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Pseudo-iteration semigroups and commuting holomorphic maps

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Geometria. — *Pseudo-iteration semigroups and commuting holomorphic maps.* Nota di GRAZIANO GENTILI e FABIO VLACCI, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — A connection between iteration theory and the study of sets of commuting holomorphic maps is investigated, in the unit disc of C. In particular, given two holomorphic maps f and g of the unit disc into itself, it is proved that if g belongs to the pseudo-iteration semigroup of f (in the sense of Cowen) then – under certain conditions on the behaviour of their iterates – the maps f and g commute.

KEY WORDS: Iteration theory; Commuting holomorphic maps; Fixed points.

RIASSUNTO. — *Semigruppi di pseudo iterazione e mappe olomorfe che commutano.* Si studia una relazione tra la teoria dell'iterazione e lo studio di famiglie di mappe che commutano, nel disco unità di C. In particolare, date due mappe olomorfe f e g del disco unità in sé, si prova che se g appartiene al semigruppo di pseudo iterazione di f (nel senso di Cowen) allora – sotto certe condizioni sul comportamento delle loro iterate – le mappe f e g commutano.

PREFACE

In the study of iteration theory for holomorphic maps, it was clear from the beginning that the behaviour of the iterates of a map depends, in a neighborhood of a fixed point, on the value of the derivative of the map at the fixed point itself (see, *e.g.* [9, 7]). In fact the study of fixed points plays a fundamental role in iteration theory. In the classical case of the unit disc $\Delta = \{z \in C: |z| < 1\}$ of C, for a holomorphic map from Δ into Δ, $f \in Hol(\Delta, \Delta)$, having a fixed point in Δ, the Schwarz and the Schwarz-Pick Lemmas completely reduce the study of the iterates of f to the knowledge of the value of the derivative of f at the fixed point. In the case of a map $f \in Hol(\Delta, \Delta)$ without fixed points in Δ, Julia, Wolff and Carathéodory proved «boundary versions» of the Schwarz and the Schwarz-Pick Lemmas, known as the Julia Lemma, the Julia-Wolff-Carathéodory Theorem and the Wolff's Lemma (Section 1). In particular the Wolff's Lemma states the existence (for any $f \in Hol(\Delta, \Delta)$ without fixed points in Δ) of a unique point $\tau \in \partial\Delta$, called the *Wolff point* of f, such that all the «horocycles» (*i.e.* all the Euclidean discs) tangent to $\partial\Delta$ at τ are mapped by f into «smaller horocycles» tangent to $\partial\Delta$ at τ. The Wolff point is fundamental in the study of the iterates of a holomorphic map f without fixed points in Δ: the Wolff-Denjoy Theorem asserts that the iterates of f converge, uniformly on compact sets, to the Wolff point τ.

On the other hand, the study of the set of fixed points of a holomorphic map $f \in Hol(\Delta, \Delta)$ turns out to be a useful tool in the determination of all holomorphic maps which commute with f (with respect to composition). The problem of finding conditions under which two (or more) given holomorphic maps commute has been investigated by many authors. For example Julia, Fatou, Ritt and Jacobsthal (see *e.g.* [4] for Bibliography) paid attention to the case of rational maps; Baker [2] and Hadamard [6]

(*) Nella seduta del 18 giugno 1993.

analyzed the case of entire functions. In the case of $Hol(\Delta, \Delta)$, Behan and Shields [3] proved that, in general (except the case of two hyperbolic automorphisms of Δ), two commuting holomorphic maps have the same (fixed point in Δ or the same) Wolff point.

In this paper we contribute to the investigation of the connection between iteration theory and the study of sets of commuting holomorphic maps in $Hol(\Delta, \Delta)$. We start from some results obtained by Cowen [4, 5] and Pommerenke [8] regarding the «pseudo-iteration semigroup of a map», which turns out to be a powerful tool in this environment (Section 2). Taking into account the results of Behan and Shields [3] mentioned above, we prove that, given two holomorphic maps f and g of the unit disc into itself with the same Wolff point, if g belongs to the pseudo-iteration semigroup of f, then – under certain conditions on the behaviour of their iterates – the maps f and g commute (Theorems 2.7 and 2.8).

1. Iteration and fixed points

Suppose $f \in Hol(\Delta, \Delta)$ is a holomorphic function of the unit disc Δ into itself. If f has a *fixed point* $z_0 \in \Delta$, that is if there exists a point z_0 in Δ such that $f(z_0) = z_0$, then the Schwarz-Pick lemma [11] implies that f maps every disc for the Poincaré metric, centered in z_0, into itself. Now, except the cases in which f is either the identity or an elliptic automorphism of Δ, (that is an automorphism with a fixed point in Δ), the sequence of iterates $\{f^k\}_{k \in N}$ converges, for all $z \in \Delta$ (and then uniformly on compact sets of Δ), to the constant map z_0. If, instead, f has no fixed points in Δ, then the Wolff's Lemma describes the behaviour of the iterates of f in terms of the «horocycles», which play the role of «Poincaré discs at the boundary of Δ». Let $\tau \in \partial\Delta$ be a point; then for all $R > 0$ the open disc of Δ tangent to $\partial\Delta$ at τ defined as $E(\tau, R) = \{z \in \Delta \colon |\tau - z|^2 / /(1 - |z|^2) < R\}$ is called a *horocycle* of center τ and radius R. We have [12, 1]:

Lemma 1.1 (Wolff). *Let $f \in Hol(\Delta, \Delta)$ be without fixed points. Then there is a unique $\tau \in \partial\Delta$ such that for all $z \in \Delta$*

$$|\tau - f(z)|^2/(1 - |f(z)|^2) \leq |\tau - z|^2/(1 - |z|^2) \tag{1.1}$$

that is

$$\forall R > 0 \quad f(E(\tau, R)) \subseteq E(\tau, R), \tag{1.2}$$

where $E(\tau, R)$ is the horocycle of center τ and radius $R > 0$. Moreover, the equality (1.1) *holds at one point* (*and hence at all points*) *if and only if f is a* (*parabolic*) *automorphism of Δ leaving τ fixed.*

If $f \in Hol(\Delta, \Delta)$ has a fixed point in Δ (and $f \neq id_\Delta$), then we denote this fixed point by $\tau(f)$. Otherwise, $\tau(f)$ denotes the point constructed in Lemma 1.1. In both cases $\tau(f)$ is called the *Wolff point* of f. The behaviour of the sequence of the iterates of a holomorphic map $f \in Hol(\Delta, \Delta)$ is described in general by the classical (see, *e.g.*, [12, 1]).

THEOREM 1.2 (Wolff-Denjoy). *If $f \in Hol(\Delta, \Delta)$ is neither an elliptic automorphism nor the identity, then the sequence of iterates $\{f^k\}_{k \in N}$ converges, uniformly on compact sets, to the Wolff point τ of f.*

As we will see in a while, when $f \in Hol(\Delta, \Delta)$ has no fixed points in Δ, in some sense its Wolff point $\tau(f)$ plays the role of a «fixed point» on the boundary of Δ. Since a map $f \in Hol(\Delta, \Delta)$ and its derivative need not be continuous in $\bar{\Delta}$, we have to explain the meaning of «fixed point on the boundary» and «derivative of f at a fixed point on the boundary».

DEFINITION 1.3. Take $\sigma \in \partial\Delta$ and $M > 1$. The set

$$K(\sigma, M) = \{z \in \Delta \mid |\sigma - z|/(1 - |z|) < M\} \tag{1.3}$$

is called *Stolz region $K(\sigma, M)$ of vertex σ and amplitude M.*

The Stolz region $K(\sigma, M)$ is an «angular region» with vertex at σ. Stolz regions are used to give the following

DEFINITION 1.4. Let $f\colon \Delta \to \bar{C}$ be a function. We say that c is the *non-tangential limit* (or *angular limit*) of f at $\sigma \in \partial\Delta$ if $f(z) \to c$ as z tends to σ within $K(\sigma, M)$, for all $M > 1$. We shall write $K\text{-}\lim_{z \to \sigma} f(z) = c$.

The definition of non-tangential limit is used in the classical (see [13]).

THEOREM 1.5 (Julia-Wolff-Carathéodory). *Let $f \in Hol(\Delta, \Delta)$ and let τ, σ be any two points in $\partial\Delta$, then*

$$K\text{-}\lim_{z \to \sigma} \frac{\tau - f(z)}{\sigma - z}$$

exists. If this K- lim *is finite, then*

$$K\text{-}\lim_{z \to \sigma} f(z) = \tau$$

and

$$K\text{-}\lim_{z \to \sigma} f'(z) = K\text{-}\lim_{z \to \sigma} \frac{\tau - f(z)}{\sigma - z}.$$

In particular, if $\tau = \sigma$ then the non-tangential limit of f' at σ is a strictly positive real number.

We say that $\tau \in \partial\Delta$ is a *fixed point of f on the boundary of Δ* if $K\text{-}\lim_{z \to \tau} f(z) = \tau$; at the same time we call *derivative of f at a fixed point τ on the boundary of Δ* the value of $K\text{-}\lim_{z \to \tau} f'(z)$. The Julia-Wolff-Carathéodory Theorem states that the derivative of a holomorphic map f at a fixed point τ on the boundary is a positive real number $f'(\tau)$. The Wolff's Lemma yields [4, 1] that, if τ is the Wolff point of f, then $f'(\tau)$ is bounded from above by 1.

The value of the derivative of a holomorphic map f at its Wolff point is important to understand the behaviour of the iterates of f. In fact the two following Lemmas hold, [4]:

LEMMA 1.6. *Suppose that a map* $f \in Hol(\Delta, \Delta)$ *has its Wolff point* $\tau(f)$ *on the boundary of* Δ. *If* $K\text{-}\lim_{z \to \tau} f'(z) < 1$, *then for any* $z \in \Delta$ *the sequence of iterates* $\{f^n(z)\}_{n \in N}$ *converges to* τ *non-tangentially (that is, converges within* $K(\tau, M)$, *for some* $M > 1$).

LEMMA 1.7. *Suppose that a map* $f \in Hol(\Delta, \Delta)$ *has its Wolff point* $\tau(f)$ *on the boundary of* Δ. *If for some* z_0 *in* Δ *the sequence of iterates* $\{f^n(z_0)\}_{n \in N}$ *converges to* τ *non-tangentially, then for any compact set* K *in* Δ, *the sequence of iterates* $\{f^n(K)\}_{n \in N}$ *converges to* τ *non-tangentially*.

As a consequence of the two Lemmas above, we obtain that, for a holomorphic map $f \in Hol(\Delta, \Delta)$, whenever there exists a point z_0 in Δ such that the sequence of iterates $\{f^n(z_0)\}_{n \in N}$ converges to the Wolff point $\tau(f)$ non-tangentially, then for any point z in Δ the sequence of iterates $\{f^k(z)\}_{k \in N}$ converges to $\tau(f)$ non-tangentially. On the other hand, if, for a holomorphic map $f \in Hol(\Delta, \Delta)$, we find a point z_1 in Δ such that the sequence of iterates $\{f^n(z_1)\}_{n \in N}$ converges to τ tangentially, then for any point z in Δ the sequence of iterates $\{f^k(z)\}_{k \in N}$ converges to τ tangentially. Let us remark that, by Lemma 1.6, a point $z_1 \in \Delta$ such that $\{f^n(z_1)\}_{n \in N}$ converges to the Wolff point $\tau(f)$ tangentially does not exist if the derivative of f at $\tau(f)$ is strictly less than 1. The converse of this statement does not hold in general: consider, for example, the map $f(z) = (1 + 3z^2)/(3 + z^2)$; we have $\tau(f) = 1$, $f'(\tau) = 1$ and $\{f^n(0)\}_{n \in N} \subset \boldsymbol{R}$ converges to 1 non-tangentially.

2. The pseudo-iteration semigroup of a map $f \in Hol(\Delta, \Delta)$

The definition of the pseudo-iteration semigroup of a map $f \in Hol(\Delta, \Delta)$ is given by Cowen [5] in two different cases:

1) the case in which the value of the derivative of the map f at the Wolff point $\tau(f)$ is 0 (by Theorem 1.5, this can occur only if $\tau(f)$ is in Δ); in this (simpler) case, for the definition of the pseudo-iteration semigroup, Cowen uses several results due to Böttcher, which are related to the study of «functional equations» (see, *e.g.* [5, 10]).

2) the case in which the value of the derivative of the map f at the Wolff point $\tau(f)$ is not 0: this is the case we are interested in.

To state the «main Theorem» of Cowen and some of its consequences, [4], on which the definition of the pseudo-iteration semigroup of a map is based, we need the following notions:

let $\sigma \in \partial\Delta$ and let T be the line containing the diameter of $\overline{\Delta}$ passing through σ. An *angular sector* of vertex σ and opening θ in Δ, is the intersection of Δ with the open angle having vertex in σ, bisectrix line T and opening θ. A *small angular sector* of vertex σ and opening θ in Δ, is the intersection of an angular sector of vertex σ and opening θ in Δ with any open, Euclidean disc of positive radius centered at σ.

An open, connected, simply connected subset V of Δ is called a *fundamental set for* $f \in Hol(\Delta, \Delta)$, if $f(V) \subset V$ and if for any compact set K in Δ, there is a positive integer n so that $f^n(K) \subset V$. The «fundamental set» of a map f is a set of points «near» the Wolff point $\tau(f)$ «small enough» that f is «well behaved» on it, and «large enough» that $f^n(z)$ belongs eventually to this set.

THEOREM 2.1 (Cowen). *Let $\varphi \in Hol(\Delta, \Delta)$ be neither a constant map nor an automorphism of Δ. Let τ be the Wolff point of φ and suppose that $\varphi'(\tau) \neq 0$. Then there exists a fundamental set V for φ in Δ on which φ is univalent.*

Furthermore there also exist:

1) *a domain Ω, which is either the complex plane $\mathbf{C}$ or the unit disc Δ,*
2) *a linear fractional transformation Φ mapping Ω onto Ω,*
3) *an analytic map σ mapping Δ into Ω,*

such that

i) *σ is univalent on V,*
ii) *$\sigma(V)$ is a fundamental set for Φ in Ω,*
iii) $\sigma \circ \varphi = \Phi \circ \sigma$.

Finally, Φ is unique up to a conjugation under a linear fractional transformation mapping Ω onto Ω, and the maps Φ and σ depend only on φ and not on the choice of the fundamental set V.

In the case in which the Wolff point $\tau(\varphi)$ is inside Δ and $\varphi'(\tau) \neq 0$ the existence of a fundamental set for φ on which φ is univalent is an obvious consequence of the local inversion Theorem. If, instead, the Wolff point is on the boundary of Δ then, by Theorem 1.5, $\varphi'(\tau) \neq 0$ and the proof of the existence of a fundamental set for φ is quite subtle and relies upon results due to Pommerenke [8]. As for the second part of the statement of the theorem, the idea of the proof is to «adjoin» to the fundamental set V (on which φ is already proved to be univalent) «points» corresponding to the «negative iterates» of φ. To do this a Riemann surface S is constructed as a quotient space of $V \times \mathbf{N}$, according to the equivalence relation $(v, n) \sim (z, m)$ if and only if there exists $k \in \mathbf{N}$, $k \geq \max(-n, -m)$ so that $\varphi^{n+k}(v) = \varphi^{m+k}(z)$. This Riemann surface S turns out to be equivalent either to the unit disc Δ or to the whole plane $\mathbf{C}$, and φ extends to a linear fractional transformation Φ of Ω.

The following Proposition, [4], establishes a geometric property of the fundamental set of a map φ without fixed points in Δ which will be used in the sequel:

PROPOSITION 2.2. *Let $\varphi \in Hol(\Delta, \Delta)$ be neither a constant map nor an automorphism of Δ and let the Wolff point $\tau(\varphi)$ belong to the boundary of Δ. If, for some point z_0 of Δ, the sequence $\{\varphi^n(z_0)\}_{n \in N}$ converges to $\tau(\varphi)$ non-tangentially, then the fundamental set V of φ (see Theorem 2.1) contains small angular sectors of vertex $\tau(\varphi)$ and opening θ, for all $\theta < \pi$.*

REMARK. One of the purpouses of Theorem 2.1 is to classify holomorphic maps without fixed points in Δ by means of «representing» linear fractional transformations of Ω. For example one can reduce the investigation of the behaviour of $\{\varphi^n\}_{n\in N}$ to the description of the (known) behaviour of $\{\Phi^n\}_{n\in N}$. This also clarifies why Theorem 2.1 does not consider the case of an automorphism of Δ.

If we are given φ, it is natural to ask what Ω and Φ are. One can show that Ω and Φ, up to conjugation, fall into one of the four cases:

1) $\Omega = C$ $\sigma(\tau) = 0$ $\Phi(z) = sz$ $0 < |s| < 1$,
2) $\Omega = \Delta$ $\sigma(\tau) = 1$, $\Phi(z) = ((1+s)z + 1 - s)/((1-s)z + 1 + s)$ $0 < s < 1$,
3) $\Omega = C$ $\sigma(\tau) = \infty$ $\Phi(z) = z + 1$,
4) $\Omega = \Delta$ $\sigma(\tau) = 1$, $\Phi(z) = ((1 \pm 2i)z - 1)/(z - 1 \pm 2i)$.

Deciding which of the four cases a particular φ falls into may be difficult, but, from the study of fixed points of φ and Φ, we can say that case 1 happens if, and only if, the Wolff point τ of φ is in Δ; moreover, in this case $\varphi'(\tau) = s$; case 2 happens if the Wolff point τ of φ is on the boundary of Δ and the value of the derivative of φ at τ is smaller than 1 and, finally, cases 3 and 4 happen when the Wolff point τ of φ is on the boundary of Δ and if the value of the derivative of φ at τ is 1.

We will now define the pseudo-iteration semigroup of a map $f \in Hol(\Delta, \Delta)$

DEFINITION 2.3. Let f and g be holomorphic maps of Δ into Δ. Let τ be the Wolff point of f and suppose that $f'(\tau) \neq 0$. Let V, Ω, σ and Φ as in Theorem 2.1, relative to f. We say that g *is in the pseudo iteration semigroup of* f if there exists a linear fractional transformation Ψ that commutes with Φ, such that $\sigma \circ g = \Psi \circ \sigma$.

It is easy to verify that the set of functions defined above is a semigroup under composition. The main results known concerning the relationship between iteration theory and the study of sets of commuting holomorphic maps on the unit disc are the following, [5],

PROPOSITION 2.4. *Let f be a holomorphic map of Δ into Δ, neither constant nor an automorphism of Δ. Let τ be its Wolff point. If g is in the pseudo-iteration semigroup of f, then there is an integer n such that $f^n \circ g$ and f commute. In particular, if $f'(\tau) = 0$ (which, by Theorem 1.5, implies $\tau \in \Delta$), then $n = 0$, that is f and g commute.*

In the case in which $f'(\tau) = 0$, the proof of Proposition 2.4 relies upon results due to Böttcher on «functional equations» (see, *e.g.*, [5, 10]). If $f'(\tau) \neq 0$ (with $\tau \in \overline{\Delta}$), then Cowen [5] proves the following result, in which he states a condition equivalent to the fact that f and g commute:

PROPOSITION 2.5. *Let f be a holomorphic map of Δ into Δ, neither constant nor an automorphism of Δ and let τ be its Wolff point. Let g be in the pseudo-iteration semigroup of f. If $f'(\tau) \neq 0$, then f and g commute if and only if, there is an open set U in Δ such that $g(U)$ and $g(f(U))$ are contained in the fundamental set V_f of f.*

Commuting holomorphic maps of Δ into Δ belong to a same pseudo-iteration semigroup, namely, [5], PROPOSITION 2.6. *Let f and g be holomorphic maps of Δ into Δ, which commute and which are not automorphisms. Then f and g are in the pseudo-iteration semigroup of $f \circ g$. In particular, if $|f'(\tau)| < 1$, where τ is the Wolff point of f, then g is in the pseudo-iteration semigroup of f.*

Recently it has been proved [3, 1] that, except the case of two hyperbolic automorphisms (case, for other reasons, well known), whenever two holomorphic maps f and g of Δ into Δ commute, they have the same Wolff point. It follows that, if we want that a map g in the same pseudo-iteration semigroup of f commutes with f, we have to ask that the Wolff points of f and g coincide. Taking into account this result, the following theorems identify a relationship between the fact that g is in the pseudo-iteration semigroup of f and the fact that f and g commute under composition.

THEOREM 2.7. *Let $f, g \in Hol(\Delta, \Delta)$ be neither constants nor automorphisms of Δ, and let g be in the pseudo-iteration semigroup of f. Suppose that g and f have the same fixed (Wolff) point $a \in \Delta$. Then f and g commute.*

PROOF. If $f'(a) = 0$, then f and g commute by Proposition 2.4. If $f'(a) \neq 0$, then f and g commute since the condition stated in Proposition 2.5 is fulfilled. q.e.d.

The case in which f and g have no fixed points in Δ is the most interesting and difficult one. We can state the following result using hypotheses on the «behaviour» of the iterates of f and g:

THEOREM 2.8. *Let $f, g \in Hol(\Delta, \Delta)$ be neither constants nor automorphisms of Δ, and let g be in the pseudo-iteration semigroup of f. Suppose that g and f have the same Wolff point $\tau \in \partial\Delta$. If there exist z_0 and w_0 in Δ so that $g^n(z_0) \to \tau$ and $f^n(w_0) \to \tau$ non-tangentially, then f and g commute.*

PROOF. By Proposition 2.5, it is sufficient to prove the existence of an open set U in Δ such that $g(U)$ and $g(f(U))$ are contained in a fundamental set V_f of f. Let A be an open set in Δ, so that $\bar{A} \subset \Delta$. By Lemma 1.7, there exists an angular sector S_α of vertex τ and opening $\alpha < \pi$ so that $\forall n > \bar{n}\ g^n(\bar{A}) \subset S_\alpha$. Now, by Proposition 2.2, there exists a horocycle O_α with center τ, so that $S_\alpha \cap O_\alpha \subset V_f$. Using (1.1), we have

$$\frac{|\tau - f(z)|}{1 - |f(z)|} \leqslant \frac{|\tau - z|}{1 - |z|} \cdot \frac{1 + |f(z)|}{1 + |z|} \cdot \frac{|\tau - z|}{|\tau - f(z)|} . \tag{2.1}$$

The right-hand member of inequality (2.1) is, by Theorem 1.5, bounded from above if we suppose that z belongs to some Stolz region $K(\tau, M)$; so we have proved that a holomorphic map f with Wolff point τ sends Stolz angles of vertex τ (*i.e.* portions of Stolz regions near τ) into Stolz angles of vertex τ. Therefore, we have $f(S_\alpha) \subset S_\beta$, $\beta < \pi$ and $g(S_\beta) \subset S_\gamma$, $\gamma < \pi$. Again, let O_γ be a horocycle of vertex τ so that $S_\gamma \cap O_\gamma \subset V_f$. Let $U = g^{n_0}(A) \subset S_\alpha \cap O_\gamma$ with $n_0 > \bar{n}$. We have $g(U) = g^{n_0+1}(A) \subset O_\gamma \cap S_\alpha$, since, by the Wolff's Lemma, $g(O_\gamma) \subseteq O_\gamma$, and since, by definition of S_α, $g(S_\alpha) \subset S_\alpha$. On the other hand, $g(f(U)) = g(f(g^{n_0}(A))) \subset g(f(S_\alpha \cap O_\gamma)) \subset S_\gamma \cap O_\gamma$, since, by the Wolff's Lemma, $g(f(O_\gamma)) \subseteq O_\gamma$ and since $g(f(S_\alpha)) \subset S_\gamma$. So $g(f(U))$ and $g(U)$ are in V_f. q.e.d.

REMARK. We still do not know whether this result is true in the general case, that is when we only suppose that $f'(\tau) = g'(\tau) = 1$ (but we do not suppose that $\{g^n(z)\}_{n \in N} \to \tau$ and $\{f^n(z)\}_{n \in N} \to \tau$ non-tangentially, for all $z \in \Delta$). In the proof of Theorem 2.8, we used in an essential way the geometric property of the fundamental set V of f stated in Proposition 2.2, so it is impossible to apply the same technique to construct a proof without the hypothesis of the existence of some z_0 in Δ such that $\{f^n(z_0)\}_{n \in N} \to \tau$ non-tangentially. In the proof of theorem 2.1, when there is not any $z_0 \in \Delta$ such that $f^n(z_0) \to \tau$ non-tangentially, Cowen himself [4] has to use a different approach to obtain a fundamental set V for f on which f is univalent. In fact Cowen has to use a result due to Pommerenke, [8], to explain which regions one has to add to a Stolz angle in order to get a set where f (mapping Δ into itself, with Wolff point 1 and angular derivative 1 at the Wolff point) is univalent.

3. FURTHER EXAMPLES AND REMARKS

The following Proposition, [5], relates the value of the derivatives of two commuting analytic maps at their (common) Wolff point.

PROPOSITION 3.1. *If f and g commute and τ is their Wolff point, then*

1) *if* $f'(\tau) = 0 \Rightarrow g'(\tau) = 0$;
2) *if* $0 < |f'(\tau)| < 1 \Rightarrow 0 < |g'(\tau)| < 1$;
3) *if* $f'(\tau) = 1 \Rightarrow g'(\tau) = 1$.

Suppose now that f and g commute and $\tau(f) = \tau(g) \in \partial\Delta$. If $0 < f'(\tau) < 1$, then, by Lemma 1.6, for all $z \in \Delta$, $f^n(z) \to \tau$ non-tangentially. Since by Proposition 3.1 also $0 < g'(\tau) < 1$, then for all $z \in \Delta$, $g^n(z) \to \tau$ non-tangentially. When $f'(\tau) = g'(\tau) = 1$, one cannot say that the iterates of two commuting maps have the same behaviour: consider, for instance $f(z) = C^{-1}(C(z) + 1)$, $g(z) = C^{-1}(C(z) + i)$, where $C\colon \Delta \to H^+ = \{w \in \mathbf{C}\colon \operatorname{Im} w > 0\}$ is the Cayley transformation which sends the Wolff point of f (and of g) to ∞ ; even though f and g commute, $\{f^n\}_{n \in N}$ converges to τ tangentially (*i.e.* $\{f^n(z)\}_{n \in N}$ converges to τ tangentially for all $z \in \Delta$), while $\{g^n\}_{n \in N}$ converges to τ non-tangentially.

Other difficulties arise because we do not know how to characterize the pseudo-iteration semigroup of a map. Something can be done when we are able to «imagine» the geometry of a fundamental set V_f of a holomorphic map f. In this case, in fact, whenever a map g has a fundamental set V_g which intersects V_f, then, if g were in the pseudo-iteration semigroup of f, by Proposition 2.5, f and g would have to commute.

Consider, for example, the family of maps from the upper half plane of $\mathbf{C}$, H^+, into itself defined by $F_a(w) = w - (1/w) + a$, $a \in \mathbf{R}$, $a > 0$. This is a family of holomorphic maps, all with the same Wolff point, having derivative 1 at the Wolff point. The iterates of each map of the family converge to the Wolff point tangentially: take $\varepsilon > 0$ and write $w = x + iy$. We have $\operatorname{Im} F_a(w) = y + y/(x^2 + y^2) < \varepsilon/2 + y$ if and only if $x^2 + y^2 - (2/\varepsilon)y > 0$, and this happens if and only if $w \in H^+$ and w does not belong to the circle D of center i/ε and radius $1/\varepsilon$. Therefore $\operatorname{Im} F_a^n(w) < n\varepsilon/2 + y$ if $F_a^n(w) \notin D$ and since the imaginary part of $F_a^n(w)$ is increasing with n, we have only to suppose that $y > 2/\varepsilon$. As for the real part, we have $\operatorname{Re} F_a(w) = x - x/(x^2 + y^2) + a \geqslant x + a/2$ if and only if $w \in H^+$ and w does not belong to the circle D' of center $1/a$ and radius $1/a$. $\operatorname{Re} F_a^n(w) \geqslant x + na/2$ if and only if $x > 2/a$. Suppose now that w is such that $\operatorname{Im} w > 2/\varepsilon$ and $\operatorname{Re} w > 2/a$; we have

$$\lim_{n\to\infty} \frac{\operatorname{Im} F_a^n(w)}{\operatorname{Re} F_a^n(w)} \leqslant \lim_{n\to\infty} \frac{\operatorname{Im} w + \frac{n\varepsilon}{2}}{\operatorname{Re} w + \frac{na}{2}} = \frac{\varepsilon}{a}.$$

Since ε is arbitrary, the tangential convergence of $\{F_a^n(w)\}$ is proved. Now, consider the family $\{f_a\}_{a \in \mathbf{R}}$, where $f_a(z) = C^{-1}(F_a(C(z)))$, C being the Cayley transformation which sends the Wolff point of f_a to the Wolff point of F_a. For all $a \in \mathbf{R}$, f_a is a holomorphic map of Δ into Δ and has derivative 1 at the Wolff point. Moreover, all the maps f_a of this family have the same Wolff point. Since any two maps F_a, F_b do not commute, then f_a, f_b do not commute. Now, the intersection of the fundamental sets of f_a and f_b is not empty and therefore the two maps cannot belong to the same pseudo-iteration semigroup.

The above example (one of the few known concerning families of holomorphic maps which converge tangentially to their Wolff point) gives an effective idea of how to use Proposition 2.5 and Theorem 2.8: it is seldom possible to find explicitly the pseudo-iteration semigroup of a map; however, sometimes we can deduce from the geometry of the fundamental set (that is, basically, from the properties of the iterates of the function) whether or not another map commutes with the given one. This approach is, in some sense, an attempt to by-pass the (unsolved) problem of describing maps in the same pseudo-iteration semigroup.

Notwithstanding these remarks, the question of deciding if Theorem 2.8 holds without the assumptions on the non-tangential convergence of $f^n(z_0)$ and $g^n(w_0)$ to the Wolff point $\tau(f) = \tau(g)$ remains open.

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