dimensionalize according to the scales

$$\begin{array}{llll} t = t^* d^2/\mu_0, & \pi = \pi^* P, & P = U\mu_0 d/k, & Pr = \mu_0/K, \\ u_i = u_i^* U, & \theta = \theta^* T^\#, & T^\# = U\sqrt{\beta\mu_0 d^2 (Kg\alpha k\rho_0)^{-1}}, & R = \sqrt{\alpha g\beta d^2 k\rho_0 (K\mu_0)^{-1}}, \\ x_i = x_i^* d, & \Gamma = \gamma\beta d, & \lambda = \tilde{\mu}k/d^2\mu_0, & U = \mu_0/d \end{array}$$

where Pr is the Prandtl number, R^2 is the Rayleigh number, Γ is a measure of the viscosity variation with temperature, and λ is a measure of the effective viscosity. If we now omit the stars, although dimensionless quantities are to be understood, the perturbation equations governing convection become

$$\pi_{,i} = -u_i(1+\Gamma z) + R^{-1}\Gamma Pr\,\theta u_i + \lambda\Delta u_i + R\delta_{i3}\theta, \tag{6}$$

$$Pr(\theta_{,t} + u_i\theta_{,i}) = \Delta\theta + Rw, \tag{7}$$

$$u_{i,i} = 0. \tag{8}$$

Throughout we also employ the notation $\boldsymbol{u} = (u, v, w)$, and we have chosen to employ μ_0, as opposed to $\tilde{\mu}$, in the definition of R to be consistent with other papers dealing with convection in porous media according to Darcy's law.

Boundary conditions. The boundary conditions which are applicable are a matter of contention in a porous medium. A clear account is provided in [6, 1.5, 1.6].

When there is no slip at the boundary we may take $u = v = w = 0$ on $z = 0,1$.

If, however, the porous medium is free at the boundary the situation is not so clear. Nevertheless, for a porosity close to unity the usual stress free boundary conditions should hold, viz. $\partial u/\partial z = \partial v/\partial z = w = 0$ on $z = 0, 1$.

We shall also assume u, v, w, θ and p are periodic on the x, y boundaries of the convection cell V. The temperature perturbation also satisfies $\theta = 0$ on $z = 0, 1$.

Energy analysis. We employ an energy analysis in order to study the nonlinear system (6)-(8). We multiply (6) by u_i, (7) by θ and integrate over the period cell V. The notation $\|\cdot\|$, $\langle\cdot\rangle$ signifies the $L^2(V)$ norm and integration over V, respectively. Upon adding the resulting equations we may derive

$$\frac{dE}{dt} = -D + RI + \frac{\Gamma Pr}{R}\langle \theta u_i u_i \rangle, \tag{9}$$

where E, D, I are defined by $E = Pr\|\theta\|^2/2$, $D = \|\boldsymbol{u}\|^2 + \Gamma\langle z u_i u_i\rangle + \lambda\|\nabla \boldsymbol{u}\|^2 + \|\nabla\theta\|^2$, $I = 2\langle \theta w\rangle$.

Due to the symmetry of the linearized system (3)-(5) we do not need a coupling parameter in (9).

Now set

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D}, \tag{10}$$

where $\mathcal{H}$ is the space of admissible solutions.

Then, from eq. (9) we may derive the inequality

$$\frac{dE}{dt} \leqslant -aD + \frac{\Gamma Pr}{R}\langle \theta u_i u_i \rangle, \tag{11}$$

where we have set $a = (R_E - R)/R_E$.

The number R_E is the critical Rayleigh number of energy stability theory and we determine this number below. It is henceforth understood that we shall be working with Rayleigh numbers such that $a > 0$. The existence of a maximizing solution to problems like (10) was first established in the fundamental papers of Rionero [12, 13]. We here concentrate on resolving the Euler-Lagrange equations associated with (10), and these are

$$\lambda \Delta u_i - (1 + \Gamma z) u_i + R_E \delta_{i3}\theta = \tilde{\omega}_{,i}, \tag{12}$$

$$\Delta\theta + R_E w = 0, \tag{13}$$

where $\tilde{\omega}$ is a Lagrange multiplier.

Note that the eigenvalue problem arising from (12) and (13) is the same as that arising from the linearized version of system (6)-(8), and so, if we can demonstrate decay of E from (11) we will have an optimum result. These circumstances arise due to the symmetry of the linear operator, cf. Straughan [15], chapter 4.

Parametric differentiation with respect to λ yields $\partial R_E/\partial\lambda = \|\nabla \boldsymbol{u}\|^2/2\langle w\theta\rangle$. Upon substitution for $\langle w\theta\rangle$ one may then obtain $\partial R_E/\partial\lambda = R_E\|\nabla\boldsymbol{u}\|^2/(2\|\nabla\theta\|^2)$. Thus $\partial R_E/\partial\lambda \geqslant 0$, a result which is useful in the numerical calculations.

To solve (12) and (13) numerically for R_E we first take the double curl of (12) and then consider the third component of the resulting equation. Normal modes reduces the system to

$$\lambda(D^2 - a^2)^2 W - (1 + \Gamma z)(D^2 - a^2) W - \Gamma DW - R_E a^2 \Theta = 0, \tag{14}$$

$$(D^2 - a^2)\Theta + R_E W = 0. \tag{15}$$

In these equations a is the wavenumber and $D \equiv d/dz$. We solve this system for

$$Ra = \min_{a^2} R_E^2, \tag{16}$$

subject to the boundary conditions

$$W = DW = \Theta = 0, \quad \text{on} \quad z = 0, 1, \tag{17}$$

for two fixed surfaces, and

$$W = D^2 W = \Theta = 0, \quad \text{on} \quad z = 0, 1, \tag{18}$$

on two stress free surfaces. The numerical results, which employ the compound matrix method as the eigenvalue solver, appear in §3.

Before proceeding to the numerical results it is instructive to consider a special case, namely, when $\Gamma = 0$, in the situation when (18) hold. In this case we may solve the system exactly by firstly eliminating Θ to find

$$\lambda(D^2 - a^2)^3 W - (D^2 - a^2)^2 W + R_E^2 a^2 W = 0. \tag{19}$$

The boundary conditions (18) together with (14) show that

$$D^{(2n)} W = 0 \quad \text{on} \quad z = 0, 1, \tag{20}$$

$n = 0, 1, 2, \ldots$, and so $W = \sin \pi z$ which yields

$$R_E^2 = (\lambda(\pi^2 + a^2)^3 + (\pi^2 + a^2)^2)/a^2. \tag{21}$$

The minimum wavenumber is then found as

$$a_c^2 = (- (1 + \lambda\pi^2) + \sqrt{(1 + \lambda\pi^2)^2 + 8\lambda\pi^2(1 + \lambda\pi^2)})/4\lambda. \tag{22}$$

It is further instructive to develop the analysis for $0 < \lambda \ll 1$. In the limit $\lambda \to 0$ we should recover the classical result for convection with Darcy's law and constant viscosity. Of course, in the classical case we must recognize that the eq. (19) is fourth order and hence the boundary condition $D^2 W = 0$ in (18) is not required. For λ small use of the binomial theorem in (22) shows

$$a_c^2 \sim \pi^2 + O(\lambda), \tag{23}$$

and then putting this in (21) gives

$$R_E^2 \sim 4\pi^2 + O(\lambda). \tag{24}$$

Expressions (23) and (24) are in complete agreement with the classical results obtained via Darcy's law, as they should be.

We now return to (11) and the energy stability analysis. In order to control the cubic term in (11) we employ the Sobolev inequality:

$$\left(\int_V (u_i u_i)^2 dV \right)^{1/2} \leqslant c \|\nabla \boldsymbol{u}\|^2;$$

for no slip boundary conditions this is the usual embedding inequality of $H_0^1(V)$ in $L^4(V)$, and it may be shown that this inequality holds also for stress free boundary conditions. In both cases the embedding constant c depends on the domain V.

The procedure is as follows,

$$\langle \theta u_i u_i \rangle \leq \left(\int_V \theta^2 dV \right)^{1/2} \left(\int_V (u_i u_i)^2 dV \right)^{1/2} \leq c \|\theta\| \|\nabla \boldsymbol{u}\|^2 . \tag{25}$$

Note that there exists a positive constant $k_1 > 0$ such that

$$D = \|\boldsymbol{u}\|^2 + \Gamma\langle z u_i u_i \rangle + \lambda \|\nabla \boldsymbol{u}\|^2 + \|\nabla \theta\|^2 \geq k_1 \|\boldsymbol{u}\|^2 + \lambda \|\nabla \boldsymbol{u}\|^2 + \|\nabla \theta\|^2 = \mathcal{D}, \text{ say.}$$

From inequality (11) we then deduce $dE/dt \leq -a\mathcal{D} + R^{-1}\Gamma Pr\langle \theta u_i u_i \rangle$, and the cubic term may be bounded, using (25), in terms of $\mathcal{D}$ and E. For, $c\|\theta\| \|\nabla \boldsymbol{u}\|^2 \leq (\sqrt{2}c/(\sqrt{Pr}\lambda))\, \mathcal{D}E^{1/2}$ and so

$$dE/dt \leq -a\mathcal{D} + A\mathcal{D}E^{1/2} , \tag{26}$$

where $A = \Gamma c\sqrt{2Pr}/R\lambda$.

Hence, provided

$$(1)\ R < R_E \quad \text{and} \quad (2)\ E^{1/2}(0) < a/A \tag{27}$$

then (26) can be integrated as in Chapter 2 of [15] to give $E(t) \leq E(0) \exp(-K\zeta t)$ where $K = a - AE^{1/2}(0)$, is a positive constant and ζ is given by the inequality $\mathcal{D} \geq \zeta E$. Thus we have established conditional nonlinear stability under conditions (27).

Remarks. 1) It is important to realise that the above analysis hinges on the presence of the Brinkman term as otherwise there is no $\|\nabla \boldsymbol{u}\|^2$ term in $\mathcal{D}$ to control the cubic term. The equivalent problem without the Brinkman term is unresolved.

2) We could have allowed the effective viscosity $\tilde{\mu}$ to also have a linear dependence on temperature, but the analysis then becomes much more complicated and a generalized energy such as that employed in [11] is necessary.

3. Numerical results

The analysis so far has been performed with a Rayleigh number which is defined in terms of the viscosity at the lower boundary. In practice, it will probably be more useful to employ a Rayleigh number which is defined in terms of an average viscosity over the layer. If we let $\bar{\mu}(\bar{T})$ be the viscosity associated with the steady solution, then

$$\bar{\mu} = \mu_0 (1 - \gamma(\bar{T} - T_0)) = \mu_0 (1 + \gamma\beta z) . \tag{28}$$

Define now an average viscosity, μ_{av}, by

$$\mu_{av} = \frac{1}{d} \int_0^d \bar{\mu}\, dz = \mu_0 \left(1 + \frac{1}{2}\Gamma \right) . \tag{29}$$

In our interpretation in this section we, therefore, propose to employ a Rayleigh number, Ra, defined as $Ra = \alpha\beta g d^2 k \rho_0 / (K\mu_{av})$ and this is easily seen to be related to the

Rayleigh number, R^2, of §2 by

$$Ra = R^2/(1 + \Gamma/2)\,. \tag{30}$$

The tables below which present the critical values of Rayleigh and wavenumbers for linear instability and nonlinear energy stability are based on (30). The results presented are all for free surface boundary conditions.

TABLE I. – *Critical Rayleigh numbers Ra, with their respective critical wavenumbers a, versus the non-dimensional viscosity coefficient Γ, for non-dimensionalized effective viscosities of* 0.1 *and* 0.2, *and free surface boundary conditions.*

$\lambda = 0.1$		$\lambda = 0.2$		Γ
Ra	a^2	*Ra*	a^2	
108.573	6.111	174.948	5.621	0.0
105.389	6.153	168.647	5.649	0.1
102.485	6.193	162.912	5.672	0.2
95.121	6.307	148.431	5.756	0.5

TABLE II. – *Critical Rayleigh numbers Ra, with their respective critical wavenumbers a, versus the non-dimensional viscosity coefficient Γ, for non-dimensionalized effective viscosities of* 0.3, 0.4 *and* 0.5, *and free surface boundary conditions.*

$\lambda = 0.3$		$\lambda = 0.4$		$\lambda = 0.5$		Γ
Ra	a^2	*Ra*	a^2	*Ra*	a^2	
240.970	5.420	306.873	5.310	372.722	5.241	0.0
231.547	5.441	294.325	5.327	357.047	5.255	0.1
222.975	5.462	282.913	5.344	342.793	5.269	0.2
201.352	5.522	254.137	5.393	306.857	5.310	0.5

From the tables it is immediately evident that the effective viscosity plays a large role in stabilizing, as one would expect. We also see that as Γ increases the critical Rayleigh number decreases which means it is easier for convection to commence; this is physically correct since increasing Γ means decreasing viscosity and one then expects convective motion to be easier.

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