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L^p-Estimates for linear elliptic systems with discontinuous coefficients

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Equazioni a derivate parziali. — *L^p-Estimates for linear elliptic systems with discontinuous coefficients.* Nota (*) di FILIPPO CHIARENZA, MICHELANGELO FRANCIOSI e MICHELE FRASCA, presentata dal Socio G. Fichera.

ABSTRACT. — In this *Note* we give L^p estimates ($1 < p < +\infty$) for the highest order derivatives of an elliptic system in non-divergence form with coefficients in VMO.

KEY WORDS: Elliptics systems; L^p-Estimates; VMO.

RIASSUNTO. — *Stime L^p per sistemi ellittici lineari con coefficienti discontinui.* In questa *Nota* si provano maggiorazioni in L^p ($1 < p < +\infty$) per le derivate di ordine massimo di un sistema ellittico in forma di non divergenza i cui coefficienti, eventualmente discontinui, appartengono a VMO.

1. INTRODUCTION

In the recent paper [6] F. Chiarenza, M. Frasca and P. Longo gave L^p-estimates, ($1 < p < +\infty$), for the second order derivatives of local solutions of the elliptic equation

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} = f$$

assuming a_{ij} in the Sarason VMO space (see n. 2).

They were able to get such a result by the use of an explicit representation formula for the second order derivatives of the solution in terms of some singular integral operators.

Similar results were previously obtained by L. Caffarelli [4] (in the case $p \geq n$) while studying some non linear problems. His deep technique relies heavily on the Pucci-Alexandroff maximum principle and then cannot be used for systems or higher order equations.

In this paper we study an elliptic system in the sense of Petrowski with coefficients in VMO, giving L^p-estimates, ($1 < p < +\infty$), for the highest order derivatives of local solutions. This was again achieved by means of an explicit representation formula, as in [6].

At our knowledge these estimates seem to be the first available for systems with discontinuous coefficients in the full range $p \in]1, +\infty[$.

Our result in the case of continuous coefficients is contained in [1, 2]. Some other similar results, valid only for p in a neighborhood of 2, have been given by [5].

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2. Preliminary facts

We recall the definition of BMO and VMO (see [10, 11]).

We say that a function f locally integrable in $\mathbf{R}^n$ is in the space BMO if

$$\sup_B \fint_B |f(x) - f_B|\, dx \equiv \|f\|_* < +\infty ,$$

where B is a ball of $\mathbf{R}^n$ and f_B is the average

$$\fint_B f(x)\, dx \equiv \frac{1}{|B|} \int_B f(x)\, dx .$$

For $f \in$ BMO we say that f is in VMO if

$$\lim_{r \to 0} \eta(r) = 0$$

where

$$\eta(r) = \sup_{\substack{B_\rho \\ \rho \leq r}} \fint_{B_\rho} |f(x) - f_{B_\rho}|\, dx ,$$

and B_s is a ball of $\mathbf{R}^n$ with radius s.

We note that uniformly continuous bounded functions in $\mathbf{R}^n$ belong to VMO as well as the functions in the space $W^{\vartheta, n/\vartheta}$, $0 < \vartheta \leq 1$. For details and properties of these spaces see [11, 12, 7].

In order to prove our result we will need the notion of Calderon-Zygmund kernel and the following theorem concerning the L^p-boundedness properties of some singular integral operators which is proved in [6].

Definition 2.1. Let k: $\mathbf{R}^n \setminus \{0\} \to \mathbf{R}$. We call k a Calderon-Zygmund kernel if

i) $k \in C^\infty(\mathbf{R}^n \setminus \{0\})$;

ii) k is homogeneous of degree $-n$;

iii) $\int_\Sigma k\, d\sigma_x = 0$, where $\Sigma = \{x \in \mathbf{R}^n :\ |x| = 1\}$.

Theorem 2.1. *Let Ω be an open subset of $\mathbf{R}^n$ and K: $\Omega \times (\mathbf{R}^n \setminus \{0\}) \to \mathbf{R}$ a function verifying*:

i) *$K(x, \cdot)$ is a Calderon-Zygmund for a.a. $x \in \Omega$*;

ii) $$\max_{|j| \leq 2n} \left\| \frac{\partial^j}{\partial z^j} K(x, z) \right\|_{L^\infty(\Omega \times \Sigma)} = M > +\infty .$$

For $f \in L^p(\Omega)$, $1 < p < +\infty$, $\varphi \in L^\infty(\mathbf{R}^n)$ *and* $x \in \Omega$, *set*

$$K_\varepsilon f(x) = \int_{\substack{|x-y|>\varepsilon \\ y\in\Omega}} K(x, x-y)\, f(y)\, dy\,,$$

$$C_\varepsilon[\varphi, f](x) = \varphi(x) K_\varepsilon f(x) - K_\varepsilon(\varphi f)(x) = \int_{\substack{|x-y|>\varepsilon \\ y\in\Omega}} K(x, x-y)\,[\varphi(x) - \varphi(y)]\, f(y)\, dy\,.$$

Then, for any $f \in L^p(\Omega)$,

$$Kf = \text{P.V.} \int_\Omega K(x, x-y)\, f(y)\, dy\,,$$

$$C[\varphi, f] = \text{P.V.} \int_\Omega K(x, x-y)[\varphi(x) - \varphi(y)]\, f(y)\, dy$$

exist in $L^p(\Omega)$ *and*

$$\lim_{\varepsilon\to 0} \|K_\varepsilon f - Kf\|_{L^p(\Omega)} = \lim_{\varepsilon\to 0} \|C_\varepsilon[\varphi, f] - C[\varphi, f]\|_{L^p(\Omega)} = 0\,.$$

Moreover, there exists a constant $c = c(n, p, M)$ *such that*

$$\|Kf\|_{L^p(\Omega)} \leqslant c\|f\|_{L^p(\Omega)}\,; \qquad \|C[\varphi, f]\|_{L^p(\Omega)} \leqslant c\|\varphi\|_*\, \|f\|_{L^p(\Omega)}\,.$$

Let Ω be an open subset of $\mathbf{R}^n$, $n \geqslant 3$. In Ω we consider the differential operators

(2.1) $$m_{ij}(x, D) = \sum_{|\alpha| = 2r} a_{ij}^{(\alpha)}(x) D^\alpha \qquad i, j = 1, \dots, N\,.$$

Here α is a multi-index and we set, as usual,

$$|\alpha| = \alpha_1 + \dots + \alpha_n\,, \qquad D^\alpha = \partial^{|\alpha|} / (\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n})\,.$$

Given $f_i \in L^q_{\text{loc}}(\Omega)$, $1 < q < +\infty$, $i = 1, \dots, N$, consider the system

(2.2) $$\sum_{j=1}^N m_{ij}(x, D)\, u_j(x) = f_i(x) \qquad i = 1, \dots, N\,.$$

A local solution of (2.2) is a vector function $u \equiv (u_1, \dots, u_N)$, $u_i \in W^{2r,q}_{\text{loc}}(\Omega)$, verifying (2.2) a.e. in Ω. Let us make the following assumption.

(2.3) $$a_{ij}^{(\alpha)} \in \text{VMO} \cap L^\infty(\mathbf{R}^n) \qquad i, j = 1, \dots, N\,, \quad |\alpha| = 2r\,.$$

We will set

(2.4) $$\tilde{M} = \max_{\substack{i,j = 1, \dots, N \\ |\alpha| = 2r}} \sup_\Omega |a_{ij}^{(\alpha)}(x)|\,.$$

For a.e. $x \in \Omega$ set $m_{ij}(x, \xi)$ for the homogeneous polynomial of degree $2r$ in ξ obtained substituting in (2.1) D^α with $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$. We suppose that system (2.2) is strongly elliptic in the following sense

(2.5) $$\exists \lambda > 0\colon L(x, \xi) \equiv \det(m_{ij}(x, \xi)) \geqslant \lambda |\xi|^{2rN}\,,$$

a.e. in Ω, for any $\xi \in \mathbf{R}^n$.

From now on we will assume without loss of generality n to be odd (see [8]).

3. Representation formula

We will now derive a representation formula for local solutions of system (2.2).

Let B a ball, $B \subseteq \Omega$, $\tilde{B}$ the subset of B where the $a_{ij}^{(\alpha)}$ are defined and (2.5) holds. Let $x_0 \in \tilde{B}$. Consider the differential operator

$$L(x_0) \equiv \det(m_{ij}(x_0, D)) . \tag{3.1}$$

$L(x_0)$ is a linear elliptic differential operator of order $2rN$.

Let $\Gamma(x_0, t)$ the F. John fundamental solution of (3.1). It is well known (see [9]) that Γ has the following form

$$\Gamma(x_0, t) = |t|^{2rN-n}\psi(x_0, t/|t|) \tag{3.2}$$

where $\psi(x_0, \cdot)$ is an analytic function.

From (3.2) it follows

$$|D^\alpha \Gamma(x_0, t)| \leqslant c|t|^{2rN-n-|\alpha|}$$

where c depends only on r, n, $|\alpha|$, λ and $\tilde{M}$.

Also it is possible to prove that $D^\alpha \Gamma(x_0, t)$ for $|\alpha| = 2rN$ is a homogeneous function of degree $-n$ with zero mean value on the sphere $|t| = 1$.

For this fundamental solution and for any $v \in C_0^\infty(B)$

$$v(x) = \int_\Omega \Gamma(x_0, x-y) L(x_0) v(y)\, dy . \tag{3.3}$$

Consider the matrix $(m_{ij}(x_0, \xi))$, the cofactor $m^{ij}(x_0, \xi)$ of the element $m_{ij}(x_0, \xi)$ and set $m^{ij}(x_0, D)$ for the corrispondent differential operator which has order $2r(N-1)$, unless $m^{ij}(x_0, D) \equiv 0$.

Assume $u = (u_1, \dots, u_N)$ to be a $C_0^\infty(B)$ vector function. Following an idea of Bureau [3] we can write

$$L(x_0) u_i = L(x_0)\left(\sum_{j=1}^N \delta_{ij} u_j\right) = \sum_{j=1}^N \delta_{ij} L(x_0) u_j = \sum_{j=1}^N \left(\sum_{k=1}^N m^{ki}(x_0, D) m_{kj}(x_0, D)\right) u_j ,$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$, $i, j = 1, \dots, N$. Then, using (3.3) we obtain

$$u_i(x) = \int_B \Gamma(x_0, x-y) \sum_{j=1}^N \left(\sum_{k=1}^N m^{ki}(x_0, D) m_{kj}(x_0, D)\right) u_j(y)\, dy =$$

$$= \sum_{k=1}^N \int_B \Gamma(x_0, x-y) m^{ki}(x_0, D) \sum_{j=1}^N m_{kj}(x_0, D) u_j(y)\, dy .$$

From this, integrating by parts, we can express u_i as a linear combination of integral of the form

$$\int_B D^\sigma \Gamma(x_0, x-y) \sum_{j=1}^N m_{kj}(x_0, D) u_j(y)\, dy ,$$

where $|\sigma| = 2r(N-1)$, which may be rewritten as

$$\int_B D^\sigma \Gamma(x_0, x-y)\left[\sum_{j=1}^N (m_{kj}(x_0, D) - m_{kj}(y, D))u_j(y) + \sum_{j=1}^N m_{kj}(y, D)u_j(y)\right]dy =$$

$$= \int_B D^\sigma \Gamma(x_0, x-y) \sum_{j=1}^N \sum_{|\alpha|=2r} [a_{kj}^{(\alpha)}(x_0) - a_{kj}^{(\alpha)}(y)] D^\alpha u_j(y)\, dy +$$

$$+ \int_B D^\sigma \Gamma(x_0, x-y) \sum_{j=1}^N m_{kj}(y, D)u_j(y)\, dy .$$

Hence, by standard arguments, one can see that the derivatives $D^\alpha u_i$, for $|\alpha| = 2r$, can be expressed as a linear combination of terms like

$$\text{P.V.}\int_B D^{\sigma+\alpha}\Gamma(x_0, x-y) \sum_{j=1}^N \sum_{|\alpha|=2r} [a_{kj}^{(\alpha)}(x_0) - a_{kj}^{(\alpha)}(y)] D^\alpha u_j(y)\, dy +$$

$$+\text{P.V.}\int_B D^{\sigma+\alpha}\Gamma(x_0, x-y) \sum_{j=1}^N m_{kj}(y, D)u_j(y)\, dy +$$

$$+c_{\sigma+\alpha}(x_0)\left(\left[\sum_{j=1}^N \sum_{|\alpha|=2r} (a_{kj}^{(\alpha)}(x_0) - a_{kj}^{(\alpha)}(x)) D^\alpha u_j(x)\right] + \sum_{j=1}^N m_{kj}(x, D)u_j(x)\right),$$

$|\sigma| = 2r(N-1)$, $|\alpha| = 2r$, $x \in B$, $x_0 \in \widetilde{B}$, where $c_{\sigma+\alpha}(x_0)$ is bounded uniformly in x_0 by a constant depending on n, r, N, λ, M and $\widetilde{M}$.

To go furher we now set $x = x_0$ and obtain that, for any $C_0^\infty(B)$ vector function $u = (u_1, \ldots, u_N)$, the derivative $D^\alpha u_i(x)$, $|\alpha| = 2r$, can be written a.e. in B as a linear combination of terms like

$$\text{P.V.}\int_B D^{\sigma+\alpha}\Gamma(x, x-y) \sum_{j=1}^N \sum_{|\alpha|=2r} [a_{kj}^{(\alpha)}(x) - a_{kj}^{(\alpha)}(y)] D^\alpha u_j(y)\, dy ,$$

$$\text{P.V.}\int_B D^{\sigma+\alpha}\Gamma(x, x-y) \sum_{j=1}^N m_{kj}(y, D)u_j(y)\, dy , \qquad \sum_{j=1}^N m_{kj}(x, D)u_j(x) .$$

4. Main result

In this section we state the main result of this paper which, given the representation formula we proved in the previous section and the recalled boundedness properties of the relevant singular operators (Theorem 2.1), can be proved much like in the paper [6].

THEOREM 4.1. *Assume (2.3), (2.5). Let $1 < q \leq p < +\infty$ and $u \in W_{\text{loc}}^{2r,q}(\Omega)$ a solution of* (2.2) *with $f_i \in L_{\text{loc}}^p$, $i = 1, \ldots, N$. Then we have $u \in W_{\text{loc}}^{2r,p}(\Omega)$.*

Also, given $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, there is a constant c not depending on u_i and f_i, $i = 1, \ldots, N$

such that

$$\sum_{\substack{j=1 \\ |\alpha|=2r}}^{N} \|D^{\alpha} u_j\|_{L^p(\Omega')} \leq c \left(\sum_{j=1}^{N} (\|u_j\|_{L^p(\Omega'')} + \|f_j\|_{L^p(\Omega'')}) \right).$$

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