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Multiplicity of homoclinic orbits for a class of asymptotically periodic Hamiltonian systems

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Analisi matematica. — *Multiplicity of homoclinic orbits for a class of asymptotically periodic Hamiltonian systems.* Nota di PIERO MONTECCHIARI, presentata (*) dal Corrisp. A. Ambrosetti.

ABSTRACT. — We prove the existence of infinitely many geometrically distinct homoclinic orbits for a class of asymptotically periodic second order Hamiltonian systems.

KEY WORDS: Hamiltonian systems; Homoclinic orbits; Multibump solutions; Minimax argument.

RIASSUNTO. — *Molteplicità di orbite omocline per sistemi hamiltoniani asintoticamente periodici.* Si dimostra l'esistenza di infinite orbite omocline geometricamente distinte per una classe di sistemi Hamiltoniani del secondo ordine asintoticamente periodici.

1. INTRODUCTION

In this work we study the problem of existence of homoclinic solutions of a second order asymptotically periodic Hamiltonian system: *find* $q \in C^2(\mathbf{R}, \mathbf{R}^m) \setminus \{0\}$ *such that*:

$$(HS) \qquad \ddot{q} = q - \nabla V(t, q), \qquad q(t) \to 0 \text{ and } \dot{q}(t) \to 0 \text{ as } |t| \to \infty$$

∇V being asymptotic, as $t \to -\infty$, to a periodic function ∇V_-. Precisely we assume that $V, V_- \in C^1(\mathbf{R} \times \mathbf{R}^m, \mathbf{R})$ satisfy

$V1)$ $|\nabla V(t,x)|$, $|\nabla V_-(t,x)| = o(x)$ *as* $x \to 0$,

$V2)$ $|\nabla V(t,\cdot)|$, $|\nabla V_-(t,\cdot)|$ *are locally lipschitz continuous functions*,

$V3)$ $\exists \mu > 2 / 0 < \mu V(t,x) \leq \nabla V(t,x)x$ *and* $0 < \mu V_-(t,x) \leq \nabla V_-(t,x)x$ $\forall x \neq 0$,

uniformly with respect to $t \in \mathbf{R}$, and

$V4)$ $\exists T_- > 0 / V_-(t + T_-, x) = V_-(t,x)$ $\forall (t,x) \in \mathbf{R} \times \mathbf{R}^m$,

$V5)$ $|\nabla V(t,x) - \nabla V_-(t,x)| \to 0$ *as* $t \to -\infty$ *unif. on the compacts of* $\mathbf{R}^m$.

This setting is a natural generalization of the case in which V is periodic in time (see [1] for a study of the asymptotically periodic problem for a class of semilinear elliptic equations on $\mathbf{R}^n$). We note that the periodic problem always admits at least one non trivial solution, see [3, 5, 8]. This is not the case for the asymptotically periodic problem which presents situations in which there are no solutions different from $q = 0$, like for example the case in which $V(t,x) = (\pi + \arctan(t)) \cdot |x|^4$. This does not happen if we make a discreteness hypothesis on the set of critical points of the functional associated to the problem at $-\infty$: $\varphi_-(u) = (1/2)\|u\|^2_{1,2} - \int_{\mathbf{R}} V_-(t,u)\,dt$, $u \in W^{1,2}(\mathbf{R}, \mathbf{R}^m)$. To be precise, letting c be the mountain pass level of φ_-, and noting that φ_- is invariant under the $\mathbf{Z}$-action:

(*) Nella seduta del 18 giugno 1993.

$j \to u(\cdot - jT_-)$ we require that

$(*)$ *there exists a* $c^* > c$ *such that* $K_-^{c^*}/\boldsymbol{Z}$ *is finite*,

where $K_-^{c^*}$ is the set of critical points of φ_- with critical value less or equal to c^*.

In this setting we are able to prove our main theorem:

THEOREM 1.1. *If* $V1$)-$V5$) *and* $(*)$ *hold then* (HS) *admits infinitely many homoclinic solutions.*

Precisely there exists a homoclinic solution $u \neq 0$ *of the equation* $\ddot{q} = q - \nabla V_-(t, q)$ *for which we have that* $\forall r > 0$ *there exists* $M = M(r) > 0$ *and* $n_0 = n_0(r) \in \boldsymbol{Z}$ *such that for each finite sequence* $\{p_1, \dots, p_k\} \subset \boldsymbol{Z}$ *that verifies* $p_j - p_{j+1} > M$, $j = 1, \dots, k-1$ *and* $p_1 < n_0$, *there exists a homoclinic solution* x *of* (HS) *such that, if we put* $p_0 = +\infty$, $p_{k+1} = -\infty$, *then* $|x(t) - u(t - p_j T_-)| < r$ $\forall t \in ((1/2)(p_j + p_{j+1})T_-, (1/2)(p_j + p_{j-1})T_-)$, $j = 1, \dots, k$.

In particular for $k = 1$ we obtain that if $p \in \boldsymbol{Z}$ is smaller than a certain value n_0, then near $u(\cdot - pT_-)$ there is a homoclinic solution of (HS). For $k > 1$ we obtain homoclinic solutions of (HS) which go away from zero and return near it, k times, staying near translates of u.

We call this type of solution k-bump solution.

The first proof of existence of 2-bump solutions, under the hypothesis $(*)$, was given in [9] for a class of first order Hamiltonian systems, and then in [4] was proved the existence of k-bump solutions for any $k \in \boldsymbol{N}$ for a class of second order Hamiltonian systems.

Independence from k of the distance of the bumps was proved by Eric Séré [10] for first order convex and periodic Hamiltonian systems and its main consequence is the existence of a new class of solutions, which seems to be related to the chaotic behavior of this type of systems. We note that in [10], instead of $(*)$, it is assumed only that the set of critical points of the functional associated to the problem, with critical value less then or equal to c^*, is denumerable.

Our result is the analogous of the Séré' one for a second order, asymptotically periodic Hamiltonian system. When V is periodic, there are no restrictions on p_1, and Theorem 1.1 strengthens the result in [4], showing that the distance between any two bumps of a k-bump solution is independent of k. In particular, from Theorem 1.1, as in [10], we deduce:

COROLLARY 1.2. *Assume* $V1$)-$V5$) *and* $(*)$. *Then for the same* u *of Theorem 1.1 we have that* $\forall r > 0$ *there exists* $M = M(r) > 0$, $n_0 = n_0(r) \in \boldsymbol{Z}$ *such that if* $\{p_j\}_{j \in \boldsymbol{N}} \subset \boldsymbol{Z}$ *satisfies* $p_1 < n_0$, $p_j - p_{j+1} \geq M$, $\forall j \in \boldsymbol{N}$ *then there exists* $x \in C^2(\boldsymbol{R}, \boldsymbol{R}^m)$ *such that* $\ddot{x}(t) = x(t) - \nabla V(t, x(t))$, $\forall t \in \boldsymbol{R}$ *and such that if we put* $p_0 = +\infty$, *then* $\forall j \in \boldsymbol{N}$ $|x(t) - u(t - p_j T_-)| < r$ $\forall t \in ((1/2)(p_j + p_{j+1})T_-, (1/2)(p_j + p_{j-1})T_-)$.

Obviously an analogous of Theorem 1.1 holds if the potential V is asymptotic at $+\infty$ in the sense of $V5$), to a certain periodic potential V_+ which satisfies also $V1$)-$V4$) and $(*)$.

2. Preliminaries

We set $X = W^{1,2}(\mathbf{R}, \mathbf{R}^m)$, $\|\cdot\| = \|\cdot\|_{1,2}$, and, for $u \in X$,

$$\varphi(u) = (1/2)\|u\|^2 - \int_R V(t,u)\,dt, \qquad \varphi_-(u) = (1/2)\|u\|^2 - \int_R V_-(t,u)\,dt.$$

We have that $\varphi, \varphi_- \in C^1(X, \mathbf{R})$ and if $K_- = \{u \in X \setminus \{0\} / \varphi'_-(u) = 0\}$, $K = \{u \in X \setminus \{0\} / \varphi'(u) = 0\}$ then $\Lambda = \inf_{K_- \cup K} \|u\| > 0$. We have that φ and φ_- satisfy the geometrical hypotheses of the Mountain Pass theorem. The Palais Smale condition, see [2], does not hold for the invariance of φ_- under the action of the non compact group of translations by integer multiples of T_-. In any case, by $V1$) and the continuity of the embedding $X \to L^\infty(\mathbf{R}, \mathbf{R}^m)$, we get that there exists $\rho_0 > 0$ such that if $\{u_n\}_{n \in N}$ is a Palais Smale sequence of φ_i with $\|u_n\| \leq 2\rho_0$, then $u_n \to 0$ as $n \to \infty$. From this simple fact and using the concentration-compactness lemma [6], if we put for A measurable $\subset \mathbf{R}$, $\|u_n\|_A^2 = \int_A |\dot{u}_n|^2 + |u_n|^2 dt$, we get:

Proposition 2.1. *Assume $V1$)-$V5$) and let $\{u_n\}_{n \in N} \subset X$ such that $\varphi(u_n) \to b$, $\varphi'(u_n) \to 0$, as $n \to \infty$ and finally $\exists R > 0$ such that $\|u_n\|_{t > R} \leq \rho_0$. Then there exist a subsequence of $\{u_n\}_{n \in N}$ (still denoted with $\{u_n\}_{n \in N}$), a critical point u of φ, an integer $k \in \mathbf{N} \cup \{0\}$, k sequences $\{t_n^i\}_{n \in N} \subset \mathbf{Z}$ and k non zero critical points of φ_-, $v_i \in K_-$, $i = 1, \ldots, k$ such that*

1) $t_n^1 \to -\infty$ *and* $t_n^j - t_n^{j-1} \to -\infty$, $\quad j = 2, \ldots, k$,

2) $u_n \to u$ *weakly in* X,

3) $\left\| u_n - u - \sum_{i=1}^k v_i(\cdot - t_n^i T_-) \right\| \to 0 \quad$ *as* $n \to \infty$,

4) $b = \varphi(u) + \sum_{i=1}^k \varphi_-(v_i)$.

In particular if a Palais Smale sequence $\{u_n\}_{n \in N}$ at a level b of φ does not converge and satisfies for an $R > 0$, $\|u_n\|_{t > R} \leq \rho_0$, then for any $R_- < 0$, we have that, up to a subsequence, $\|u_n\|_{t < R_-} > (1/2)\Lambda$ for n sufficiently large.

Lemma 2.2. *Assume $V1$)-$V5$) and let $r' = (1/2)\min\{\Lambda, \rho_0\}$. Then any Palais Smale sequence $\{u_n\}_{n \in N}$ at a level b of φ such that there exists $R > 0$ with $\|u_n\|_{|t| \geq R} \leq r'$ $\forall n \in \mathbf{N}$ admits a converging subsequence.*

By the concentration-compactness lemma it is also possible to characterize the Palais Smale sequences of φ_-. This characterization together with the hypothesis $(*)$, allow us to bound from below $|\varphi'_-(u)|$ in certain regions of X even if φ_- does not satisfy the Palais Smale condition. In fact by $(*)$ we get that there exists a $\rho_1 > 0$ which is smaller than the distance between any two point of $K_-^{c^*}$. If for $r > 0$ we set $N_r(K_-^{c^*}) = \{x \in X / \inf_{y \in K_-^{c^*}} \|x - y\| \leq r\}$ and if $r'' = \min\{r', \rho_1/3\}$ then it is possible to prove that:

Lemma 2.3. *Assume V1)-V5) and (*). Then* $\forall r_1 < r_2 \in (0, r'')$, $\exists \mu_1 = \mu_1(r_1, r_2) > 0$ *such that*: $q \in N_{r_2}(K_-^{c^*}) \setminus N_{r_1}(K_-^{c^*})$ *and* $\varphi_-(q) < c^* \Rightarrow |\varphi'_-(q)| \geq \mu_1$.

Another important consequence of the hypothesis (*) together with the characterization of the Palais Smale sequence of φ_- is that the critical levels of φ_- are isolated points of the set of asymptotic critical level of φ_- (we say that $b \in \boldsymbol{R}$ is an asymptotic critical level of φ_- if there exists at this level a Palais Smale sequence of φ_-):

Lemma 2.4. *Assume V1)-V5) and (*). Then for any critical level (of* φ_-*)* $b < c^*$ *there exists* $\lambda_0 = \lambda_0(b) \in (0, c^* - b)$ *such that* $(b - \lambda_0, b + \lambda_0)$ *does not contain asymptotic critical levels different from b.*

From this we get that if $b \in \varphi_-(K_-^{c^*})$, $b < c^*$, and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in (0, \lambda_0(b))$, $\lambda_1 < \lambda_2$, $\lambda_3 < \lambda_4$, then there exists $\mu_2 = \mu_2(\lambda_1, \lambda_2, \lambda_3, \lambda_4) > 0$ such that

$$x \in \varphi_-^{-1}((b - \lambda_4, b - \lambda_3) \cup (b + \lambda_1, b + \lambda_2)) \Rightarrow \|\varphi'_-(x)\| \geq \mu_2 . \tag{2.5}$$

The last property we give here is connected with the asymptotic assumption on V:

Lemma 2.6. $\forall \varepsilon > 0$, $\forall C > 0$ *there exists* $n_0 = n_0(\varepsilon, C) \in \boldsymbol{Z}$ *such that*: $u \in B(0, C)$, $u(t) = 0$ $\forall t \geq n_0 \Rightarrow \|\varphi'(u) - \varphi'_-(u)\| \leq \varepsilon$.

3. Sketch of the Proof of Theorem 1.1.

From now on we will assume for simplicity that $T_- = 1$ and if f: $X \to R$ and $a, b \in \boldsymbol{R}$ we set $f^a = \{x \in X / f(x) \leq a\}$, $f_a = (-f)^{-a}$, $f_a^b = f^b \cap f_a$. Also if $s \in \boldsymbol{R}$ and $x \in X$ we put $s * x = x(\cdot - s)$.

Given $n \in \boldsymbol{Z}, k, N \in \boldsymbol{N}$, we say that $\boldsymbol{p} = (p_0, p_1, \ldots, p_k, p_{k+1}) \in P(k, n, N)$ if $p_0 = +\infty$, $p_{k+1} = -\infty$, $p_j \in \boldsymbol{Z}$, $1 \leq j \leq k$, $p_j - p_{j+1} \geq 2N(N + 3/2)$, $1 \leq j \leq k$, and finally $p_1 < n - N(N+1)$. If $\boldsymbol{p} \in P(k, n, N)$, then for $i = 1, \ldots, k$ we set $\mathcal{U}_i = ((p_i + p_{i+1})/2, \ (p_i + p_{i-1})/2)$ and we define the functionals, $\varphi_{-,i}(x) = (1/2)\|x\|_{\mathcal{U}_i}^2 - \int_{\mathcal{U}_i} V_-(t, x)\,dt$, $x \in X$, which are in $C^1(X, \boldsymbol{R})$. Also, if $r_2 > r_1 \geq 0$, $u \in X$ and $\boldsymbol{p} \in P(k, n, N)$, we set $B_{\boldsymbol{p}}^u(r_2, r_1) = \{x \in X / r_1 \leq \max_{i=1,\ldots,k} \|x - p_i * u\|_{\mathcal{U}_i} < r_2\}$. Putting $K_-(c) = K_- \cap \{x \in X / \varphi_-(x) = c\}$, Theorem 1.1 will be proved if we show that

Theorem 3.1. *Assume V1)-V5) and (*). Then there exists* $u \in K_-(c)$ *such that* $\forall r > 0$ $\exists N = N(r) > 0$, $n = n(r) \in \boldsymbol{Z}$ *such that* $K \cap B_{\boldsymbol{p}}^u(r, 0) \neq \emptyset$, $\forall k \in \boldsymbol{N}$, $\forall \boldsymbol{p} \in P(k, n, N)$.

Proof. We give first two technical lemmas.

From Prop. 2.22 of [4] and Lemma 2.3, we can prove that $\exists r''' \in (0, r'')$ for which

Lemma 3.2. $\exists u \in K_-(c)$ *for which* $\forall r \in (0, r''')$, $\forall h_+ > 0$, $\exists h_- = h_-(r) > 0$, $\exists R = R(r, h_+) > 0$ *and* $\exists \bar{g} \in C([0, 1], X)$ *such that*:

1) $\operatorname{supp}(\bar{g}(t)) \subset (-R, R)$ $\forall t \in [0, 1]$,

2) $\bar{g}(0), \bar{g}(1) \in \partial B(u, r)$ *and* $\bar{g}(t) \in B(u, r)$ $\forall t \in [0, 1]$,

3) $\max_{t \in [0,1]} \varphi_-(\bar{g}(t)) < c + h_+$,

4) $\bar{g}(t) \notin B(u, r/2) \Rightarrow \varphi_-(\bar{g}(t)) \leq c - h_-$,

5) $\forall g \in C([0,1], X)$ *with* $g(0) = \bar{g}(0)$, $g(1) = \bar{g}(1)$ *we have* $\max_{[0,1]} \varphi_-(g(t)) \geq c$.

We claim that Theorem 3.1 holds with this u. In fact by Lemmas 2.3, 2.4, 2.6 and by (2.5), we can prove also that

LEMMA 3.3. *$\forall r_1 < r_2 < r_3 \in (0, r''')$ there exists $\mu_1 = \mu_1(r_1, r_3) > 0$ and, if we fix $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in (0, \lambda_0(c))$, $\lambda_1 < \lambda_2$, $\lambda_3 < \lambda_4$, there exists $n_0 \in \mathbf{Z}$, $\varepsilon_1 > 0$, such that $\forall \varepsilon \in (0, \varepsilon_1)$, $\exists N_\varepsilon \in \mathbf{N}$, for which $\forall k \in \mathbf{N}$ and $\boldsymbol{p} \in P(k, n_0, N_\varepsilon)$ there exists a locally lipschitz continuous function $\mathcal{V}: X \to X$ such that $\mathcal{V}(x) \in B_p^0(2, 0)$ $\forall x \in X$, $\mathcal{V}(x) = 0$ $\forall x \in X \setminus B_p^u(r_3, 0)$ and*

1) $x \in B_p^u(r_2, r_1) \Rightarrow \varphi'(x)\,\mathcal{V}(x) \geq \mu_1$; $\|x - p_i * u\|_{\mathcal{U}_i} \in (r_1, r_2) \Rightarrow \varphi'_{-,i}(x)\,\mathcal{V}(x) \geq \mu_1$,

2) $x \in B_p^u(r_3, r_2) \Rightarrow \varphi'(x)\,\mathcal{V}(x) > 0$; $\|x - p_i * u\|_{\mathcal{U}_i} \in (r_2, r_3) \Rightarrow \varphi'_{-,i}(x)\,\mathcal{V}(x) > 0$,

3) $x \in B_p^u(r_3, 0) \cap ((\varphi_{-,i})_{b+\lambda_1}^{b+\lambda_2} \cup (\varphi_{-,i})_{b-\lambda_4}^{b-\lambda_3}) \Rightarrow \varphi'_{-,i}(x)\,\mathcal{V}(x) > 0$,

4) $x \in B_p^u(r_3, 0)$ *and* $\max_{0 \leq l \leq K} \|x\|_{E_l}^2 \geq 4\varepsilon \Rightarrow \langle x, \mathcal{V}(x)\rangle_{E_l} > 0$ $l = 0, \dots, k$,

where $E_l = (p_{l+1} + N(N+1),\ p_l - N(N+1))$ *and* $\langle x, \mathcal{V}(x)\rangle_{E_l} = \int_{E_l} \dot{x}\,\dot{\mathcal{V}}(x) + x\,\mathcal{V}(x)\,dt$. *Moreover if $K \cap B_p^u(r_1, 0) = \emptyset$, then $\exists \mu_p > 0$ such that*

5) $x \in B_p^u(r_1, 0) \Rightarrow \varphi'(x)\,\mathcal{V}(x) \geq \mu_p$.

If we consider the flow associated to this pseudogradient field, we call it $\eta(\cdot, x)$, we get that, if $K \cap B_p^u(r_1, 0) = \emptyset$, then φ is always decreasing along the trajectories of $\mathcal{V}$ and, if for an $i \in \{1, \dots, k\}$, $\|\eta(s, x) - p_i * u\|_{\mathcal{U}_i} \geq r_1$ $\forall s \in [t_0, t_1]$, then also the function $s \to \varphi_{-,i}(\eta(s, x))$ is decreasing on $[t_0, t_1]$. Moreover, thanks to (3) of Lemma 3.3, we have that

(3.4) $\qquad \varphi_{-,i}^{c+\lambda_1}, \varphi_{-,i}^{c-\lambda_4}$ *are positively invariant sets*,

that is $\eta(t, \varphi_{-,i}^{c+\lambda_1}) \subset \varphi_{-,i}^{c+\lambda_1}$, $\eta(t, \varphi_{-,i}^{c-\lambda_4}) \subset \varphi_{-,i}^{c-\lambda_4}$, $\forall t \geq 0$.

Setting $\mathcal{E} = \{x \in X / \max_{0 \leq l \leq k} \|x\|_{E_l}^2 \leq 4\varepsilon\}$ by (4) of Lemma 3.3, we get also that

(3.5) $\qquad \mathcal{E}$ *is a positively invariant set*.

Assume now by contradiction that there exists $\bar{r} > 0$, such that $\forall N > 0$, $\forall n \in \mathbf{Z}$ there exist $k \in \mathbf{N}$ and $\boldsymbol{p} \in P(k, n, N)$ for which $K \cap B_p^u(\bar{r}, 0) = \emptyset$. Fixing $r_0 = (1/2)\min\{r''', \bar{r}\}$, we can use Lemma 3.2 with $h_+ = (1/3)\min\{\lambda_0(c), (1/12)\mu_1 r_0\}$ and $r = r_0$ getting that $\exists h_- \in (0, h_+)$, $R > 0$, $\bar{g} \in C^1([0,1], X)$, which satisfy the listed properties (1)-(5).

Put also $r_1 = r_0/2$, $r_2 = 2r_0/3$, $r_3 = 5r_0/6$, $\lambda_1 = (4/3)h_+$, $\lambda_2 = (5/3)h_+$, $\lambda_4 = (1/2)h_-$, $\lambda_3 = (1/3)h_-$ and fix a suitable small ε. By the contradiction hypothesis there exist $N > \max\{R, N_\varepsilon\}$, $n < n_0$, $k \in \mathbf{N}$, $\boldsymbol{p} \in P(k, n, N) \subset P(k, n_0, N_\varepsilon)$, for which $K \cap B_p^u(r_0, 0) = \emptyset$, so by Lemma 3.3, we get a field $\mathcal{V}$ which satisfies the properties (1)-(6) with this k and $\boldsymbol{p}$.

Consider the function $G: [0, 1]^k \to X$, $G(\theta) = \sum_{i=1}^{k} p_i * \bar{g}(\theta_i)$.

For any $\theta \in [0, 1]^k$ we have $\operatorname{supp}(G(\theta)) \subset \boldsymbol{R} \setminus \left(\bigcup_{l=0}^{k} E_l \right)$ therefore $G(\theta) \in \mathcal{E}$. Moreover, by construction, $G(\theta) \in B_p^u(r_0, 0) \cap \left(\bigcap_{i=1}^{k} (\varphi_{-,i})^{c+\lambda_1} \right)$ and if for a $\theta \in [0, 1]^k$ we have $G(\theta) \in X \setminus B_p^u(r_1, 0)$ then there exists $i_\theta \in \{1, \ldots, k\}$ such that $G(\theta) \in$ $\in (\varphi_{-,i_\theta})^{c-\lambda_4}$.

From this, using the pseudogradient flow, if ε was chosen sufficiently small, it is possible to prove that

LEMMA 3.6. $\theta \in \partial[0, 1]^k \Rightarrow \eta(t, G(\theta)) = G(\theta)$ $\forall t > 0$.

LEMMA 3.7. $\exists \mathcal{T} > 0$: $\forall \theta \in [0, 1]^k$ $\exists i_\theta \in \{1, \ldots, k\} / \varphi_{-,i_\theta}(\eta(\mathcal{T}, G(\theta))) \leq c - \lambda_4$.

From Lemma 3.7, if $0_i = \{\theta \in [0, 1]^k / \theta_i = 0\}$, $1_i = \{\theta \in [0, 1]^k / \theta_i = 1\}$, $i = 1, \ldots, k$, and if we put $\bar{G}(\theta) = \eta(\mathcal{T}, G(\theta))$, $\theta \in [0, 1]^k$ we get

LEMMA 3.8. $\exists i_0 \in \{1, \ldots, k\}$ $\exists \alpha \in C([0, 1], [0, 1]^k) / \alpha(0) \in 0_{i_0}$, $\alpha(1) \in 1_{i_0}$, $\bar{G}(\alpha(s)) \in (\varphi_{-,i_0})^{c-\lambda_4/2}$ $\forall s \in [0, 1]$.

Defining the cutoff function $\beta \in C(\boldsymbol{R}, \boldsymbol{R})$, such that $\beta(t) = 0$ if $t \notin \mathcal{U}_{i_0}$, $\beta(t) = 1$ if $t \in \mathcal{U}_{i_0} \setminus (E_{i_0} \cup E_{i_0 - 1})$ and in such a way it is linear on the intervals $\mathcal{U}_{i_0} \cap E_{i_0-1}$, $\mathcal{U}_{i_0} \cap$ $\cap E_{i_0}$, we set $\gamma(s) = \beta \bar{G}(\alpha(s))$, $s \in [0, 1]$. By Lemma 3.6 we have that $\gamma(0) = p_{i_0} * \bar{g}(0)$ and $\gamma(1) = p_{i_0} * g(1)$; moreover, by (3.5), $\bar{G}(\alpha(s)) \in \mathcal{E}$ for any $s \in [0, 1]$, therefore $|\varphi_{-,i_0}(\gamma(s)) - \varphi_{-,i_0}(\bar{G}(\alpha(s)))| \leq C\varepsilon$ $\forall s \in [0, 1]$, with $C = C(r''') > 0$. From this, if ε was chosen such that $C\varepsilon \leq (1/4)\lambda_4$, we get

$$\varphi_{-}(\gamma(s)) = \varphi_{-,i_0}(\gamma(s)) \leq \varphi_{-,i_0}(\bar{G}(\alpha(s))) + \lambda_4/4 \leq c - \lambda_4/4, \qquad \forall s \in [0, 1]$$

which is in contradiction with Lemma 3.2. q.e.d.

The complete proofs and other results are contained in [7].

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